HEAT KERNEL FOR NON-LOCAL OPERATORS WITH VARIABLE ORDER

XIN CHEN, ZHEN-QING CHEN AND JIAN WANG

Abstract. Let $\alpha(x)$ be a measurable function taking values in $[\alpha_1, \alpha_2]$ for $0 < \alpha_1 \leqslant \alpha_2 < 2$, and $\kappa(x, z)$ be a positive measurable function that is symmetric in z and bounded between two positive constants. Under uniform Hölder continuous assumptions on $\alpha(x)$ and $x \mapsto \kappa(x, z)$, we obtain existence, upper and lower bounds, and regularity properties of the heat kernel associated with the following non-local operator of variable order

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} \big(f(x + z) - f(x) - \langle \nabla f(x), z\rangle \mathbb{1}_{\{|z|\leqslant 1\}}\big) \frac{\kappa(x, z)}{|z|^{d+\alpha(x)}}\, dz.$$

In particular, we show that the operator $\mathcal{L}$ generates a conservative Feller process on $\mathbb{R}^d$ having strong Feller property, which is usually assumed a priori in the literature to study analytic properties of $\mathcal{L}$ via probabilistic approaches. Our near-diagonal estimates and lower bound estimates of the heat kernel depend on the local behavior of index function $\alpha(x)$. When $\alpha(x) \equiv \alpha \in (0, 2)$, our results recover some results by Chen and Kumagai (2003) and Chen and Zhang (2016).

Keywords: non-local operator with variable order; stable-like process; heat kernel; Levi's method

MSC 2010: 60G51; 60G52; 60J25; 60J75.

1. Introduction and Main Results

1.1. **Goal and Setting.** In this paper, we study existence, uniqueness and two-sided global estimates for the fundamental solutions of a class of state-dependent non-local operators on $\mathbb{R}^d$ that contains the following fractional Laplacian of variable orders as special cases:

$$\mathcal{L} = -(-\Delta)^{\alpha(x)/2}, \tag{1.1}$$

where $\alpha(x)$ is a Hölder continuous function on $\mathbb{R}^d$ that takes values in a compact subset of $(0, 2)$. The non-local operator $\mathcal{L}$ in (1.1) can be rigorously defined as a pseudo-differential operator $-p(x, D)$ with symbol $p(x, \xi) = -|\xi|^{\alpha(x)}$ via Fourier transform; that is,

$$\mathcal{L}f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\langle ix,\xi\rangle} p(x, \xi) \widehat{f}(\xi)\, d\xi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{i\langle x-y,\xi\rangle} p(x, \xi) f(y)\, dy\, d\xi.$$

When $\alpha(x)$ is a constant $\alpha \in (0, 2)$, $\mathcal{L} = -(-\Delta)^{\alpha/2}$ is the usual fractional Laplacian of order α, which is also the infinitesimal generator of the rotationally symmetric α-stable Lévy process in $\mathbb{R}^d$. We call $\alpha(x)$ the order of the non-local operator $\mathcal{L}$ of (1.1) at the point $x \in \mathbb{R}^d$. One can write $(-\Delta)^{\alpha(x)/2}$ as an integro-differential operator:

$$(-\Delta)^{\alpha(x)/2} f(x) = \int_{\mathbb{R}^d} \big(f(x + z) - f(x) - \langle \nabla f(x), z\rangle \mathbb{1}_{\{|z|\leqslant 1\}}\big) \frac{\kappa(x)}{|z|^{d+\alpha(x)}}\, dz, \quad f \in C_c^2(\mathbb{R}^d),$$

where $\kappa(x) = \frac{\alpha(x)2^{\alpha(x)-1}\Gamma((\alpha(x)+d)/2)}{\pi^{d/2}\,\Gamma(1-(\alpha(x)/2))}$.

The non-local operators with variable order we will consider in this paper are extensions of above and are given by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} \big(f(x + z) - f(x) - \langle \nabla f(x), z\rangle \mathbb{1}_{\{|z|\leqslant 1\}}\big) \frac{\kappa(x, z)}{|z|^{d+\alpha(x)}}\, dz, \quad f \in C_c^2(\mathbb{R}^d), \tag{1.2}$$

where $\alpha : \mathbb{R}^d \to (0, 2)$ is a Hölder continuous function such that

$$\begin{aligned} &0 < \alpha_1 \leqslant \alpha(x) \leqslant \alpha_2 < 2 \quad \text{for } x \in \mathbb{R}^d,\\ &|\alpha(x) - \alpha(y)| \leqslant c_1(|x - y|^{\beta_1} \wedge 1) \quad \text{for } x, y \in \mathbb{R}^d, \end{aligned} \tag{1.3}$$

X. Chen: Department of Mathematics, Shanghai Jiao Tong University, 200240, Shanghai, P.R. China. Email: `chenxin217@sjtu.edu.cn`.

Z.-Q. Chen: Department of Mathematics, University of Washington, Seattle, WA 98195, USA. Email: `zqchen@uw.edu`.

J. Wang: College of Mathematics and Informatics & Fujian Key Laboratory of Mathematical Analysis and Applications (FJKLMAA), Fujian Normal University, 350007 Fuzhou, P.R. China. Email: `jianwang@fjnu.edu.cn`.

for some constants $c_1 \in (0,\infty)$ and $\beta_1 \in (0,1]$, and $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ is a measurable function satisfying

$$\begin{aligned} &\kappa(x,z) = \kappa(x,-z) \quad \text{for } x,z \in \mathbb{R}^d,\\ &0 < \kappa_1 \leqslant \kappa(x,z) \leqslant \kappa_2 < \infty \quad \text{for } x,z \in \mathbb{R}^d,\\ &|\kappa(x,z) - \kappa(y,z)| \leqslant c_2(|x-y|^{\beta_2} \wedge 1) \quad \text{for } x,y,z \in \mathbb{R}^d, \end{aligned} \tag{1.4}$$

for some constants $c_2 \in (0,\infty)$ and $\beta_2 \in (0,1]$. Set $\beta_0 = \beta_1 \wedge \beta_2$. Clearly (1.3) and (1.4) hold with β_0 in place of β_1 and β_2, respectively.

1.2. **Background.** Non-local operators arise naturally in the study of stochastic processes with jumps. Various properties of the non-local operator $\mathcal{L}$ have been intensively investigated both from the analytic and the probabilistic point of view.

We first recall some known results for the constant order case, i.e. $\alpha(x) \equiv \alpha \in (0,2)$ for all $x \in \mathbb{R}^d$. The regularity properties of the operator $\mathcal{L}$ of (1.2) including the Hölder continuity of harmonic functions, the Schauder estimates and the L^p estimates were studied in [2, 10, 11, 12, 18, 29, 47, 48, 52]. When the operator $\mathcal{L}$ is symmetric with respect to the Lebesgue measure, existence and two-sided estimates of the heat kernel associated with $\mathcal{L}$ were obtained in [16, 17], and the corresponding parabolic Harnack inequality was also established there. For general $\kappa(x,z)$ satisfying (1.4), the associated heat kernel was constructed by Levi's method in [20], where the corresponding two-sided estimates and gradient estimates were given, and the parabolic equation (see (1.5) below) was also verified. Recently the extension of [20] from the Lévy kernel $1/|z|^{d+\alpha}$ to that of a class of subordinate Brownian motions was presented in [32]. See also [9, 26, 31, 33, 34, 36, 38, 40, 49] for the construction of certain Lévy type processes via Levi's (parametrix) method. Besides, existence and uniqueness (i.e. well-posedness) of the martingale problem for $\mathcal{L}$ were proved in [5, 23, 28] under some mild continuous conditions on $\kappa(x,z)$.

Existence and well-posedness of the martingale solution for variable order fractional Laplacian $\mathcal{L} = -(-\Delta)^{\alpha(x)/2}$ were established in Bass [1], which immediately yields that there is a strong Markov process corresponding to $-(-\Delta)^{\alpha(x)/2}$. (This process was called the stable-like process in [1]). A few properties for such stable-like process (including existence of heat kernel, sample path properties and the long-time behaviors) have been investigated in [6, 25, 30, 44, 45, 46] by using the theory of pseudo-differential operators and Fourier analysis. When $\kappa(x,z) = \kappa(x)$ is independent of z and both $\alpha(x)$ and $\kappa(x)$ have uniformly bounded continuous derivatives, the heat kernel for $\mathcal{L}$ was constructed formally in [35, Section 5] via Duhamel's formula, and some upper bound estimates for heat kernel were also given there. However it seems that there are some problems in [35, Section 2], which are used in the rest of that paper. (The issue is that for symmetric stable Lévy processes, the estimate (3.6) in [35, Proposition 3.1] would imply that the Lévy measure $\mu(dz)$ has a density $j(z)$ with respect to the Lebesgue measure that is comparable to $|z|^{-(d+\alpha)}$. This is stronger than and certainly can not be deduced from the assumption (2.4) in [35] as claimed there, even if one assumes a priori that the Lévy measure $\mu(dz)$ has a smooth density.) Duhamel's formula has also been adopted in [19, 50] to study the fundamental solution to fractional Laplacian or Laplacian perturbed by non-local operators. We also mention that there are some results for the case that $\kappa(x,z)$ depends only on z. For instance, the regularity for harmonic functions or the semigroups associated with $\mathcal{L}$ were studied in [3, 41, 47], and the elliptic Harnack inequality was obtained in [4]. Recently, the existence and uniqueness of the martingale problem, as well as the existence of Feller process enjoying the strong Feller property have been investigated in [39] for a class of locally α-stable Lévy-type operators.

From all the results mentioned above, we can see that there are already a lot of developments related to non-local operators of variable order. However, the following questions, which should be fundamental and interesting, were still unknown.

(1) When $\alpha(x)$ is not a constant, how can we construct a fundamental solution of $\mathcal{L}$ that really satisfies the parabolic equation (1.5) below? What are upper and lower bound estimates for this solution (if it exists)? We point out that it had not been established that, even under additional smoothness assumptions on $\alpha(x)$ and $\kappa(x)$, the heat kernel constructed in [35] by Duhamel formula is actually a solution to the equation (1.5).

(2) When $\alpha(x)$ is not a constant and $\kappa(x,z)$ depends on z, is there a strong Markov process associated with $\mathcal{L}$? In literature, the existence of strong Markov process was always assumed a priori in the study of regularity of harmonic functions and elliptic Harnack inequalities for such non-local operator $\mathcal{L}$, see [3, 4, 41] for examples.

(3) In existing literature, regularity for the solution to parabolic equation (1.5) associated with $\mathcal{L}$ usually depends on the uniform bounds α_1 and α_2 in (1.3). Can one establish regularity of the fundamental solution in some neighborhood of x_0 (such as gradient estimate) in terms of $\alpha(x_0)$ under some special settings?

1.3. **Main results.** The aim of this paper is to address these questions. We show that there is a Feller process associated with $\mathcal{L}$ and it has strong Feller property. We establish the existence and uniqueness of fundamental solutions to $\mathcal{L}$ as well as their regularity properties.

For $x_0 \in \mathbb{R}^d$ and $r > 0$, define $B(x_0, r) := \{x \in \mathbb{R}^d : |x - x_0| < r_0\}$. Let $C_{b,u}(\mathbb{R}^d)$ denote the set of all bounded and uniformly continuous functions on $\mathbb{R}^d$. The following are two of the main results of our paper.

Theorem 1.1. *Suppose that conditions* (1.3) *and* (1.4) *hold. If $\kappa(x,z) = \kappa(x)$ is independent of z, then there exists a jointly continuous non-negative function $p : (0,1] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+ := [0,\infty)$ such that for every $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$\frac{\partial p(t,x,y)}{\partial t} = \mathcal{L}p(t,\cdot,y)(x), \tag{1.5}$$

and has the following properties.

(i) **(Upper bounds)** *For every $\gamma, c_0 > 0$, there exist positive constants $c_1 = c_1(\alpha,\kappa,c_0)$ and $c_2 = c_2(\alpha,\kappa,\gamma,c_0)$ such that for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$p(t,x,y) \leqslant \begin{cases} c_1 t^{-d/\alpha(x)}, & |x-y| < c_0 t^{1/\alpha(x)}, \\ \frac{c_1 t}{|x-y|^{d+\alpha_2}} \wedge \frac{c_2 t^{1-\gamma}}{|x-y|^{d+\alpha(x)}}, & c_0 t^{1/\alpha(x)} \leqslant |x-y| < 1, \\ \frac{c_1 t}{|x-y|^{d+\alpha_1}}, & |x-y| \geqslant 1. \end{cases} \tag{1.6}$$

If, in addition, there are some $x_0 \in \mathbb{R}^d$ and $r_0 \in (0,\infty]$ such that $\alpha(z) = \alpha(x_0)$ for all $z \in B(x_0,r_0)$, then there is a positive constant $c_3 = c_3(\alpha,\kappa,r_0)$ such that for every $t \in (0,1]$ and $y \in \mathbb{R}^d$,

$$p(t,x_0,y) \leqslant \begin{cases} \frac{c_3 t}{(t^{1/\alpha(x_0)} + |y-x_0|)^{d+\alpha(x_0)}}, & |y-x_0| < r_0/2, \\ \frac{c_3 t}{|y-x_0|^{d+\alpha_1}}, & |y-x_0| \geqslant r_0/2; \end{cases} \tag{1.7}$$

(Lower bounds) *There exists a constant $c_4 = c_4(\alpha,\kappa) > 0$ such that for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$p(t,x,y) \geqslant \frac{c_4 t}{(t^{1/\alpha(x)} + |x-y|)^{d+\alpha(x)}}. \tag{1.8}$$

(ii) **(Hölder regularity and gradient estimates)** *For any $\gamma \in (0,\alpha_2]$, there exist positive constants $c_5 = c_5(\alpha,\kappa,\gamma)$ and $R_1 = R_1(\alpha,\kappa,\gamma)$ such that for all $t \in (0,1]$ and $x, x', y \in \mathbb{R}^d$ with $|x - x'| \leqslant R_1$,*

$$\big|p(t,x,y) - p(t,x',y)\big| \leqslant c_5 |x-x'|^{(\alpha(x)-\gamma)_+ \wedge 1} \big(\rho_{\gamma_0}^{y,0}(t,x-y) + \rho_{\gamma_0}^{y,0}(t,x'-y)\big), \tag{1.9}$$

where $\gamma_0 = \gamma/(2\alpha_2)$ and

$$\rho_\gamma^{y,\beta}(t,x) = t^\gamma(|x|^\beta \wedge 1) \begin{cases} \frac{1}{(t^{1/\alpha(y)} + |x|)^{d+\alpha(y)}}, & |x| \leqslant 1, \\ \frac{1}{|x|^{d+\alpha_1}}, & |x| > 1. \end{cases} \tag{1.10}$$

If moreover $\tilde\beta_0(x_0) := \big(\alpha_1\beta_0/\alpha(x_0)\big) \wedge \alpha_1 > 1 - \alpha(x_0)$ for some $x_0 \in \mathbb{R}^d$, then for every fixed $t \in (0,1]$ and $y \in \mathbb{R}^d$, $p(t,\cdot,y)$ is differentiable at $x = x_0$. In this case, for every $\gamma > 0$, $t \in (0,1]$ and $y \in \mathbb{R}^d$,

$$|\nabla p(t,\cdot,y)(x_0)| \leqslant c_6\, \rho^{y,0}_{1-(1/\alpha(x_0))+(\beta_0^*/\alpha_2)-(\beta_0^*/\alpha_1)-\gamma}(t,x_0-y) \tag{1.11}$$

for some $c_6 = c_6(\alpha,\kappa,\gamma,x_0) > 0$, where $\beta_0^* := \beta_0 \wedge \alpha_2$.

(iii) **(Chapman-Kolmogorov equation)** *For every $s, t \in (0,1]$ with $s + t \leqslant 1$,*

$$\int_{\mathbb{R}^d} p(s,x,z)p(t,z,y)\,dz = p(s+t,x,y), \quad x, y \in \mathbb{R}^d. \tag{1.12}$$

(iv) **(Conservativeness)** *For every $(t,x) \in (0,1] \times \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} p(t,x,y)\,dy = 1. \tag{1.13}$$

(v) **(Strong continuity and generator)** *For every $f \in C_{b,u}(\mathbb{R}^d)$, let*

$$u_f(t,x) := \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy.$$

Then

$$\mathcal{L}u_f(t,\cdot)(x) \text{ exists pointwise}, \quad t \mapsto \mathcal{L}u_f(t,)(x) \text{ is continuous}, \tag{1.14}$$

$$\frac{\partial u_f(t,x)}{\partial t} = \mathcal{L}u_f(t,\cdot)(x), \quad t \in (0,1],\ x \in \mathbb{R}^d, \tag{1.15}$$

and

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}|u_f(t,x)-f(x)|=0. \tag{1.16}$$

For every $f\in C^2_{b,u}(\mathbb{R}^d):=C_{b,u}(\mathbb{R}^d)\cap C^2(\mathbb{R}^d)$, *we also have*

$$\frac{\partial u_f(t,x)}{\partial t}=\int_{\mathbb{R}^d}p(t,x,y)\mathcal{L}f(y)\,dy,\quad t\in(0,1],\ x\in\mathbb{R}^d \tag{1.17}$$

and

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\Big|\frac{1}{t}\cdot\big(u_f(t,x)-f(x)\big)-\mathcal{L}f(x)\Big|=0. \tag{1.18}$$

(vi) **(Uniqueness)** *Jointly continuous function* $p(t,x,y)$ *on* $(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$ *that is bounded for each* $t>0$ *and satisfies* (1.9) *and* (1.14)-(1.16) *is unique.*

Remark 1.2. (1) From (1.6) and (1.8), we know that both upper and lower bounds of near-diagonal estimates for $p(t,x,y)$ enjoy the same order $t^{-d/\alpha(x)}$, which typically depends on x. In particular, along with (1.8), when $r_0=\infty$ in (1.7) (i.e. $\alpha(x)\equiv\alpha\in(0,2)$ for all $x\in\mathbb{R}^d$), we arrive at the two-sided heat kernel estimates obtained in [20]. Furthermore, by (1.9) and (1.11), the regularity of $p(t,\cdot,y)$ at $x=x_0$ only depends on x_0.

(2) Compared with these results yielded by Duhamel's formula in [35], Theorem 1.1(v) shows that the function $u_f(t,x):=\int_{\mathbb{R}^d}p(t,x,y)f(y)\,dy$ solves the Cauchy problem for $\mathcal{L}$. In particular, by (1.5) and (1.16), we have

$$\begin{cases}\frac{\partial}{\partial t}p(t,x,y)=\mathcal{L}p(t,\cdot,y)(x)\\ \lim_{t\downarrow 0}p(t,x,y)=\delta_y(x),\end{cases}$$

which means that $p(t,x,y)$ is the fundamental solution associated with $\mathcal{L}$. It is easy to deduce from (1.9), (1.12), (1.13), (1.15) and (1.17) that there is a conservative Feller process having the strong Feller property associated with the operator $\mathcal{L}$; cf. Proposition 5.7 below. Recall that a Markov process on $\mathbb{R}^d$ is said to be a Feller process if its transition semigroup is a strongly continuous semigroup in the Banach space of continuous functions that vanish at infinity equipped with the uniform norm. A Markov process is said to have the strong Feller property if its transition semigroup maps bounded measurable functions to bounded continuous functions.

Theorem 1.3. *Let* $\beta_0^{**}\in(0,\beta_0]\cap(0,\alpha_2/2)$. *For general* $\kappa(x,z)$ *satisfying* (1.3) *and* (1.4), *if* $(\alpha_2/\alpha_1)-1<\beta_0^{**}/\alpha_2$, *there exists a jointly continuous non-negative function* $p:(0,1]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}_+$ *such that all the conclusions in Theorem* 1.1 *hold except that the upper bounds* (1.6)-(1.7) *in* (i) *and Hölder regularity and gradient estimates* (1.9) *and* (1.11) *in* (ii) *are to be replaced by the following estimates.*

(i) **(Upper bounds)** *For any* $\gamma,c_0>0$, *there exist constants* $c_1=c_1(\alpha,\kappa,c_0)>0$ *and* $c_2=c_1(\alpha,\kappa,\gamma,c_0)>0$ *such that for all* $t\in(0,1]$ *and* $x,y\in\mathbb{R}^d$,

$$p(t,x,y)\leqslant\begin{cases}c_1t^{-d/\alpha(x)}, & |x-y|<c_0t^{1/\alpha(x)},\\ \frac{c_1t^{2-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_2}}\wedge\frac{c_2t^{2-(\alpha_2/\alpha_1)-\gamma}}{|x-y|^{d+\alpha(x)}}, & c_0t^{1/\alpha(x)}\leqslant|x-y|<1,\\ \frac{c_1t^{2-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_1}}, & |x-y|\geqslant 1.\end{cases} \tag{1.19}$$

If, in addition, there are some $x_0\in\mathbb{R}^d$ *and* $r_0\in(0,\infty]$ *such that* $\alpha(z)=\alpha(x_0)$ *for all* $z\in B(x_0,r_0)$, *then for any* $c_0>0$ *there is a positive constant* $c_3:=c_3(\alpha,\kappa,r_0,c_0)$ *so that for every* $t\in(0,1]$ *and* $y\in\mathbb{R}^d$,

$$p(t,x_0,y)\leqslant\begin{cases}c_3t^{-d/\alpha(x_0)}, & |y-x_0|<c_0t^{1/\alpha(x_0)},\\ \frac{c_3t^{2-(\alpha_2/\alpha_1)}}{|y-x_0|^{d+\alpha(x_0)}}, & c_0t^{1/\alpha(x_0)}\leqslant|y-x_0|<r_0/2,\\ \frac{c_3t^{2-(\alpha_2/\alpha_1)}}{|x-y_0|^{d+\alpha_1}}, & |y-x_0|\geqslant r_0/2.\end{cases} \tag{1.20}$$

(ii) **(Hölder regularity and gradient estimates)** *For every* $\gamma\in(0,\alpha_1]$, *there exist constants* $c_5=c_5(\alpha,\kappa,\gamma)$ *and* $R_1=R_1(\alpha,\kappa,\gamma)>0$ *such that for all* $t\in(0,1]$ *and* $x,x',y\in\mathbb{R}^d$ *with* $|x-x'|\leqslant R_1$,

$$\big|p(t,x,y)-p(t,x',y)\big|\leqslant c_5|x-x'|^{(\alpha_1-\gamma)_+\wedge 1}\big(\rho^{y,0}_{\gamma_1}(t,x-y)+\rho^{y,0}_{\gamma_1}(t,x'-y)\big), \tag{1.21}$$

where $\gamma_1:=1-(\alpha_2/\alpha_1)+(\gamma/(2\alpha_2))$. *Let* β_0^* *and* $\tilde\beta_0(x_0)$ *be the same constants in Theorem* 1.1 *(ii). If moreover* $\tilde\beta_0(x_0)-(\alpha_1\alpha_2/\alpha(x_0))\big((\alpha_2/\alpha_1)-1\big)>1-\alpha_1$ *for some* $x_0\in\mathbb{R}^d$, *then for every fixed*

$t \in (0,1]$ and $y \in \mathbb{R}^d$, $p(t,\cdot,y)$ is differentiable at $x = x_0$. In this case, for every $\gamma > 0$, there exists a constant $c_6 := c_6(\alpha,\kappa,\gamma,x_0)$ such that for all $t \in (0,1]$ and $y \in \mathbb{R}^d$,

$$|\nabla p(t,\cdot,y)(x_0)| \leqslant c_6 \rho^{x,0}_{1-(1/\alpha_1)+(\beta_0^*/\alpha_2)-(\beta_0^*/\alpha_1)-\gamma_2}(t, x_0 - y), \tag{1.22}$$

where $\gamma_2 = (\alpha_2/\alpha_1) - 1 + \gamma$.

Remark 1.4. (1) According to Theorem 1.3 above, when $\alpha(x)$ is not a constant function and $\kappa(x,z)$ depends on z, we can construct heat kernel associated with the operator $\mathcal{L}$ when $(\alpha_2/\alpha_1)-1 < \beta_0^{**}/\alpha_2$, which indicates that the oscillation of the index function $\alpha(x)$ could not be too large. We note that in [4], the elliptic Harnack inequality for $\mathcal{L}$ was established under similar assumptions on the index function $\alpha(x)$. Note that by (1.20) and (1.8), when $\alpha(x) \equiv \alpha \in (0,2)$ for all $x \in \mathbb{R}^d$, two-sided estimates of heat kernel in Theorem 1.3 are reduced to those in [20].

(2) As mentioned in Remark 1.2(2), Theorem 1.3 indeed provides us some sufficient conditions for the existence of a strong Markov process (in fact a Feller process having the strong Feller property) associated with $\mathcal{L}$ when $\kappa(x,z)$ depends on z. Note that, the existence of such strong Markov process is required and assumed a prior in the proofs of the Hölder continuity of harmonic functions in [3] and the elliptic Harnack inequality for $\mathcal{L}$ in [4].

We conclude this part with the following remarks on some possible extensions of Theorems 1.1 and 1.3.

Remark 1.5. (1) By carefully checking the arguments of Theorem 1.1, one can see that Theorem 1.1 still holds true if the condition that $\kappa(x,z)$ is independent of z is replaced by the following two assumptions on $\kappa(x,z)$:

(i) For every fixed $x \in \mathbb{R}^d$, $\kappa(x,\cdot) : \mathbb{R}^d \to \mathbb{R}_+$ is a radial function, i.e., $\kappa(x,z)$ only depends on $|z|$.

(ii) Let $j(x,r) = \widetilde{\kappa}(x,r) r^{-d-\alpha(x)}$ for $x \in \mathbb{R}^d$ and $r > 0$, where $\widetilde{\kappa}(x,r) := \kappa(x,z)$ with $|z| = r$. For every $x \in \mathbb{R}^d$, $j(x,\cdot)$ is non-increasing and differentiable on $(0,\infty)$ such that the function $r \mapsto -\frac{1}{r}\frac{\partial j(x,r)}{\partial r}$ is non-increasing.

(2) Let $\widetilde{\mathcal{L}} f(x) = \mathcal{L} f(x) + \langle b(x), \nabla f(x)\rangle$, where $b : \mathbb{R}^d \to \mathbb{R}^d$ is bounded and uniformly Hölder continuous. It is possible to extend Theorems 1.1 and 1.3 to $\widetilde{\mathcal{L}}$ under the assumption that $\alpha_1 > 1$, When $\alpha_1 > 1$, the gradient perturbation $\langle b(x), \nabla\rangle$ is of lower order than $\mathcal{L}$. We refer the readers to [8, 13, 14, 27, 42, 43, 51] on heat kernel estimates associated with non-local operators under gradient perturbation. For heat kernel estimates associated with non-local operators under non-local perturbations, see [19, 50, 15].

(3) By combining with the approach from [23], it is possible to remove the symmetry assumption of $\kappa(x,z)$ in z. But we will not pursue this extension in this paper.

1.4. **Idea of the proofs: Levi's method.** In this paper, we will apply the Levi's method to construct the fundamental solution $p(t,x,y)$ for $\mathcal{L}$. Some ideas of our approach are inspired by those in [20, 35].

We first introduce some notation which will be frequently used in this article, and then briefly mention the ideas of our approach. Due to the symmetry assumption $\kappa(x,z) = \kappa(x,-z)$, it is easy to see that

$$\mathcal{L} f(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_{\{|z|>\varepsilon\}} \big(f(x+z) + f(x-z) - 2f(x)\big) \frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\, dz. \tag{1.23}$$

For simplicity, throughout this paper we write $\mathcal{L} f(x)$ as

$$\mathcal{L} f(x) = \frac{1}{2} \int_{\mathbb{R}^d} \big(f(x+z) + f(x-z) - 2f(x)\big) \frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\, dz.$$

Note that the above integral is absolutely convergent for bounded C^2 functions.

For fixed $y \in \mathbb{R}^d$, define

$$\mathcal{L}^y f(x) := \int_{\mathbb{R}^d} \big(f(x+z) - f(x) - \langle \nabla f(x), z\rangle \mathbb{1}_{\{|z|\leqslant 1\}}\big) \frac{\kappa(y,z)}{|z|^{d+\alpha(y)}}\, dz. \tag{1.24}$$

Then $\mathcal{L}^y$ is the generator of a pure jump symmetric Lévy process $X^y := (X^y_t)_{t\geqslant 0}$ with jump measure $\nu^y(dz) = \frac{\kappa(y,z)}{|z|^{d+\alpha(y)}}\, dz$. We denote the fundamental solution for $\mathcal{L}^y$ by $p^y : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}_+$, which is just the transition density of the process X^y. The fundamental solution p^y satisfies that

$$\frac{\partial p^y(t,x)}{\partial t} = \mathcal{L}^y p^y(t,\cdot)(x) \quad \text{for every } (t,x) \in (0,1] \times \mathbb{R}^d.$$

We remark here that although the operator $\mathcal{L}^y$ is clearly well defined on $C^2_c(\mathbb{R}^d)$. It is also pointwisely well defined for the function $x \mapsto p^y(t,x)$; see the estimates in [20, Theorem 2.4].

Throughout the paper, we define for $f:\mathbb{R}^d\to\mathbb{R}$,

$$\delta_f(x;z):=f(x+z)+f(x-z)-2f(x),\qquad x,z\in\mathbb{R}^d.$$

$$\delta_{p^y}(t,x;z):=p^y(t,x+z)+p^y(t,x-z)-2p^y(t,x),\quad t\in(0,\infty),\ x,y,z\in\mathbb{R}^d.$$

Then, for every $x,y,w\in\mathbb{R}^d$,

$$\mathcal{L}^w p^y(t,\cdot)(x)=\frac{1}{2}\int_{\mathbb{R}^d}\delta_{p^y}(t,x;z)\frac{\kappa(w,z)}{|z|^{d+\alpha(w)}}\,dz. \tag{1.25}$$

According to the Levi's method (cf. [24, pp. 310–311]), we look for the fundamental solution to (1.5) of the following form:

$$p(t,x,y)=p^y(t,x-y)+\int_0^t\int_{\mathbb{R}^d}p^z(t-s,x-z)q(s,z,y)\,dz\,ds, \tag{1.26}$$

where $q(t,x,y)$ solves

$$q(t,x,y)=q_0(t,x,y)+\int_0^t\int_{\mathbb{R}^d}q_0(t-s,x,z)q(s,z,y)\,dz\,ds \tag{1.27}$$

with $q_0(t,x,y)=(\mathcal{L}^x-\mathcal{L}^y)p^y(t,\cdot)(x-y)$. So the main task of the remainder of this paper is to solve equation (1.26), show it is indeed the unique fundamental solution of $\mathcal{L}$, and derive its various properties. The next three sections are devoted to the estimates for $q_0(t,x,y)$, $q(t,x,y)$ and $p(t,x,y)$ respectively. The proofs of main results are presented in Section 5.

We will mainly follow the approach of [20], where the jumping kernel is of type $\kappa(x,z)/|z|^{d+\alpha}$ for some $\alpha\in(0,2)$. However, due to the variable order nature of the operator $\mathcal{L}$ given by (1.2), there are many new challenges and difficulties. In order to obtain good estimates and regularity of heat kernel for $\mathcal{L}$ in terms of the local behavior of the index function $\alpha(x)$, we need to introduce the key function $\rho_\gamma^{y,\beta}$ (see (1.10)), which involves the variable order $\alpha(x)$. This brings us a lot of difficulties from the beginning of applying the Levi's method. In comparison to [20, Section 2.1], we need take into account the variable index function $\alpha(x)$ in some key convolution inequalities, see Section 3.1 for more details. In the derivation of explicit upper bounds for $q_0(t,x,y)$ in Proposition 2.5, we need consider the variation of both $\kappa(x,z)$ and $\alpha(x)$ in dealing with the difference between $\mathcal{L}^x p^y(t,\cdot)$ and $\mathcal{L}^y p^y(t,\cdot)$. These causes a lot of complications in proofs for the crucial Proposition 3.4, Proposition 4.6 and Proposition 4.7.

Notation For any $a,b\in\mathbb{R}_+$, $a\wedge b:=\min\{a,b\}$ and $a\vee b:=\max\{a,b\}$. For every measurable function $f,g:(0,1]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}_+$, the notation $f\asymp g$ means that there exists a constant $1\leqslant c_0<\infty$ such that $c_0^{-1}g(t,x,y)\leqslant f(t,x,y)\leqslant c_0 g(t,x,y)$ for every $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$, and the notation $f\preceq g$ (resp. $f\succeq g$) means that there exists a constant $0<c_1<\infty$ such that $f(t,x,y)\leqslant c_1 g(t,x,y)$ (resp. $f(t,x,y)\geqslant c_1 g(t,x,y)$) for every $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$.

2. Estimates for $q_0(t,x,y)$

2.1. **Preliminary estimates.** For each fixed $y\in\mathbb{R}^d$, $\beta\in[0,\infty)$, $\gamma\in\mathbb{R}$ and $R\in(0,\infty)$, we define a function $\rho_{\gamma,R}^{y,\beta}:(0,1]\times\mathbb{R}^d\to(0,\infty)$ as follows

$$\rho_{\gamma,R}^{y,\beta}(t,x):=t^\gamma\left(|x|^\beta\wedge 1\right)\begin{cases}\left(t^{1/\alpha(y)}+|x|\right)^{-(d+\alpha(y))}, & |x|\leqslant R,\\ |x|^{-(d+\alpha_1)}, & |x|>R,\end{cases}$$

where $\alpha(x)$ is the index function in (1.2) and α_1 is the lower bound of $\alpha(x)$ in (1.3). It is easy to see that for any $0<R_1\leqslant R_2<\infty$, there is a constant $c>1$ such that $c^{-1}\rho_{\gamma,R_1}^{y,\beta}\leqslant\rho_{\gamma,R_2}^{y,\beta}\leqslant c\rho_{\gamma,R_1}^{y,\beta}$ for all $y\in\mathbb{R}^d$ and $\beta\in[0,\infty)$. Therefore, without loss of generality, we may and do assume that $R=1$ in the definition of $\rho_{\gamma,R}^{y,\beta}$, and we write $\rho_{\gamma,1}^{y,\beta}$ as $\rho_\gamma^{y,\beta}$, which is exactly the one defined by (1.10).

For every $y\in\mathbb{R}^d$ and $s>0$, define $\overline{\alpha}(y;s):=\sup\{\alpha(z):|z-y|\leqslant s\}$ and $\underline{\alpha}(y;s):=\inf\{\alpha(z):|z-y|\leqslant s\}$. For simplicity, we write $\overline{\alpha}(y;|x-y|)$ and $\underline{\alpha}(y;|x-y|)$ as $\overline{\alpha}(y;x)$ and $\underline{\alpha}(y;x)$, respectively.

We begin with the following simple lemma.

Lemma 2.1. *For every $\beta\in[0,\alpha_2)$, $t\in(0,1]$ and $x\in\mathbb{R}^d$,*

$$\int_{\mathbb{R}^d}\rho_0^{z,\beta}(t,x-z)\,dz\preceq\left(t^{(\beta/\alpha(x))-1}\vee 1\right)\left(1+|\log t|\mathbb{1}_{\{\beta=\alpha(x)\}}\right)\preceq t^{(\beta/\alpha_2)-1}\vee 1. \tag{2.1}$$

Proof. Since $\beta \in (0,\alpha_2)$, it suffices to prove the first inequality in (2.1).

For every $x,z\in\mathbb{R}^d$ with $|x-z|\leqslant 1$,

$$\begin{aligned}|x-z|^{-\alpha(z)}&=|x-z|^{-\alpha(x)}|x-z|^{\alpha(x)-\alpha(z)}\preceq|x-z|^{-\alpha(x)}\cdot\exp\left(|\log|x-z||\,|\alpha(x)-\alpha(z)|\right)\\&\preceq|x-z|^{-\alpha(x)}\cdot\exp\left(C\,|\log|x-z||\cdot|x-z|^{\beta_0}\right)\preceq|x-z|^{-\alpha(x)},\end{aligned}\tag{2.2}$$

where in the last inequality we have used

$$\sup_{z\in\mathbb{R}^d:|z|\leqslant 1}\exp\left(C|\log|z||\cdot|z|^{\beta_0}\right)<\infty.$$

Then, for all $t\in(0,1]$ and $x,z\in\mathbb{R}^d$,

$$\begin{aligned}\frac{1}{(t^{1/\alpha(z)}+|x-z|)^{d+\alpha(z)}}&\preceq\begin{cases}|x-z|^{-d-\alpha(z)}, & |x-z|\leqslant 1,\\ t^{-1-(d/\underline{\alpha}(x;t^{1/\alpha_2}))}, & |x-z|\leqslant t^{1/\alpha(x)},\end{cases}\\&\preceq\begin{cases}|x-z|^{-d-\alpha(x)}, & |x-z|\leqslant 1,\\ t^{-1-(d/\underline{\alpha}(x;t^{1/\alpha_2}))}, & |x-z|\leqslant t^{1/\alpha(x)}.\end{cases}\end{aligned}$$

Therefore, by the definition of $\rho_0^{z,\beta}$ (see (1.10)), for all $t\in(0,1]$ and $x\in\mathbb{R}^d$, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\rho_0^{z,\beta}(t,x-z)\,dz&\preceq\int_{\{|z-x|>1\}}\frac{1}{|x-z|^{d+\alpha_1}}\,dz+\int_{\{t^{1/\alpha(x)}<|z-x|\leqslant 1\}}\frac{|x-z|^\beta}{|x-z|^{d+\alpha(x)}}\,dz\\&\quad+t^{-1-(d/\underline{\alpha}(x;t^{1/\alpha_2}))}\int_{\{|z-x|\leqslant t^{1/\alpha(x)}\}}|x-z|^\beta\,dz\\&=:J_1+J_2+J_3.\end{aligned}$$

It is easy to verify that $J_1\preceq 1$ and $J_2\preceq\left(t^{(\beta-\alpha(x))/\alpha(x)}\vee 1\right)\left(1+|\log t|\mathbb{1}_{\{\beta=\alpha(x)\}}\right)$, due to the assumption that $\beta\in[0,\alpha_2)$. At the same time,

$$\begin{aligned}J_3&\preceq t^{-1-(d/\underline{\alpha}(x;t^{1/\alpha_2}))}\cdot t^{(\beta+d)/\alpha(x)}\preceq t^{(\beta-\alpha(x))/\alpha(x)}\exp\left(d|\log t|\frac{\alpha(x)-\underline{\alpha}(x;t^{1/\alpha_2})}{\alpha(x)\underline{\alpha}(x;t^{1/\alpha_2})}\right)\\&\preceq t^{(\beta-\alpha(x))/\alpha(x)}\exp\left(c|\log t|t^{\beta_0/\alpha_2}\right)\preceq t^{(\beta-\alpha(x))/\alpha(x)},\end{aligned}$$

where in the forth inequality we have used

$$\frac{\alpha(x)-\underline{\alpha}(x;t^{1/\alpha_2})}{\alpha(x)\underline{\alpha}(x;t^{1/\alpha_2})}\preceq t^{\beta_0/\alpha_2},$$

thanks to (1.3). Combining all the estimates above, we get the first inequality in (2.1). The proof is complete. □

Remark 2.2. By the proof of [20, Lemma 2.1(i)], we can obtain that for every $\beta\in[0,\alpha_2)$, $t\in(0,1]$ and $x\in\mathbb{R}^d$,

$$\int_{\mathbb{R}^d}\rho_0^{x,\beta}(t,x-z)\,dz\preceq\left(t^{(\beta/\alpha(x))-1}\vee 1\right)\left(1+|\log t|\mathbb{1}_{\{\beta=\alpha(x)\}}\right)\preceq t^{(\beta/\alpha_2)-1}\vee 1.\tag{2.3}$$

Note that although the definition of $\rho_0^{x,\beta}(t,x)$ here is a little different from that in [20] when $|x|>1$, the proof of [20, Lemma 2.1] still works for the first inequality in (2.3). We emphasize that in the present setting we need estimate (2.1), where in the integrand $\rho_0^{z,\beta}$ the index $\alpha(z)$ depends on z.

Lemma 2.3. *For every $t\in(0,1]$ and $x,y,w\in\mathbb{R}^d$, define*

$$\begin{aligned}I_1(t,x,y,w)&:=\int_{\{|z|\leqslant 1\}}\left[\left(t^{-2/\alpha(y)}|z|^2\right)\wedge 1\right]\rho_1^{y,0}(t,w+z)\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz,\\I_2(t,x,y,w)&:=\int_{\{|z|>1\}}\rho_1^{y,0}(t,w+z)\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(y;x)}}\,dz,\\I_3(t,x,y)&:=\int_{\mathbb{R}^d}\left[\left(t^{-2/\alpha(y)}|z|^2\right)\wedge 1\right]\left(\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\mathbb{1}_{\{|z|\leqslant 1\}}+\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(y;x)}}\mathbb{1}_{\{|z|>1\}}\right)dz.\end{aligned}\tag{2.4}$$

Then there exists a constant $c_1:=c_1(\alpha,\kappa)$ such that for every $x,y,w\in\mathbb{R}^d$ and $t\in(0,1]$,

$$I_1(t,x,y,w)\leqslant\begin{cases}c_1(1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\rho_0^{y,0}(t,w), & |w|\leqslant 1/2,\\ c_1\left[1+|\log|w||+(1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\right]\rho_0^{y,0}(t,w), & |w|>1/2;\end{cases}\tag{2.5}$$

$$I_2(t,x,y,w) \leqslant \begin{cases} c_1\rho_1^{y,0}(t,w), & |w|\leqslant 1/2, \\ c_1\big(1+|\log|w||\big)\rho_0^{y,0}(t,w), & |w|>1/2; \end{cases} \tag{2.6}$$

and

$$I_3(t,x,y) \leqslant c_1(1+|\log t|)t^{-(\overline{\alpha}(y;x)/\alpha(y))}. \tag{2.7}$$

Proof. We only prove (2.5) and (2.6), since the proof of (2.7) is similar and more direct. We denote $I_1(t,x,y,w)$ and $I_2(t,x,y,w)$ by I_1 and I_2, respectively. Write

$$\begin{aligned} I_1 &= \int_{\{|z|\leqslant t^{1/\alpha(y)}\}} \big(t^{-2/\alpha(y)}|z|^2\big)\rho_1^{y,0}(t,w{+}z)\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz + \int_{\{t^{1/\alpha(y)}<|z|\leqslant 1\}} \rho_1^{y,0}(t,w{+}z)\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz \\ &=: I_{11}+I_{12}. \end{aligned}$$

We will divide the proof into the following three subcases.

Case (a): $0\leqslant |w|\leqslant 2t^{1/\alpha(y)}$.

Since, by (1.10),

$$\rho_1^{y,0}(t,w{+}z) \preceq t^{-d/\alpha(y)} \qquad \text{for all } t\in(0,1] \text{ and } y,z,w\in\mathbb{R}^d, \tag{2.8}$$

we have

$$\begin{aligned} I_{11} &\preceq t^{-(d+2)/\alpha(y)}\int_{\{|z|\leqslant t^{1/\alpha(y)}\}} \frac{(1+|\log|z|)|z|^2}{|z|^{d+\overline{\alpha}(y;x)}}\,dz \\ &\preceq (1+|\log t|)t^{-(d+\overline{\alpha}(y;x))/\alpha(y)} \preceq (1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\rho_0^{y,0}(t,w), \end{aligned}$$

where in the last inequality we have used the fact that

$$t^{-d/\alpha(y)} \preceq \rho_1^{y,0}(t,w) \qquad \text{when } |w|\leqslant 2t^{1/\alpha(y)}. \tag{2.9}$$

Applying (2.8) and (2.9) to I_{12} and I_2 again, we get

$$\begin{aligned} I_{12} &\preceq t^{-d/\alpha(y)}\int_{\{t^{1/\alpha(y)}<|z|\leqslant 1\}} \frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz \preceq (1+|\log t|)t^{-(d+\overline{\alpha}(y;x))/\alpha(y)} \\ &\preceq (1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\rho_0^{y,0}(t,w) \end{aligned}$$

and

$$I_2 \preceq t^{-d/\alpha(y)}\int_{\{|z|>1\}} \frac{1+\log|z|}{|z|^{d+\alpha_1}}\,dz \preceq \rho_1^{y,0}(t,w).$$

Case (b): $2t^{1/\alpha(y)}<|w|\leqslant 1/2$.

When $|w|>2t^{1/\alpha(y)}$ and $|z|\leqslant t^{1/\alpha(y)}$, $|w{+}z|\geqslant |w|-|w|/2\geqslant |w|/2$, which along with (1.10) implies that

$$\rho_0^{y,0}(t,w{+}z) \preceq \frac{1}{(t^{1/\alpha(y)}+|w{+}z|)^{d+\alpha(y)}} \preceq \frac{1}{(t^{1/\alpha(y)}+|w|)^{d+\alpha(y)}}. \tag{2.10}$$

Therefore,

$$\begin{aligned} I_{11} &\preceq \frac{t^{1-2/\alpha(y)}}{(t^{1/\alpha(y)}+|w|)^{d+\alpha(y)}}\int_{\{|z|\leqslant t^{1/\alpha(y)}\}} \frac{(1+|\log|z||)|z|^2}{|z|^{d+\overline{\alpha}(y;x)}}\,dz \\ &\preceq (1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\frac{1}{(t^{1/\alpha(y)}+|w|)^{d+\alpha(y)}} \preceq (1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\rho_0^{y,0}(t,w), \end{aligned}$$

where the last step is due to the fact that

$$\frac{1}{(t^{1/\alpha(y)}+|w|)^{d+\alpha(y)}} \preceq \rho_0^{y,0}(t,w) \qquad \text{when } |w|>2t^{1/\alpha(y)}. \tag{2.11}$$

On the other hand, we have the following decomposition for I_{12}:

$$\begin{aligned} I_{12} &= \int_{\{t^{1/\alpha(y)}<|z|\leqslant |w|/2\}} \rho_1^{y,0}(t,w{+}z)\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz + \int_{\{|w|/2\leqslant|z|\leqslant 1\}} \rho_1^{y,0}(t,w{+}z)\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz \\ &=: I_{121}+I_{122}. \end{aligned}$$

If $|z|\leqslant |w|/2$, then $|w{+}z|\geqslant |w|/2$, and so (2.10) still holds. Hence, we have

$$\begin{aligned} I_{121} &\preceq \frac{t}{(t^{1/\alpha(y)}+|w|)^{d+\alpha(y)}}\int_{\{|z|>t^{1/\alpha(y)}\}} \frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz \\ &\preceq (1+|\log t|)t^{-\overline{\alpha}(y;x)/\alpha(y)}\frac{t}{(t^{1/\alpha(y)}+|w|)^{d+\alpha(y)}} \preceq (1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\rho_0^{y,0}(t,w), \end{aligned}$$

where the last step follows from (2.11). Meanwhile,

$$\begin{aligned} I_{122} &\preceq \frac{1+|\log|w||}{|w|^{d+\overline{\alpha}(y;x)}}\int_{\mathbb{R}^d}\rho_1^{y,0}(t,w+z)\,dz \preceq \frac{1+|\log|w||}{|w|^{d+\overline{\alpha}(y;x)}} \\ &\preceq \frac{(1+|\log|w||)|w|^{\alpha(y)-\overline{\alpha}(y;x)}}{(t^{1/\alpha(y)}+|w|)^{d+\alpha(y)}} \preceq (1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\rho_0^{y,0}(t,w), \end{aligned}$$

where the second inequality is due to (2.3), in the third inequality we have used the fact that

$$|w|^{-d-\alpha(y)} \preceq (t^{1/\alpha(y)}+|w|)^{-d-\alpha(y)} \quad \text{when } 2t^{1/\alpha(y)}<|w|\leqslant 1/2,$$

and the fourth inequality follows from

$$(1+|\log|w||)|w|^{\alpha(y)-\overline{\alpha}(y;x)} \preceq (1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))} \quad \text{when } 2t^{1/\alpha(y)}<|w|\leqslant 1/2.$$

Combining with all the estimates above, we obtain

$$I_1 \preceq (1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\rho_0^{y,0}(t,w).$$

Furthermore, note that if $|w|\leqslant 1/2$ and $|z|>1$, then $|w+z|\geqslant |z|-|w|\geqslant 2|w|-|w|=|w|$. Therefore, for every $y,z,w\in\mathbb{R}^d$ and $t\in(0,1]$ with $|z|>1$ and $2t^{1/\alpha(y)}\leqslant|w|\leqslant 1/2$,

$$\begin{aligned} \rho_0^{y,0}(t,w+z) &\preceq \begin{cases} |w|^{-d-\alpha(y)}, & |w+z|\leqslant 1,\\ |w|^{-d-\alpha_1}, & |w+z|>1 \end{cases} \\ &\preceq |w|^{-d-\alpha(y)}, \end{aligned}$$

which implies that

$$I_2 \preceq \frac{t}{|w|^{d+\alpha(y)}}\int_{\{|z|\geqslant 1\}}\frac{1+\log|z|}{|z|^{d+\alpha_1}}\,dz \preceq \frac{t}{|w|^{d+\alpha(y)}} \preceq \rho_1^{y,0}(t,w).$$

Case (c): $|w|>1/2$.

By adjusting the constants properly, it is easy to verify that

$$\begin{aligned} I_1+I_2 &\preceq \int_{\{|z|\leqslant 1/4\}}\big[\big(t^{-2/\alpha(y)}|z|^2\big)\wedge 1\big]\rho_1^{y,0}(t,w+z)\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz+\int_{\{|z|>1/4\}}\rho_1^{y,0}(t,w+z)\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(y;x)}}\,dz \\ &=: J_1+J_2. \end{aligned}$$

Note that when $|w|>1/2$ and $|z|\leqslant 1/4$, $|w+z|\geqslant|w|-|z|\geqslant|w|/2>1/4$, which implies that

$$\rho_1^{y,0}(t,w+z)\preceq\frac{t}{|w|^{d+\alpha_1}}.$$

On the other hand,

$$\begin{aligned} &\int_{\{|z|\leqslant 1\}}\big[\big(t^{-2/\alpha(y)}|z|^2\big)\wedge 1\big]\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz \\ &\preceq \int_{\{|z|\leqslant t^{1/\alpha(y)}\}}\big(t^{-2/\alpha(y)}|z|^2\big)\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz+\int_{\{t^{1/\alpha(y)}<|z|\leqslant 1\}}\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz \\ &\preceq (1+|\log t|)t^{-\overline{\alpha}(y;x)/\alpha(y)}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} J_1 &\preceq \frac{t}{|w|^{d+\alpha_1}}\int_{\{|z|\leqslant 1/4\}}\big[\big(t^{-2/\alpha(y)}|z|^2\big)\wedge 1\big]\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\,dz \\ &\preceq \frac{\big(1+|\log t|\big)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}}{|w|^{d+\alpha_1}} \preceq \big(1+|\log t|\big)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\rho_0^{y,0}(t,w). \end{aligned}$$

Meanwhile,

$$\begin{aligned} J_2 &\preceq \int_{\{|z|>|w|/2\}}\rho_1^{y,0}(t,w+z)\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(y;x)}}\,dz+\int_{\{1/4<|z|<|w|/2\}}\rho_1^{y,0}(t,w+z)\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(y;x)}}\,dz \\ &\preceq \frac{1+|\log|w||}{|w|^{d+\underline{\alpha}(y;x)}}\int_{\{|z|>|w|/2\}}\rho_1^{y,0}(t,w+z)\,dz+\frac{t}{|w|^{d+\alpha_1}}\int_{\{|z|>1/4\}}\frac{1+|\log|z||}{|z|^{d+\alpha_1}}\,dz \\ &\preceq \frac{1+|\log|w||}{|w|^{d+\alpha_1}} \preceq (1+|\log|w||)\rho_0^{y,0}(t,w), \end{aligned}$$

where the second step above follows from the following property

$$\rho_0^{y,0}(t,w+z) \preceq |w|^{-d-\alpha_1} \quad \text{when } |z|<|w|/2 \text{ and } |w|>1/2,$$

and the third step is due to (2.3). Hence, for all $w\in\mathbb{R}^d$ with $|w|>1/2$,

$$I_1 \preceq J_1+J_2 \preceq \big[1+|\log|w||+(1+|\log t|)t^{-\overline{\alpha}(y;x)/\alpha(y)}\big]\rho_0^{y,0}(t,w)$$

and

$$I_2 \preceq J_2 \preceq (1+|\log|w||)\rho_0^{y,0}(t,w).$$

Combining all the three cases together, we finish the proof. □

Remark 2.4. Following the arguments of (2.5) and (2.6) for case (c) above, we can get the following for every $c_0>0$, $t\in(0,1]$ and $x,y,w\in\mathbb{R}^d$ with $|w|>2c_0$:

(i)

$$\int_{\{|z|\leqslant c_0\}}\big[\big(t^{-2/\alpha(y)}|z|^2\big)\wedge 1\big]\big(\rho_1^{y,0}(t,w+z)+\rho_1^{y,0}(t,w)\big)\frac{1}{|z|^{d+\alpha_2}}\,dz \preceq \frac{t^{1-\alpha_2/\alpha_1}}{|w|^{d+\alpha_1}} \tag{2.12}$$

and

$$\begin{aligned}&\int_{\{|z|\leqslant c_0\}}\big[\big(t^{-2/\alpha(y)}|z|^2\big)\wedge 1\big]\big(\rho_1^{y,0}(t,w+z)+\rho_1^{y,0}(t,w)\big)\frac{1+|\log|z||}{|z|^{d+\alpha_2}}\,dz\\ &\preceq \big(1+|\log t|+|\log|w||\big)\frac{t^{1-\alpha_2/\alpha_1}}{|w|^{d+\alpha_1}}.\end{aligned} \tag{2.13}$$

(ii)

$$\int_{\{|z|>c_0\}}\big(\rho_1^{y,0}(t,w\pm z)+\rho_1^{y,0}(t,w)\big)\frac{1}{|z|^{d+\alpha_1}}\,dz \preceq \frac{1}{|w|^{d+\alpha_1}} \tag{2.14}$$

and

$$\int_{\{|z|>c_0\}}\big(\rho_1^{y,0}(t,w+z)+\rho_1^{y,0}(t,w)\big)\frac{1+|\log|z||}{|z|^{d+\alpha_1}}\,dz \preceq \log(1+|w|)\frac{1}{|w|^{d+\alpha_1}}. \tag{2.15}$$

2.2. Upper bounds for $q_0(t,x,y)$.

Proposition 2.5. *(1) Suppose $\kappa(x,z)=\kappa(x)$ is independent of z. Then for any $\gamma>0$, there exist constants $R_0:=R_0(\alpha,\kappa,\gamma)\in(0,1)$ and $c_0:=c_0(\alpha,\kappa,\gamma)>0$ such that for all $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,*

$$\begin{aligned}|q_0(t,x,y)| &\leqslant c_0\big(|x-y|^{\beta_0}\wedge 1\big)\cdot\begin{cases}\frac{t^{-\gamma}}{(t^{1/\alpha(y)}+|x-y|)^{d+\alpha(y)}}, & |x-y|<R_0,\\ \frac{1}{|x-y|^{d+\alpha_1}}, & |x-y|\geqslant R_0\end{cases}\\ &\preceq \rho_{-\gamma}^{y,\beta_0}(t,x-y).\end{aligned} \tag{2.16}$$

Suppose, in addition, that there are some $y_0\in\mathbb{R}^d$ and $r_0\in(0,\infty]$ such that $\alpha(z)=\alpha(y_0)$ for all $z\in B(y_0,r_0)$. Then there exists a constant $c_1:=c_1(\alpha,\kappa,\gamma,r_0)>0$ such that for every $t\in(0,1]$ and $x\in\mathbb{R}^d$,

$$|q_0(t,x,y_0)|\leqslant c_1\big(|x-y_0|^{\beta_0}\wedge 1\big)\cdot\begin{cases}\frac{1}{(t^{1/\alpha(y_0)}+|x-y_0|)^{d+\alpha(y_0)}}, & |x-y_0|<r_0,\\ \frac{1}{|x-y_0|^{d+\alpha_1}}, & |x-y_0|\geqslant r_0.\end{cases} \tag{2.17}$$

(2) For general $\kappa(x,z)$ and for any $\gamma>0$, there are constants $R_0:=R_0(\alpha,\kappa,\gamma)\in(0,1)$ and $c_0:=c_0(\alpha,\kappa,\gamma)>0$ such that for all $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,

$$\begin{aligned}|q_0(t,x,y)| &\leqslant c_0\big(|x-y|^{\beta_0}\wedge 1\big)\cdot\begin{cases}\frac{t^{-\gamma}}{(t^{1/\alpha(y)}+|x-y|)^{d+\alpha(y)}}, & |x-y|<R_0,\\ \frac{t^{1-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_1}}, & |x-y|\geqslant R_0\end{cases}\\ &\preceq \rho_{1-(\alpha_2/\alpha_1)-\gamma}^{y,\beta_0}(t,x-y).\end{aligned} \tag{2.18}$$

Suppose, in addition, that there are some $y_0\in\mathbb{R}^d$ and $r_0\in(0,\infty]$ such that $\alpha(z)=\alpha(y_0)$ for all $z\in B(y_0,r_0)$. Then for every $t\in(0,1]$ and $x\in\mathbb{R}^d$,

$$|q_0(t,x,y_0)|\leqslant c_1\big(|x-y_0|^{\beta_0}\wedge 1\big)\cdot\begin{cases}\frac{1}{(t^{1/\alpha(y_0)}+|x-y_0|)^{d+\alpha(y_0)}}, & |x-y_0|<r_0,\\ \frac{t^{1-(\alpha_2/\alpha_1)}}{|x-y_0|^{d+\alpha_1}}, & |x-y_0|\geqslant r_0.\end{cases} \tag{2.19}$$

To prove Proposition 2.5, we need some regularity estimates for $p^y(t,\cdot)$ taken from [20]. For simplicity, we use the following abbreviation for a function f:

$$f(x\pm y):=f(x+y)+f(x-y).$$

Lemma 2.6. ([20, Lemmas 2.2 and 2.3]) *The following statements hold.*

(1) If $\kappa(x,z)=\kappa(x)$ is independent of z, then for any $j\in\mathbb{Z}_+:=\{0,1,\dots\}$, there exists a constant $c_1:=c_1(\alpha,\kappa,j)>0$ such that for every $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,

$$|\nabla^j p^y(t,\cdot)(x)|\leqslant c_1 t\big(t^{1/\alpha(y)}+|x|\big)^{-d-\alpha(y)-j}. \tag{2.20}$$

(2) For general $\kappa(x,z)$, there exists a constant $c_2:=c_2(\alpha,\kappa)>0$ such that for every $t\in(0,1]$ and $x,x',y,z\in\mathbb{R}^d$,

$$p^y(t,x)\leqslant c_2\rho_1^{y,0}(t,x), \tag{2.21}$$

$$|\delta_{p^y}(t,x;z)|\leqslant c_2\big[\big(t^{-2/\alpha(y)}|z|^2\big)\wedge 1\big]\cdot\big(\rho_1^{y,0}(t,x\pm z)+\rho_1^{y,0}(t,x)\big), \tag{2.22}$$

$$|p^y(t,x)-p^y(t,x')|\leqslant c_2\big[\big(t^{-1/\alpha(y)}|x-x'|\big)\wedge 1\big]\cdot\big(\rho_1^{y,0}(t,x')+\rho_1^{y,0}(t,x)\big), \tag{2.23}$$

and

$$\begin{aligned}|\delta_{p^y}(t,x;z)-\delta_{p^y}(t,x';z)|\leqslant c_2&\big[\big(t^{-1/\alpha(y)}|x-x'|\big)\wedge 1\big]\cdot\big[\big(t^{-2/\alpha(y)}|z|^2\big)\wedge 1\big]\\ &\times\big(\rho_1^{y,0}(t,x\pm z)+\rho_1^{y,0}(t,x)+\rho_1^{y,0}(t,x'\pm z)+\rho_1^{y,0}(t,x')\big).\end{aligned} \tag{2.24}$$

Lemma 2.7. *For every $\gamma>0$, there exist constants $R_1:=R_1(\alpha,\kappa,\gamma)\in(0,1/2)$ and $c_1:=c_1(\alpha,\kappa,\gamma)>0$ such that for every $t\in(0,1]$ and $x,y,w\in\mathbb{R}^d$ with $|x-y|\leqslant R_1$*

$$\big|\big(\mathcal{L}^x-\mathcal{L}^y\big)p^y(t,w)\big|\leqslant c_1\big(|x-y|^{\beta_0}\wedge 1\big)\cdot\begin{cases}t^{-\gamma}\rho_0^{y,0}(t,w), & |w|\leqslant 1/2,\\ (t^{-\gamma}+|\log|w||)\rho_0^{y,0}(t,w), & |w|>1/2.\end{cases} \tag{2.25}$$

In particular, for every $t\in(0,1]$ and $x,y\in\mathbb{R}^d$ with $|x-y|\leqslant R_1$,

$$\big|q_0(t,x,y)\big|\preceq\rho_{-\gamma}^{y,\beta_0}(t,x-y). \tag{2.26}$$

If, in addition, there are some $y_0\in\mathbb{R}^d$ and $r_0\in(0,\infty]$ such that $\alpha(z)=\alpha(y_0)$ for all $z\in B(y_0,r_0)$, then there exists a constant $c_2:=c_2(\alpha,\kappa,\gamma,r_0)>0$ such that for every $t\in(0,1]$ and $x\in\mathbb{R}^d$ with $|x-y_0|<r_0$,

$$|q_0(t,x,y_0)|\leqslant c_2\big(|x-y_0|^{\beta_0}\wedge 1\big)\cdot\frac{1}{(t^{1/\alpha(y_0)}+|x-y_0|)^{d+\alpha(y_0)}}. \tag{2.27}$$

Proof. (i) Applying the mean value theorem to the function $r\mapsto a^r$, we find that for every $x,y,z\in\mathbb{R}^d$ with $|z|>0$,

$$\begin{aligned}&\left|\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}-\frac{\kappa(y,z)}{|z|^{d+\alpha(y)}}\right|\\ &\preceq|\alpha(x)-\alpha(y)|\left(\frac{|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\mathbb{1}_{\{|z|\leqslant 1\}}+\frac{\log|z|}{|z|^{d+\underline{\alpha}(y;x)}}\mathbb{1}_{\{|z|>1\}}\right)+\frac{|\kappa(x,z)-\kappa(y,z)|}{|z|^{d+\alpha(y)}}\\ &\preceq\big(|x-y|^{\beta_0}\wedge 1\big)\left(\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\mathbb{1}_{\{|z|\leqslant 1\}}+\frac{1+\log|z|}{|z|^{d+\underline{\alpha}(y;x)}}\mathbb{1}_{\{|z|>1\}}\right),\end{aligned} \tag{2.28}$$

where in the last inequality we have used (1.4), (1.3) and the fact that

$$\frac{1}{|z|^{d+\alpha(y)}}\preceq\frac{1}{|z|^{d+\overline{\alpha}(y;x)}}\mathbb{1}_{\{|z|\leqslant 1\}}+\frac{1}{|z|^{d+\underline{\alpha}(y;x)}}\mathbb{1}_{\{|z|>1\}}.$$

Combining (2.28) with (1.25), we can obtain that

$$\begin{aligned}\big|\big(\mathcal{L}^x-\mathcal{L}^y\big)p^y(t,w)\big|&\preceq\big(|x-y|^{\beta_0}\wedge 1\big)\int_{\mathbb{R}^d}|\delta_{p^y}(t,w;z)|\left(\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\mathbb{1}_{\{|z|\leqslant 1\}}+\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(y;x)}}\mathbb{1}_{\{|z|>1\}}\right)dz\\ &\preceq\big(|x-y|^{\beta_0}\wedge 1\big)\cdot\big(I_1(t,x,y,w)+I_2(t,x,y,w)+\rho_1^{y,0}(t,w)I_3(t,x,y)\big),\end{aligned}$$

where in the second inequality we have used (2.22), and I_1, I_2 and I_3 are defined in (2.4). Hence, Lemma 2.3 yields that

$$\begin{aligned}&\big|\big(\mathcal{L}^x-\mathcal{L}^y\big)p^y(t,w)\big|\\ &\preceq\big(|x-y|^{\beta_0}\wedge 1\big)\cdot\begin{cases}(1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\rho_0^{y,0}(t,w), & |w|\leqslant 1/2,\\ \big(1+|\log|w||+(1+|\log t|)t^{1-(\overline{\alpha}(y;x)/\alpha(y))}\big)\rho_0^{y,0}(t,w), & |w|>1/2.\end{cases}\end{aligned} \tag{2.29}$$

Note that due to (1.3),

$$\overline{\alpha}(y;x) \leqslant \alpha(y) + c_0|x-y|^{\beta_0} \leqslant \alpha(y) + c_0 R_1^{\beta_0} \quad \text{for all } x, y \in \mathbb{R}^d \text{ with } |x-y| \leqslant R_1.$$

Therefore, choosing $R_1 \in (0, 1/2)$ small enough such that $c_0 R_1^{\beta_0}/\alpha_1 < \gamma$, we have

$$(1 + |\log t|) t^{1-(\overline{\alpha}(y;x)/\alpha(y))} \preceq t^{-\gamma} \quad \text{when } |x-y| \leqslant R_1.$$

This, along with (2.29), immediately implies (2.25).

Estimate (2.26) follows from (2.25) by taking $w = x - y$.

(ii) Suppose, in addition, that there are some $y_0 \in \mathbb{R}^d$ and $r_0 \in (0, \infty]$ such that $\alpha(x) = \alpha(y_0)$ for all $x \in B(y_0, r_0)$. Then (2.28) is reduced to

$$\left| \frac{\kappa(x,z)}{|z|^{d+\alpha(x)}} - \frac{\kappa(y_0,z)}{|z|^{d+\alpha(y_0)}} \right| \preceq \left(|x-y|^{\beta_0} \wedge 1\right) \cdot \left(\frac{1}{|z|^{d+\alpha(y_0)}} \mathbb{1}_{\{|z| \leqslant 1\}} + \frac{1}{|z|^{d+\alpha(y_0)}} \mathbb{1}_{\{|z|>1\}} \right)$$

for all $x \in B(y_0, r_0)$. Having this at hand and repeating the same argument as above, we can easily see that the second assertion holds in this case. □

Now, we are in the position to present the

Proof of Proposition 2.5. Let $R_1 = R_1(\alpha, \kappa, \gamma)$, where $R_1(\alpha, \kappa, \gamma)$ is the constant in Lemma 2.7. According to Lemma 2.7, we only need to treat the case that $|x - y| > R_1$.

(1) We first assume that $\kappa(x, z)$ is independent of z.

By (1.25), for every $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$ with $|x - y| > R_1$, we have

$$\begin{aligned} |\mathcal{L}^x p^y(t,\cdot)(x-y)| &\preceq \int_{\{|z| \leqslant R_1/2\}} |\delta_{p^y}(t, x-y; z)| \frac{1}{|z|^{d+\alpha(x)}}\, dz \\ &\quad + \int_{\{|z|>R_1/2\}} \left(p^y(t, x-y \pm z) + p^y(t, x-y)\right) \frac{1}{|z|^{d+\alpha(x)}}\, dz \\ &=: J_1 + J_2. \end{aligned} \tag{2.30}$$

Applying (2.20) and using the mean value theorem, we can get

$$\begin{aligned} J_1 &\preceq \int_{\{|z| \leqslant R_1/2\}} |\nabla^2 p^y(t,\cdot)(x-y+\theta_{x,y,z}z)| |z|^{2-d-\alpha(x)}\, dz \\ &\preceq \int_{\{|z| \leqslant R_1/2\}} \frac{t}{(t^{1/\alpha(y)} + |x-y+\theta_{x,y,z}z|)^{d+\alpha(y)+2}} |z|^{2-d-\alpha(x)}\, dz \\ &\preceq \frac{t}{|x-y|^{d+\alpha(y)+2}} \int_{\{|z| \leqslant R_1/2\}} |z|^{2-d-\alpha(x)}\, dz \preceq \frac{t}{|x-y|^{d+\alpha(y)+2}} \preceq \frac{1}{|x-y|^{d+\alpha_1}} = \rho_0^{y,0}(t, x-y), \end{aligned}$$

where in the first inequality $\theta_{x,y,z}$ is a constant depending on x, y, z such that $|\theta_{x,y,z}| \leqslant 1$, and the third inequality follows from the fact that

$$|x-y+\theta_{x,y,z}z| \geqslant |x-y|/2 \geqslant R_1/2 \geqslant C_0 t^{1/\alpha(y)} \quad \text{for any } z \in \mathbb{R}^d \text{ with } |z| \leqslant R_1/2.$$

On the other hand, according to (2.21) and (2.14), we have

$$J_2 \preceq \int_{\{|z|>R_1/2\}} \rho_1^{y,0}(t, x-y \pm z) \frac{1}{|z|^{d+\alpha(x)}}\, dz + \rho_1^{y,0}(t, x-y) \int_{\{|z|>R_1/2\}} \frac{1}{|z|^{d+\alpha(x)}}\, dz \preceq \rho_0^{y,0}(t, x-y). \tag{2.31}$$

Hence, for every $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$ with $|x - y| > R_1$,

$$|\mathcal{L}^x p^y(t,\cdot)(x-y)| \preceq \rho_0^{y,0}(t, x-y).$$

Following the same argument above, we have for every $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$ with $|x - y| > R_1$,

$$|\mathcal{L}^y p^y(t,\cdot)(x-y)| \preceq \frac{1}{|x-y|^{d+\alpha_1}} \preceq \rho_0^{y,0}(t, x-y). \tag{2.32}$$

Therefore, for every $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$ with $|x - y| > R_1$,

$$|q_0(t,x,y)| \leqslant |\mathcal{L}^x p^y(t,\cdot)(x-y)| + |\mathcal{L}^y p^y(t,\cdot)(x-y)| \preceq \rho_0^{y,0}(t, x-y).$$

By now we have shown the second case in the estimate (2.16). From (2.27) and (2.16), we arrive at (2.17) immediately.

(2) Now we suppose the general case that $\kappa(x,z)$ may depend on z. We still define J_1 and J_2 by those in (2.30). It is not difficult to verify that (2.31) also holds for this case. On the other hand, according to (2.22), it holds that

$$\begin{aligned} J_1 &\preceq \int_{\{|z|\leqslant R_1/2\}} \left[(t^{-2/\alpha(y)}|z|^2)\wedge 1\right]\cdot \rho_1^{y,0}(t,x-y\pm z)\frac{1}{|z|^{d+\alpha(x)}}\,dz \\ &\quad + \rho_1^{y,0}(t,x-y)\int_{\{|z|\leqslant R_1/2\}} \left[(t^{-2/\alpha(y)}|z|^2)\wedge 1\right]\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz \\ &\preceq \frac{t^{1-2/\alpha(y)}}{|x-y|^{d+\alpha_1}}\int_{\{|z|\leqslant t^{1/\alpha(y)}\}}|z|^{2-d-\alpha(x)}\,dz + \frac{t}{|x-y|^{d+\alpha_1}}\int_{\{t^{1/\alpha(y)}\leqslant|z|\leqslant R_1/2\}}|z|^{-d-\alpha(x)}\,dz \\ &\quad + \frac{t}{|x-y|^{d+\alpha_1}}\int_{\{|z|\leqslant R_1/2\}}\left[(t^{-2/\alpha(y)}|z|^2)\wedge 1\right]|z|^{-d-\alpha(x)}\,dz \\ &\preceq \frac{t^{1-(\overline{\alpha}(y;x)/\alpha(y))}}{|x-y|^{d+\alpha_1}} \leqslant \frac{t^{1-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_1}}. \end{aligned} \tag{2.33}$$

Here the second step follows from

$$\rho_0^{y,0}(t,x-y+z)\preceq\frac{1}{|x-y|^{d+\alpha_1}}\qquad\text{when } |x-y|>R_1 \text{ and } |z|\leqslant R_1/2,$$

which is due to the fact that for $|x-y|>R_1$ and $|z|\leqslant R_1/2$

$$|x-y+z|\geqslant|x-y|-|z|\geqslant|x-y|/2\geqslant R_1/2\geqslant C_0t^{1/\alpha(y)}.$$

Note also that (2.32) is still true for this case. Combining all the estimates above, we obtain

$$|q_0(t,x,y)|\preceq|\mathcal{L}^xp^y(t,x-y)|+|\mathcal{L}^yp^y(t,x-y)|\preceq J_1+J_2+|\mathcal{L}^yp^y(t,x-y)|\preceq t^{1-(\alpha_2/\alpha_1)}\rho_0^{y,0}(t,x-y),$$

where in the last step we used the fact that $t^{-\overline{\alpha}(y;x)/\alpha(y)}\leqslant t^{-\alpha_2/\alpha_1}$ for all $t\in(0,1]$. Thus, the second case of (2.18) is proved. □

According to the proof of Proposition 2.5, we also have the following estimates for $q_0(t,x,y)$.

Proposition 2.8. *(1) Suppose $\kappa(x,z)$ is independent of z. Then there exists a constant $c_1:=c_1(\alpha,\kappa)>0$ such that for every $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,*

$$\begin{aligned} |q_0(t,x,y)| &\leqslant \begin{cases} c_1t^{-1-(d/\alpha(y))}, & |x-y|\leqslant t^{1/\alpha(y)},\\ \frac{c_1}{|x-y|^{d+\alpha(x)}}+\frac{c_1}{|x-y|^{d+\alpha(y)}}, & |x-y|>t^{1/\alpha(y)} \end{cases} \\ &\leqslant \begin{cases} c_1t^{-1-(d/\alpha(y))}, & |x-y|\leqslant t^{1/\alpha(y)},\\ \frac{c_1}{|x-y|^{d+\alpha_2}\wedge|x-y|^{d+\alpha_1}}, & |x-y|>t^{1/\alpha(y)}. \end{cases} \end{aligned} \tag{2.34}$$

(2) Suppose $\kappa(x,z)$ depends on z and $\beta_0/\alpha_2>(\alpha_2/\alpha_1)-1$. Then there exists a constant $c_2:=c_2(\alpha,\kappa)>0$ such that for every $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,

$$|q_0(t,x,y)|\leqslant\begin{cases} c_2t^{-1-(d/\alpha(y))}, & |x-y|\leqslant t^{1/\alpha(y)},\\ \frac{c_1t^{1-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_2}\wedge|x-y|^{d+\alpha_1}}, & |x-y|>t^{1/\alpha(y)}. \end{cases} \tag{2.35}$$

Proof. (1) We first suppose that $\kappa(x,z)$ is independent of z.

Case (a): $|x-y|\leqslant t^{1/\alpha(y)}$.

According to (2.26), we can choose $0<\gamma<\beta_0/\alpha_2$ and adjust the constants properly to get that

$$\begin{aligned} |q_0(t,x-y)| &\preceq \rho_{-\gamma}^{y,\beta_0}(t,x-y)\preceq t^{-\gamma}|x-y|^{\beta_0}\cdot\rho_0^{y,0}(t,x-y) \\ &\preceq t^{(\beta_0/\alpha_2)-\gamma}\rho_0^{y,0}(t,x-y)\preceq\rho_0^{y,0}(t,x-y)\preceq t^{-(d/\alpha(y))-1}, \end{aligned} \tag{2.36}$$

where in the third inequality we have used the fact that $|x-y|\leqslant t^{1/\alpha(y)}\leqslant t^{1/\alpha_2}$, and the fourth inequality follows from $\gamma<\beta_0/\alpha_2$.

Case (b): $|x-y|>t^{1/\alpha(y)}$.

By (1.25), it holds that

$$\begin{aligned}|\mathcal{L}^x p^y(t,\cdot)(x-y)| &\preceq \int_{\{|z|\leqslant|x-y|/2\}} |\delta_{p^y}(t,x-y;z)|\frac{1}{|z|^{d+\alpha(x)}}\,dz\\&\quad+\int_{\{|z|>|x-y|/2\}} |\delta_{p^y}(t,x-y;z)|\frac{1}{|z|^{d+\alpha(x)}}\,dz\\&=: J_1+J_2.\end{aligned}\tag{2.37}$$

Then, using the mean value theorem, we have

$$\begin{aligned}J_1 &\preceq \int_{\{|z|\leqslant|x-y|/2\}} \left|\nabla^2 p^y(t,x-y+\theta_{x,y,z}z)\right|\cdot|z|^2\frac{1}{|z|^{d+\alpha(x)}}\,dz\\&\preceq \int_{\{|z|\leqslant|x-y|/2\}} \frac{t}{\left(t^{1/\alpha(y)}+|x-y+\theta_{x,y,z}z|\right)^{d+\alpha(y)+2}}\cdot|z|^{2-d-\alpha(x)}\,dz\\&\preceq \frac{t}{\left(t^{1/\alpha(y)}+|x-y|\right)^{d+\alpha(y)+2}}\int_{\{|z|\leqslant|x-y|/2\}}|z|^{2-d-\alpha(x)}\,dz\\&\preceq \frac{|x-y|^{\alpha(y)}}{|x-y|^{d+\alpha(y)+2}}\cdot|x-y|^{2-\alpha(x)}\preceq\frac{1}{|x-y|^{d+\alpha(x)}},\end{aligned}$$

where in the first inequality $\theta_{x,y,z}$ is a constant depending on x,y,z such that $|\theta_{x,y,z}|\leqslant 1$, in the second inequality we have used (2.20), the third inequality follows from the fact that $|x-y+\theta_{x,y,z}z|\geqslant|x-y|-|z|\geqslant |x-y|/2$, and the fourth inequality is due to the fact that $t\leqslant|x-y|^{\alpha(y)}$.

On the other hand, by (2.22) and the fact that $|z|>|x-y|/2\geqslant t^{1/\alpha(y)}/2$, we have $|\delta_{p^y}(t,x-y;z)|\preceq \rho_1^{y,0}(t,x-y\pm z)+\rho_1^{y,0}(t,x-y)$, and so

$$J_2\preceq\int_{\{|z|>|x-y|/2\}}\rho_1^{y,0}(t,x-y\pm z)\frac{1}{|z|^{d+\alpha(x)}}\,dz+\rho_1^{y,0}(t,x-y)\int_{\{|z|>|x-y|/2\}}\frac{1}{|z|^{d+\alpha(x)}}\,dz=:J_{21}+J_{22}.$$

It holds that

$$J_{21}\preceq\frac{1}{|x-y|^{d+\alpha(x)}}\int_{\{|z|>|x-y|/2\}}\rho_1^{y,0}(t,x-y\pm z)\,dz\preceq\frac{1}{|x-y|^{d+\alpha(x)}}\int\rho_1^{y,0}(t,x-y\pm z)\,dz\preceq\frac{1}{|x-y|^{d+\alpha(x)}},$$

where in the first inequality we have used following fact

$$\frac{1}{|z|^{d+\alpha(x)}}\preceq\frac{1}{|x-y|^{d+\alpha(x)}}\quad\text{for } |z|>|x-y|/2,$$

and the last inequality follows from (2.3). Furthermore, it is easy to verify that

$$J_{22}\preceq\rho_1^{y,0}(t,x-y)\int_{\{|z|>|x-y|/2\}}\frac{1}{|z|^{d+\alpha(x)}}\,dz\preceq\frac{t}{|x-y|^{d+\alpha(x)+\alpha(y)}}\preceq\frac{1}{|x-y|^{d+\alpha(x)}},$$

where in the last inequality we used $|x-y|\leqslant t^{1/\alpha(y)}$.

Combining all these estimates above, we arrive at

$$|\mathcal{L}^x p^y(t,\cdot)(x-y)|\preceq\frac{1}{|x-y|^{d+\alpha(x)}}.$$

Following the same arguments as above, we also have

$$|\mathcal{L}^y p^y(t,\cdot)(x-y)|\preceq\frac{1}{|x-y|^{d+\alpha(y)}}.$$

Hence,

$$|q_0(t,x,y)|\preceq|\mathcal{L}^x p^y(t,\cdot)(x-y)|+|\mathcal{L}^y p^y(t,\cdot)(x-y)|\preceq\frac{1}{|x-y|^{d+\alpha(x)}}+\frac{1}{|x-y|^{d+\alpha(y)}},$$

from which we can get (2.34) immediately.

(2) Next we consider the case that $\kappa(x,z)$ depends on z. With the estimate (2.18) and the condition $\beta_0/\alpha_2>(\alpha_2/\alpha_1)-1$ at hand, we can follow the same argument as in (2.36) to verify the upper bound in (2.35) for the case that $|x-y|\leqslant t^{1/\alpha(y)}$.

When $|x-y|>t^{1/\alpha(y)}$, we still define J_1 and J_2 via (2.37). By carefully tracking the proof above, we find that the argument for the estimates of J_2 in part (1) still works. So it remains to consider upper bound for J_1. According to (2.22), we have

$$\begin{aligned}J_1 &\preceq \int_{\{|z|\leqslant|x-y|/2\}}\left[\left(t^{-2/\alpha(y)}|z|^2\right)\wedge 1\right]\cdot\left(\rho_1^{y,0}(t,x-y\pm z)+\rho_1^{y,0}(t,x-y)\right)\frac{1}{|z|^{d+\alpha(x)}}\,dz\\ &\preceq \frac{t}{|x-y|^{d+\alpha(y)}}\int\left[\left(t^{-2/\alpha(y)}|z|^2\right)\wedge 1\right]\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz\\ &\preceq \frac{t}{|x-y|^{d+\alpha(y)}}\cdot\left(\int_{\{|z|\leqslant t^{1/\alpha(y)}\}}t^{-2/\alpha(y)}|z|^{2-d-\alpha(x)}\,dz+\int_{\{|z|>t^{1/\alpha(y)}\}}|z|^{-d-\alpha(x)}\,dz\right)\\ &\preceq t^{1-(\alpha(x)/\alpha(y))}\cdot\frac{1}{|x-y|^{d+\alpha(y)}}\preceq t^{1-(\alpha_2/\alpha_1)}\cdot\frac{1}{|x-y|^{d+\alpha(y)}},\end{aligned}$$

where the second inequality above follows from

$$\rho_1^{y,0}(t,x-y\pm z)\preceq\frac{t}{|x-y|^{d+\alpha(y)}}\qquad\text{for any } |z|\leqslant|x-y|/2.$$

Combining this with the estimate of J_2 in part (1) yields

$$|q_0(t,x,y)|\preceq|\mathcal{L}^x p^y(t,\cdot)(x-y)|+|\mathcal{L}^x p^y(t,\cdot)(x-y)|\preceq t^{1-(\alpha_2/\alpha_1)}\cdot\left(\frac{1}{|x-y|^{d+\alpha(x)}}+\frac{1}{|x-y|^{d+\alpha(y)}}\right),$$

which implies that (2.35) holds for every $x,y\in\mathbb{R}^d$ with $|x-y|>t^{1/\alpha(y)}$. The proof is complete. □

2.3. Continuity of $q_0(t,x,y)$.

Proposition 2.9. *(1) If $\kappa(x,z)=\kappa(x)$ is independent of z, then for any $\gamma,\theta>0$ with $\gamma<\theta<\beta_0/\alpha_2$ and any $\varepsilon>0$, there exists a positive constant $c_1:=c_1(\alpha,\kappa,\gamma,\theta,\varepsilon)$ such that for every $(t,x,x',y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$,*

$$\begin{aligned}&|q_0(t,x,y)-q_0(t,x',y)|\\ &\leqslant c_1\left(|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\wedge 1\right)\\ &\quad\times\Big[\left(\rho_{\theta-\gamma}^{y,0}+\rho_{\theta-\gamma-(\beta_0/\alpha(x))}^{y,\beta_0}\right)(t,x-y)+\left(\rho_{\theta-\gamma}^{y,0}+\rho_{\theta-\gamma-(\beta_0/\alpha(x'))}^{y,\beta_0}\right)(t,x'-y)\\ &\qquad+|x-y|^{\varepsilon}\mathbb{1}_{\{|x-y|>R_0\}}\rho_{\theta-\gamma}^{y,0}(t,x-y)+|x'-y|^{\varepsilon}\mathbb{1}_{\{|x'-y|>R_0\}}\rho_{\theta-\gamma}^{y,0}(t,x'-y)\Big].\end{aligned}\tag{2.38}$$

(2) For general $\kappa(x,z)$, if $(\alpha_2/\alpha_1)-1<\beta_0/\alpha_2$, then for any $\gamma,\theta>0$ such that $(\alpha_2/\alpha_1)-1+\gamma=:\gamma_2<\theta<\beta_0/\alpha_2$ and $\varepsilon>0$, there exists a positive constant $c_2:=c_2(\alpha,\kappa,\gamma,\theta,\varepsilon)$ such that for every $(t,x,x',y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$,

$$\begin{aligned}&|q_0(t,x,y)-q_0(t,x',y)|\\ &\leqslant c_2\left(|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\wedge 1\right)\Big[\left(\rho_{\theta-\gamma_2}^{y,0}+\rho_{\theta-\gamma_2-(\beta_0/\alpha(x))}^{y,\beta_0}\right)(t,x-y)\\ &\qquad+\left(\rho_{\theta-\gamma_2}^{y,0}+\rho_{\theta-\gamma_2-(\beta_0/\alpha(x'))}^{y,\beta_0}\right)(t,x'-y)+|x-y|^{\varepsilon}\mathbb{1}_{\{|x-y|>R_0\}}\rho_{\theta-\gamma_2}^{y,0}(t,x-y)\\ &\qquad+|x'-y|^{\varepsilon}\mathbb{1}_{\{|x'-y|>R_0\}}\rho_{\theta-\gamma_2}^{y,0}(t,x'-y)\Big].\end{aligned}\tag{2.39}$$

Proof. In the remainder of this paper, we denote $\nabla p^y(t,\cdot)(x)$ by $\nabla p^y(t,x)$ for simplicity.

(1) We first consider the case that $\kappa(x,z)$ is independent of z. Let $R_1:=R_1(\alpha,\kappa,\gamma)$ be a positive constant to be determined later. The proof is split into the following five different cases.

Case (a): $|x-x'|>R_1$.

By (2.16) and the condition $\beta_0>\alpha_2\theta$, it holds that

$$\begin{aligned}|q_0(t,x,y)-q_0(t,x',y)|&\leqslant|q_0(t,x,y)|+|q_0(t,x',y)|\preceq\rho_{-\gamma}^{y,\beta_0}(t,x-y)+\rho_{-\gamma}^{y,\beta_0}(t,x'-y)\\ &\preceq\left(|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\wedge 1\right)\left(\rho_{-\gamma}^{y,\beta_0}(t,x-y)+\rho_{-\gamma}^{y,\beta_0}(t,x'-y)\right).\end{aligned}$$

Case (b): $C_0t^{1/\alpha(y)}\leqslant|x-x'|\leqslant R_1$ **for some small constant** $C_0>0$.

According to (2.16) again, we have

$$\begin{aligned}|q_0(t,x,y)|&\preceq\rho_{-\gamma}^{y,\beta_0}(t,x-y)\preceq t^{(\beta_0-\alpha(x)\theta)/\alpha(x)}\rho_{-\gamma+\theta-(\beta_0/\alpha(x))}^{y,\beta_0}(t,x-y)\\ &\preceq|x-x'|^{\alpha(y)(\beta_0-\alpha(x)\theta)/\alpha(x)}\rho_{-\gamma+\theta-(\beta_0/\alpha(x))}^{y,\beta_0}(t,x-y)\end{aligned}$$

$$\preceq |x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\rho^{y,\beta_0}_{-\gamma+\theta-(\beta_0/\alpha(x))}(t,x-y),$$

where the third inequality is due to $t \preceq |x-x'|^{\alpha(y)}$. Then

$$\begin{aligned}
&|q_0(t,x,y)-q_0(t,x',y)|\\
&\leqslant |q_0(t,x,y)|+|q_0(t,x',y)|\\
&\preceq |x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\rho^{y,\beta_0}_{-\gamma+\theta-(\beta_0/\alpha(x))}(t,x-y)+|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x')}\rho^{y,\beta_0}_{-\gamma+\theta-(\beta_0/\alpha(x'))}(t,x'-y)\\
&\preceq |x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\Big(\rho^{y,\beta_0}_{-\gamma+\theta-(\beta_0/\alpha(x))}(t,x-y)+\rho^{y,\beta_0}_{-\gamma+\theta-(\beta_0/\alpha(x'))}(t,x'-y)\Big).
\end{aligned}$$

Here in the last inequality we have used the fact that for $x,x'\in\mathbb{R}^d$ with $|x-x'|\leqslant R_1$,

$$|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x')}\preceq |x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}, \tag{2.40}$$

which can be verified by following the argument of (2.2).

Next, we mainly treat the case that $|x-x'|\leqslant C_0t^{1/\alpha(y)}$, which we divide into three cases.

Case (c): $|x-x'|\leqslant C_0t^{1/\alpha(y)}$ and $|x-y|\geqslant R_1$.

By the definition of $q_0(t,x,y)$,

$$\begin{aligned}
|q_0(t,x,y)-q_0(t,x',y)| &= \frac12\bigg|\int_{\mathbb{R}^d}\delta_{p^y}(t,x-y;z)\bigg(\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}-\frac{\kappa(y,z)}{|z|^{d+\alpha(y)}}\bigg)\,dz\\
&\qquad -\int_{\mathbb{R}^d}\delta_{p^y}(t,x'-y;z)\bigg(\frac{\kappa(x',z)}{|z|^{d+\alpha(x')}}-\frac{\kappa(y,z)}{|z|^{d+\alpha(y)}}\bigg)\,dz\bigg|\\
&\leqslant \frac12\int_{\mathbb{R}^d}|\delta_{p^y}(t,x-y;z)-\delta_{p^y}(t,x'-y;z)|\bigg|\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}-\frac{\kappa(y,z)}{|z|^{d+\alpha(y)}}\bigg|\,dz\\
&\quad+\frac12\int_{\mathbb{R}^d}|\delta_{p^y}(t,x'-y;z)|\bigg|\frac{\kappa(x',z)}{|z|^{d+\alpha(x')}}-\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\bigg|\,dz\\
&=: J_1+J_2.
\end{aligned}\tag{2.41}$$

Note that

$$\begin{aligned}
\delta_{p^y}(t,x;z) &= p^y(t,x+z)+p^y(t,x-z)-2p^y(t,x)\\
&=\int_0^1\frac{d}{d\theta}\Big(p^y(t,x+\theta z)+p^y(t,x-\theta z)\Big)\,d\theta\\
&=\int_0^1\langle\nabla p^y(t,x+\theta z)-\nabla p^y(t,x-\theta z),z\rangle\,d\theta\\
&=\int_0^1\Big\langle\int_{-1}^1\frac{d}{d\theta'}\left(\nabla p^y(t,x+\theta'\theta z)\right)d\theta',z\Big\rangle\,d\theta\\
&=\int_0^1\int_{-1}^1\nabla^2p^y(t,x+\theta'\theta z)(\theta z,z)d\theta'\,d\theta,
\end{aligned}\tag{2.42}$$

and so

$$\begin{aligned}
\delta_{p^y}(t,x;z)-\delta_{p^y}(t,x';z) &= \int_0^1\int_{-1}^1\left(\nabla^2p^y(t,x+\theta'\theta z)-\nabla^2p^y(t,x'+\theta'\theta z)\right)(\theta z,z)\,d\theta'\,d\theta\\
&=\int_0^1\int_{-1}^1\int_0^1\frac{d}{d\theta''}\Big(\nabla^2p^y(t,x'+\theta''(x-x')+\theta'\theta z)\Big)(\theta z,z)\,d\theta''d\theta'\,d\theta\\
&=\int_0^1\int_{-1}^1\int_0^1\nabla^3p^y(t,x'+\theta''(x-x')+\theta'\theta z)(x-x',\theta z,z)\,d\theta''d\theta'\,d\theta,
\end{aligned}$$

which implies that

$$|\delta_{p^y}(t,x;z)-\delta_{p^y}(t,x';z)|\leqslant\Big(\int_0^1\int_{-1}^1\int_0^1|\nabla^3p^y(t,x'+\theta''(x-x')+\theta'\theta z)|\,d\theta''d\theta'\,d\theta\Big)\cdot|z|^2|x-x'|.$$

For every $x,x',z\in\mathbb{R}^d$ with $|x-x'|\leqslant C_0t^{1/\alpha(y)}$, $|x|\geqslant cR_1$, $|x'|\geqslant cR_1$ and $|z|\leqslant cR_1/4$ for C_0 small enough, there exists a constant $c'>0$ such that for every $\theta\in(0,1)$, $\theta'\in(-1,1)$ and $\theta''\in(0,1)$, we have $|x'+\theta''(x-x')+\theta'\theta z|\geqslant c'|x'|\geqslant c''R_1$, which along with (2.20) yields that

$$|\nabla^3p^y(t,x'+\theta''(x-x')+\theta'\theta z)|\preceq\frac{t}{|x'|^{d+\alpha(y)+3}}.$$

Combining with all the estimates above, we arrive at that for $|x-x'|\leqslant C_0t^{1/\alpha(y)}$, $|x|\geqslant cR_1$, $|x'|\geqslant cR_1$ and $|z|\leqslant cR_1/4$,

$$|\delta_{p^y}(t,x;z)-\delta_{p^y}(t,x';z)|\preceq \frac{t|z|^2|x-x'|}{|x'|^{d+\alpha(y)+3}}. \tag{2.43}$$

Since $|x-x'|\leqslant C_0t^{1/\alpha(y)}\leqslant C_0$, $|x'-y|\geqslant cR_1$ holds for some constant $c>0$. By taking $x=x-y$ and $x'=x'-y$ in (2.43), we obtain that for all $z\in\mathbb{R}^d$ with $|z|\leqslant cR_1/4$,

$$|\delta_{p^y}(t,x-y;z)-\delta_{p^y}(t,x'-y;z)|\preceq \frac{t|z|^2|x-x'|}{|x'-y|^{d+\alpha(y)+3}}. \tag{2.44}$$

Let

$$\begin{aligned} J_1 =&\frac{1}{2}\int_{\{|z|\leqslant cR_1/4\}}|\delta_{p^y}(t,x-y;z)-\delta_{p^y}(t,x'-y;z)|\left|\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}-\frac{\kappa(y,z)}{|z|^{d+\alpha(y)}}\right|dz\\ &+\frac{1}{2}\int_{\{|z|>cR_1/4\}}|\delta_{p^y}(t,x-y;z)-\delta_{p^y}(t,x'-y;z)|\left|\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}-\frac{\kappa(y,z)}{|z|^{d+\alpha(y)}}\right|dz\\ =&:J_{11}+J_{12}. \end{aligned} \tag{2.45}$$

By (2.44),

$$\begin{aligned} J_{11}\preceq&|x-x'|\frac{t}{|x'-y|^{d+\alpha(y)+3}}\int_{\{|z|\leqslant cR_1/4\}}\left(\frac{|z|^2}{|z|^{d+\alpha(x)}}+\frac{|z|^2}{|z|^{d+\alpha(y)}}\right)dz\\ \preceq&|x-x'|\frac{t}{|x'-y|^{d+\alpha(y)+3}}\preceq|x-x'|\frac{1}{|x'-y|^{d+\alpha_1}}, \end{aligned}$$

where the last inequality is due to $|x'-y|>cR_1$ and $t\leqslant 1$. On the other hand, according to (2.14), we have for any $x,y\in\mathbb{R}^d$ with $|x-y|>R_1$,

$$\begin{aligned} &\int_{\{|z|>cR_1/4\}}\rho_1^{y,0}(t,x-y\pm z)\frac{1}{|z|^{d+\alpha(x)}}\,dz\preceq\frac{1}{|x-y|^{d+\alpha_1}},\\ &\int_{\{|z|>cR_1/4\}}\rho_1^{y,0}(t,x-y)\frac{1}{|z|^{d+\alpha(x)}}\,dz\preceq\frac{1}{|x-y|^{d+\alpha_1}}. \end{aligned} \tag{2.46}$$

Combining (2.24) with (2.46) and the fact that $|z|>cR_1/4$ implies $|z|>C_1t^{1/\alpha(y)}$, we arrive at

$$\begin{aligned} J_{12}&\preceq t^{-1/\alpha(y)}|x-x'|\int_{\{|z|>cR_1/4\}}\Big(\rho_1^{y,0}(t,x-y\pm z)+\rho_1^{y,0}(t,x-y)\\ &\qquad+\rho_1^{y,0}(t,x'-y\pm z)+\rho_1^{y,0}(t,x'-y)\Big)\left(\frac{1}{|z|^{d+\alpha(x)}}+\frac{1}{|z|^{d+\alpha(y)}}\right)dz\\ &\preceq t^{-1/\alpha(y)}|x-x'|\left(\frac{1}{|x-y|^{d+\alpha_1}}+\frac{1}{|x'-y|^{d+\alpha_1}}\right). \end{aligned}$$

By both of the estimates above, we obtain that

$$J_1\preceq t^{-1/\alpha(y)}|x-x'|\big(\rho_0^{y,0}(t,x-y)+\rho_0^{y,0}(t,x'-y)\big)\preceq|x-x'|\big(\rho_{-1/\alpha(y)}^{y,\beta_0}(t,x-y)+\rho_{-1/\alpha(y)}^{y,\beta_0}(t,x'-y)\big),$$

where the last inequality follows from the facts that $|x-y|\geqslant cR_1$ and $|x'-y|\geqslant cR_1$.

Let

$$\begin{aligned} J_2 =&\int_{\{|z|\leqslant cR_1/4\}}|\delta_{p^y}(t,x'-y;z)|\left|\frac{\kappa(x',z)}{|z|^{d+\alpha(x')}}-\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\right|dz\\ &+\int_{\{|z|>cR_1/4\}}|\delta_{p^y}(t,x'-y;z)|\left|\frac{\kappa(x',z)}{|z|^{d+\alpha(x')}}-\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\right|dz\\ =&:J_{21}+J_{22}. \end{aligned} \tag{2.47}$$

Since $|x'-y|>cR_1$, by (2.42) and (2.20), we find that for any $|z|<cR_1/4$,

$$|\delta_{p^y}(t,x'-y;z)|\leqslant|z|^2\int_0^1\int_{-1}^1|\nabla^2p^y(t,x'-y+\theta'\theta z)|\,d\theta'\,d\theta\preceq|z|^2\frac{t}{|x'-y|^{d+\alpha(y)+2}}.$$

Thus, due to (2.28),

$$J_{21}\preceq|x-x'|^{\beta_0}\frac{t}{|x'-y|^{d+\alpha(y)+2}}\int_{\{|z|\leqslant cR_1/4\}}\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(x;x')}}|z|^2\,dz\preceq|x-x'|^{\beta_0}\frac{1}{|x'-y|^{d+\alpha_1}},$$

where we have used the facts that $|x'-y|>cR_1$ and $t<1$ in the second inequality. On the other hand, when $|z|>cR_1/4$,

$$|\delta_{p^y}(t,x'-y;z)|\preceq p^y(t,x'-y\pm z)+p^y(t,x'-y).$$

Having this at hand and using (2.15) and (2.21), we know that

$$\int_{\{|z|>cR_1/4\}}|\delta_{p^y}(t,x'-y;z)|\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(x;x')}}\,dz\preceq\frac{\log(1+|x'-y|)}{|x'-y|^{d+\alpha_1}}.$$

Then, by (2.28), we find that

$$J_{22}\preceq|x-x'|^{\beta_0}\cdot\int_{\{|z|>cR_1/4\}}|\delta_{p^y}(t,x'-y;z)|\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(x;x')}}\,dz\preceq|x-x'|^{\beta_0}\frac{\log(1+|x'-y|)}{|x'-y|^{d+\alpha_1}}.$$

By the estimates for J_{21} and J_{22}, we have

$$J_2\preceq|x-x'|^{\beta_0}\log(1+|x'-y|)\rho_0^{y,0}(t,x'-y).$$

Putting the estimates of J_1 and J_2 together, we finally arrive at

$$\begin{aligned}|q_0(t,x,y)-q_0(t,x',y)|\preceq&|x-x'|\big(\rho^{y,\beta_0}_{-1/\alpha(y)}(t,x-y)+\rho^{y,\beta_0}_{-1/\alpha(y)}(t,x'-y)\big)\\&+|x-x'|^{\beta_0}\log(1+|x'-y|)\rho_0^{y,0}(t,x'-y).\end{aligned}$$

Case (d): $|x-x'|\leqslant C_0t^{1/\alpha(y)}$ and $t^{1/\alpha(y)}\leqslant|x-y|<R_1$.
We still define J_1 and J_2 by those in (2.41). Combining (2.24) with (2.28), we arrive at

$$\begin{aligned}J_1\preceq&t^{-1/\alpha(y)}|x-x'||x-y|^{\beta_0}\Bigg[\int_{\mathbb{R}^d}\left[\left(t^{-2/\alpha(y)}|z|^2\right)\wedge1\right]\\&\times\left(\rho_1^{y,0}(t,x-y)+\rho_1^{y,0}(t,x'-y)+\rho_1^{y,0}(t,x-y\pm z)+\rho_1^{y,0}(t,x'-y\pm z)\right)\\&\times\left(\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\mathbb{1}_{\{|z|\leqslant1\}}+\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(y;x)}}\mathbb{1}_{\{|z|>1\}}\right)dz\Bigg].\end{aligned}$$

Noting that $t^{1/\alpha(y)}\leqslant|x-y|<R_1$ and using (2.5)–(2.7), we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^d}\left[\left(t^{-2/\alpha(y)}|z|^2\right)\wedge1\right]\left(\rho_1^{y,0}(t,x-y\pm z)+\rho_1^{y,0}(t,x-y)\right)\\&\quad\times\left(\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(y;x)}}\mathbb{1}_{\{|z|\leqslant1\}}+\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(y;x)}}\mathbb{1}_{\{|z|>1\}}\right)dz\\&\preceq(1+|\log t|)\,t^{-(\overline{\alpha}(y;x)-\alpha(y))/\alpha(y)}\rho_0^{y,0}(t,x-y)\preceq\rho_{-\gamma}^{y,0}(t,x-y),\end{aligned}\tag{2.48}$$

where the last step is due to the fact that we can choose the constant R_1 small enough such that

$$\frac{\overline{\alpha}(y;x)-\alpha(y)}{\alpha(y)}\leqslant\frac{C|x-y|^{\beta_0}}{\alpha_1}<\gamma\quad\text{ for every }|x-y|\leqslant R_1.$$

Since $|x-x'|\leqslant C_0t^{1/\alpha(y)}$ for some C_0 small enough, $C_2t^{1/\alpha(y)}\leqslant|x'-y|\leqslant C_3R_1$ holds for some positive constants C_2 and C_3. Hence, (2.48) still holds with x replaced by x'. Therefore, combining both the estimates above together, we arrive at

$$\begin{aligned}J_1&\preceq|x-x'|\cdot|x-y|^{\beta_0}\cdot\left(\rho^{y,0}_{-\gamma-(1/\alpha(y))}(t,x-y)+\rho^{y,0}_{-\gamma-(1/\alpha(y))}(t,x'-y)\right)\\&\preceq|x-x'|\left(\rho^{y,\beta_0}_{-\gamma-(1/\alpha(y))}(t,x-y)+\rho^{y,\beta_0}_{-\gamma-(1/\alpha(y))}(t,x'-y)\right),\end{aligned}$$

where in the last step we have used the fact that

$$|x'-y|\geqslant|x-y|-|x-x'|\geqslant|x-y|-C_0t^{1/\alpha(y)}\geqslant(1-C_0)|x-y|.$$

Now we are going to estimate J_2. (2.22) along with (2.28) yields that

$$\begin{aligned}J_2\preceq|x-x'|^{\beta_0}&\int_{\mathbb{R}^d}\left[\left(t^{-2/\alpha(y)}|z|^2\right)\wedge1\right]\cdot\left(\rho_1^{y,0}(t,x'-y)+\rho_1^{y,0}(t,x'-y\pm z)\right)\\&\times\left(\frac{|\log|z||+1}{|z|^{d+\overline{\alpha}(x;x')}}\mathbb{1}_{\{|z|\leqslant1\}}+\frac{|\log|z||+1}{|z|^{d+\underline{\alpha}(x;x')}}\mathbb{1}_{\{|z|>1\}}\right)dz.\end{aligned}\tag{2.49}$$

As noted above, it holds that $C_2t^{1/\alpha(y)}\leqslant |x'-y|\leqslant C_3R_1$. Although the indexes $\overline{\alpha}(y;x)$ and $\underline{\alpha}(y;x)$ are replaced by $\overline{\alpha}(x;x')$ and $\underline{\alpha}(x;x')$ respectively, we can still follow the proof of Lemma 2.3 to obtain that

$$\begin{aligned}\int_{\mathbb{R}^d}&\left[\left(t^{-2/\alpha(y)}|z|^2\right)\wedge 1\right]\cdot\left(\rho_1^{y,0}(t,x'-y)+\rho_1^{y,0}(t,x'-y\pm z)\right)\\&\times\left(\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(x;x')}}\mathbb{1}_{\{|z|\leqslant 1\}}+\frac{1+|\log|z||}{|z|^{d+\underline{\alpha}(x;x')}}\mathbb{1}_{\{|z|>1\}}\right)dz\\&\preceq(1+|\log t|)\,t^{1-(\overline{\alpha}(x;x')/\alpha(y))}\rho_0^{y,0}(t,x'-y).\end{aligned}\tag{2.50}$$

Note that when $|x-y|\leqslant R_1$ and $|x-x'|\leqslant C_0t^{1/\alpha(y)}$,

$$|\overline{\alpha}(x;x')-\alpha(y)|\leqslant|\overline{\alpha}(x;x')-\alpha(x)|+|\alpha(x)-\alpha(y)|\leqslant C_4(t^{\beta_0/\alpha(y)}+R_1^{\beta_0}).$$

So, by changing R_1 properly such that

$$t^{-(\overline{\alpha}(x;x')-\alpha(y))/\alpha(y)}\preceq\exp(C_5t|\log t|)\cdot t^{-C_4R_1^{\beta_0}}\preceq t^{-\gamma/2}$$

and putting all these estimates into (2.49), we arrive at

$$J_2\preceq|x-x'|^{\beta_0}\rho_{-\gamma}^{y,0}(t,x'-y).$$

Therefore, according to the estimates for J_1 and J_2, we have

$$|q_0(t,x,y)-q_0(t,x',y)|\preceq|x-x'|\left(\rho_{-\gamma-(1/\alpha(y))}^{y,\beta_0}(t,x-y)+\rho_{-\gamma-(1/\alpha(y))}^{y,\beta_0}(t,x'-y)\right)+|x-x'|^{\beta_0}\rho_{-\gamma}^{y,0}(t,x'-y).$$

Case (e): $|x-x'|\leqslant C_0t^{1/\alpha(y)}$ **and** $|x-y|\leqslant t^{1/\alpha(y)}$**.**

We still define J_1 and J_2 by those in (2.41). It is easy to see that (2.48) and (2.50) still hold for such case. Note that $|x'-y|\leqslant|x-y|+|x-x'|\leqslant(1+C_0)t^{1/\alpha(y)}$. Using (2.48) and (2.50), and repeating the argument in **Case (d)**, we have

$$\begin{aligned}J_1&\preceq|x-x'|\cdot|x-y|^{\beta_0}\cdot\left(\rho_{-\gamma-(1/\alpha(y))}^{y,0}(t,x-y)+\rho_{-\gamma-(1/\alpha(y))}^{y,0}(t,x'-y)\right),\\J_2&\preceq|x-x'|^{\beta_0}\rho_{-\gamma}^{y,0}(t,x'-y).\end{aligned}$$

Then, we arrive at

$$\begin{aligned}&|q_0(t,x,y)-q_0(t,x',y)|\\&\preceq|x-x'|\rho_{-\gamma-(1/\alpha(y))}^{y,\beta_0}(t,x-y)+|x-x'||x-y|^{\beta_0}\rho_{-\gamma-(1/\alpha(y))}^{y,0}(t,x'-y)+|x-x'|^{\beta_0}\rho_{-\gamma}^{y,0}(t,x'-y)\\&\preceq|x-x'|\rho_{-\gamma-(1/\alpha(y))}^{y,\beta_0}(t,x-y)+|x-x'|^{\beta_0}\rho_{-\gamma}^{y,0}(t,x'-y),\end{aligned}$$

where the last step follows from the property that

$$|x-x'||x-y|^{\beta_0}=|x-x'|^{\beta_0}|x-x'|^{1-\beta_0}|x-y|^{\beta}\preceq t^{1/\alpha(y)}|x-x'|^{\beta_0}.$$

Combining all **Cases (c)–(e)** together, we can find a constant $C_0>0$ small enough such that for every $|x-x'|\leqslant C_0t^{1/\alpha(y)}$

$$\begin{aligned}|q_0(t,x,y)-q_0(t,x',y)|\preceq{}&|x-x'|\big(\rho_{-\gamma-(1/\alpha(y))}^{y,\beta_0}(t,x-y)+\rho_{-\gamma-(1/\alpha(y))}^{y,\beta_0}(t,x'-y)\big)\\&+|x-x'|^{\beta_0}\big(1+\log(1+|x'-y|)\mathbb{1}_{\{|x'-y|>R_0\}}\big)\rho_{-\gamma}^{y,0}(t,x'-y).\end{aligned}$$

Furthermore, for $0<\gamma<\theta<\beta_0/\alpha_2$, we have

$$\begin{aligned}|x-x'|^{\beta_0}\rho_{-\gamma}^{y,0}(t,x'-y)&=|x-x'|^{\beta_0-\alpha(y)\theta}|x-x'|^{\alpha(y)\theta}\rho_{-\gamma}^{y,0}(t,x'-y)\\&\preceq|x-x'|^{\beta_0-\alpha(y)\theta}\rho_{\theta-\gamma}^{y,0}(t,x'-y)\preceq|x-x'|^{\beta_0-\alpha_2\theta}\rho_{\theta-\gamma}^{y,0}(t,x'-y),\end{aligned}$$

where the second step above follows from $|x-x'|\leqslant C_0t^{1/\alpha(y)}$. On the other hand, it holds that

$$\begin{aligned}|x-x'|\rho_{-\gamma-(1/\alpha(y))}^{y,\beta_0}(t,x'-y)&=|x-x'|^{\alpha_1(\beta_0-\alpha(x')\theta)/\alpha(x')}|x-x'|^{1-(\alpha_1(\beta_0-\alpha(x')\theta)/\alpha(x'))}\rho_{-\gamma-(1/\alpha(y))}^{y,\beta_0}(t,x'-y)\\&\preceq|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x')}t^{(1/\alpha(y))-(\alpha_1(\beta_0-\alpha(x')\theta)/(\alpha(x')\alpha(y)))}\rho_{-\gamma-(1/\alpha(y))}^{y,\beta_0}(t,x'-y)\\&\preceq|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\rho_{-\gamma-((\alpha_1/\alpha(y))((\beta_0/\alpha(x'))-\theta))}^{y,\beta_0}(t,x'-y)\\&\preceq|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\rho_{-\gamma+\theta-(\beta_0/\alpha(x'))}^{y,\beta_0}(t,x'-y),\end{aligned}$$

where in the first inequality we have used again that $|x-x'|\leqslant C_0t^{1/\alpha(y)}$, and the second inequality follows from (2.40). Similarly, it holds that

$$|x-x'|\rho_{-\gamma-(1/\alpha(y))}^{y,\beta_0}(t,x-y)\preceq|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\rho_{-\gamma+\theta-(\beta_0/\alpha(x))}^{y,\beta_0}(t,x-y).$$

Hence, we arrive at that for every $|x-x'|\leqslant C_0t^{1/\alpha(y)}$ and $\varepsilon>0$

$$\begin{aligned}&|q_0(t,x,y)-q_0(t,x',y)|\\&\preceq|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\Big(\rho^{y,\beta_0}_{-\gamma+\theta-(\beta_0/\alpha(x))}(t,x-y)+\rho^{y,\beta_0}_{-\gamma+\theta-(\beta_0/\alpha(x'))}(t,x'-y)\\&\quad+\rho^{y,0}_{\theta-\gamma}(t,x'-y)+|x'-y|^{\varepsilon}\mathbb{1}_{\{|x'-y|>R_0\}}\rho^{y,0}_{\theta-\gamma}(t,x'-y)\Big).\end{aligned}\tag{2.51}$$

Therefore, according to all the five different cases above, (2.38) holds for $q_0(t,x,y)$.

(2) Now we study the estimate for $|q_0(t,x,y)-q_0(t,x',y)|$ under the assumption that $\kappa(x,z)$ depends on z. Recall that, by (2.18), we have

$$|q_0(t,x,y)|\preceq\rho^{y,\beta_0}_{-\gamma_2}(t,x-y),\quad (t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d,\tag{2.52}$$

where $\gamma_2=:\gamma+(\alpha_2/\alpha_1)-1$.

Using (2.52) and following the arguments in **Case (a)** and **Case (b)** of part (1) above, we derive that for every $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$ such that $|x-x'|\geqslant C_0t^{1/\alpha(y)}$ (which includes both $|x-x'|>R_1$ and $C_0t^{1/\alpha(y)}\leqslant|x-x'|\leqslant R_1$, and where as in (1) the constant C_0 is chosen to be small enough) and $\gamma_2<\theta<\beta_0/\alpha_2$,

$$\begin{aligned}|q_0(t,x,y)-q_0(t,x',y)|\preceq&\big(|x-x'|^{\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)}\wedge 1\big)\\&\times\big(\rho^{y,\beta_0}_{-\gamma_2+\theta-(\beta_0/\alpha(x))}(t,x-y)+\rho^{y,\beta_0}_{-\gamma_2+\theta-(\beta_0/\alpha(x'))}(t,x'-y)\big).\end{aligned}$$

Note that in the proofs of **Case (d)** and **Case (e)** above, we do not need the assumption that $\kappa(x,z)$ is independent of z, so the conclusions there are still true, which in particular means that (2.39) holds for **Case (d)** and **Case (e)**. Now we turn to the case that $|x-x'|\leqslant C_0t^{1/\alpha(y)}$ and $|x-y|>R_1$, i.e., the **Case (c)** above. We still define J_{11}, J_{12}, J_{21} and J_{22} as those in (2.41), (2.45) and (2.47), respectively.

For J_{11}, we apply (2.24) instead of (2.44) in part (1), and derive that

$$\begin{aligned}J_{11}\preceq&\, t^{-1/\alpha(y)}|x-x'|\int_{\{|z|\leqslant cR_1/4\}}[(t^{-2/\alpha(y)}|z|^2)\wedge1]\\&\times\Big(\rho^{y,0}_1(t,x-y)+\rho^{y,0}_1(t,x'-y)+\rho^{y,0}_1(t,x-y\pm z)+\rho^{y,0}_1(t,x'-y\pm z)\Big)\cdot\Big(\frac{1}{|z|^{d+\alpha(x)}}+\frac{1}{|z|^{d+\alpha(y)}}\Big)\,dz.\end{aligned}$$

Note that $|x-y|>R_1$. By (2.12), we arrive at

$$\int_{\{|z|\leqslant cR_1/4\}}[(t^{-2/\alpha(y)}|z|^2)\wedge1]\big(\rho^{y,0}_1(t,x-y)+\rho^{y,0}_1(t,x-y\pm z)\big)\Big(\frac{1}{|z|^{d+\alpha(x)}}+\frac{1}{|z|^{d+\alpha(y)}}\Big)\,dz\preceq\frac{t^{1-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_1}},$$

which yields that

$$J_{11}\preceq|x-x'|\big(\rho^{y,\beta_0}_{-\gamma_2-(1/\alpha(y))}(t,x-y)+\rho^{y,\beta_0}_{-\gamma_2-(1/\alpha(y))}(t,x'-y)\big).$$

Similarly, using (2.22) instead of (2.20) in the estimate of J_{21} and applying (2.13), we obtain

$$\begin{aligned}J_{21}&\preceq|x-x'|^{\beta_0}\cdot\Big(\int_{\{|z|\leqslant cR_1/4\}}[(t^{-2/\alpha(y)}|z|^2)\wedge1]\big[\rho^{y,0}_1(t,x'-y\pm z)+\rho^{y,0}_1(t,x'-y)\big]\frac{1+|\log|z||}{|z|^{d+\overline{\alpha}(x;x')}}\,dz\Big)\\&\preceq|x-x'|^{\beta_0}(1+|\log t|+|\log|x'-y||)\rho^{y,0}_{1-(\alpha_2/\alpha_1)}(t,x'-y).\end{aligned}$$

As explained before, the proofs for estimates of J_{12} and J_{22} in (1) do not require the condition that $\kappa(x,z)$ is independent of z, so those estimates still hold here.

By the estimates for J_{11}, J_{12}, J_{21} and J_{22}, we know that for every $|x-x'|\leqslant C_0t^{1/\alpha(y)}$ and $|x-y|>R_1$,

$$\begin{aligned}|q_0(t,x,y)-q_0(t,x',y)|\preceq&|x-x'|\cdot\big[\rho^{y,\beta_0}_{-\gamma_2-(1/\alpha(y))}(t,x-y)+\rho^{y,\beta_0}_{-\gamma_2-(1/\alpha(y))}(t,x'-y)\big]\\&+|x-x'|^{\beta_0}\big(1+\log(1+|x'-y|)\big)\rho^{y,0}_{-\gamma_2}(t,x'-y).\end{aligned}$$

Combining the estimates above for all the cases, we can find a constant $C_0>0$ small enough such that for every $|x-x'|\leqslant C_0t^{1/\alpha(y)}$ and $\gamma,\varepsilon>0$,

$$\begin{aligned}|q_0(t,x,y)-q_0(t,x',y)|\preceq&|x-x'|\cdot\big[\rho^{y,\beta_0}_{-\gamma_2-(1/\alpha(y))}(t,x-y)+\rho^{y,\beta_0}_{-\gamma_2-(1/\alpha(y))}(t,x'-y)\big]\\&+|x-x'|^{\beta_0}|x-y|^{\varepsilon}\mathbb{1}_{\{|x-y|>R_0\}}\rho^{y,0}_{-\gamma_2}(t,x-y)\\&+|x-x'|^{\beta_0}|x'-y|^{\varepsilon}\mathbb{1}_{\{|x'-y|>R_0\}}\rho^{y,0}_{-\gamma_2}(t,x'-y).\end{aligned}$$

Then, by the same argument for (2.51) and all the conclusions above, we can immediately show that (2.39) holds. □

3. Estimates for $q(t,x,y)$

3.1. **Convolution inequalities.** In this subsection, we will establish a convolution inequality involving $\rho_\gamma^{y,\beta}$, which will be frequently used in the remainder of our paper. Although the proofs in this subsection are partly inspired by those of [20, 35], essential and non-trivial modifications are needed to adapt them to non-local operators with variable orders.

Lemma 3.1. *(1) For every $\theta_1,\theta_2\in\mathbb{R}$ and $\beta_1,\beta_2\in(0,1)$, it holds for all $0<s<t\leqslant 1$ and $x,y\in\mathbb{R}^d$ that*

$$\begin{aligned}&\int_{\mathbb{R}^d}\rho_{\theta_1}^{z,\beta_1}(t-s,x-z)\rho_{\theta_2}^{y,\beta_2}(s,z-y)\,dz\\&\preceq\Big[(t-s)^{\theta_1+[((\beta_1+\beta_2)/\alpha(x))\wedge 1]-1}s^{\theta_2}\big(1+|\log(t-s)|\mathbb{1}_{\{\beta_1+\beta_2=\alpha(x)\}}\big)\\&\quad+(t-s)^{\theta_1}s^{\theta_2+[((\beta_1+\beta_2)/\alpha(y))\wedge 1]-1}\big(1+|\log s|\mathbb{1}_{\{\beta_1+\beta_2=\alpha(y)\}}\big)\Big]\rho_0^{y,0}(t,x-y)\\&\quad+(t-s)^{\theta_1+[(\beta_1/\alpha(x))\wedge 1]-1}s^{\theta_2}\big(1+|\log(t-s)|\mathbb{1}_{\{\beta_1=\alpha(x)\}}\big)\rho_0^{y,\beta_2}(t,x-y)\\&\quad+(t-s)^{\theta_1}s^{\theta_2+(\beta_2/\alpha(y))\wedge 1-1}\big(1+|\log s|\mathbb{1}_{\{\beta_2=\alpha(y)\}}\big)\rho_0^{y,\beta_1}(t,x-y).\end{aligned}\tag{3.1}$$

(2) For every $\theta_1,\theta_2\in\mathbb{R}$ and $\beta_1,\beta_2\in(0,1)$, it holds for every $x,y\in\mathbb{R}^d$ and $0<s<t\leqslant 1$ that

$$\begin{aligned}&\int_{\mathbb{R}^d}\rho_{\theta_1}^{z,\beta_1}(t-s,x-z)\rho_{\theta_2}^{y,\beta_2}(s,z-y)\,dz\\&\preceq\Big[(t-s)^{\theta_1+[((\beta_1+\beta_2)/\alpha_2)\wedge 1]-1}s^{\theta_2}\big(1+|\log(t-s)|\mathbb{1}_{\{\beta_1+\beta_2=\alpha_2\}}\big)\\&\quad+(t-s)^{\theta_1}s^{\theta_2+[((\beta_1+\beta_2)/\alpha_2)\wedge 1]-1}\big(1+|\log s|\mathbb{1}_{\{\beta_1+\beta_2=\alpha_2\}}\big)\Big]\rho_0^{y,0}(t,x-y)\\&\quad+(t-s)^{\theta_1+[(\beta_1/\alpha_2)\wedge 1]-1}s^{\theta_2}\big(1+|\log(t-s)|\mathbb{1}_{\{\beta_1=\alpha_2\}}\big)\rho_0^{y,\beta_2}(t,x-y)\\&\quad+(t-s)^{\theta_1}s^{\theta_2+[(\beta_2/\alpha_2)\wedge 1]-1}\big(1+|\log s|\mathbb{1}_{\{\beta_2=\alpha_2\}}\big)\rho_0^{y,\beta_1}(t,x-y).\end{aligned}\tag{3.2}$$

Proof. (1) The proof is split into three cases.

Case (a): $|x-y|>2$.

It holds that

$$\begin{aligned}\int_{\mathbb{R}^d}\rho_{\theta_1}^{z,\beta_1}(t-s,x-z)\rho_{\theta_2}^{y,\beta_2}(s,z-y)\,dz&\leqslant\int_{\{|z-y|\geqslant|x-y|/2\}}\rho_{\theta_1}^{z,\beta_1}(t-s,x-z)\rho_{\theta_2}^{y,\beta_2}(s,z-y)\,dz\\&\quad+\int_{\{|z-y|\leqslant|x-y|/2\}}\rho_{\theta_1}^{z,\beta_1}(t-s,x-z)\rho_{\theta_2}^{y,\beta_2}(s,z-y)\,dz\\&=:J_1+J_2.\end{aligned}\tag{3.3}$$

If $|z-y|\geqslant|x-y|/2\geqslant 1\geqslant s^{1/\alpha(y)}$, then

$$\rho_0^{y,0}(s,z-y)\preceq\frac{1}{|z-y|^{d+\alpha_1}}\preceq\frac{1}{|x-y|^{d+\alpha_1}}\preceq\rho_0^{y,0}(t,x-y).\tag{3.4}$$

Observe that (see e.g. the proof of [20, Lemma 2.2]) for any $x,y,z\in\mathbb{R}^d$,

$$\big(|x-z|^{\beta_1}\wedge 1\big)\big(|z-y|^{\beta_2}\wedge 1\big)\leqslant\big(|x-z|^{\beta_1+\beta_2}\wedge 1\big)+\big(|x-z|^{\beta_1}\wedge 1\big)\big(|x-y|^{\beta_2}\wedge 1\big).\tag{3.5}$$

Combining this with (3.4) and the first inequality in (2.1), we get that for $\beta_1,\beta_2\in(0,1)$,

$$\begin{aligned}J_1&\preceq(t-s)^{\theta_1}s^{\theta_2}\rho_0^{y,0}(t,x-y)\int_{\mathbb{R}^d}\rho_0^{z,\beta_1+\beta_2}(t-s,x-z)\,dz\\&\quad+(t-s)^{\theta_1}s^{\theta_2}\rho_0^{y,\beta_2}(t,x-y)\int_{\mathbb{R}^d}\rho_0^{z,\beta_1}(t-s,x-z)\,dz\\&\preceq(t-s)^{\theta_1+[((\beta_1+\beta_2)/\alpha(x))\wedge 1]-1}s^{\theta_2}\big(1+|\log(t-s)|\mathbb{1}_{\{\beta_1+\beta_2=\alpha(x)\}}\big)\rho_0^{y,0}(t,x-y)\\&\quad+(t-s)^{[\theta_1+(\beta_1/\alpha(x))\wedge 1]-1}s^{\theta_2}\big(1+|\log(t-s)|\mathbb{1}_{\{\beta_1=\alpha(x)\}}\big)\rho_0^{y,\beta_2}(t,x-y).\end{aligned}\tag{3.6}$$

If $|z-y|\leqslant|x-y|/2$, then $|x-z|\geqslant|x-y|/2\geqslant 1\geqslant(t-s)^{1/\alpha(z)}$, and so

$$\rho_0^{z,0}(t-s,x-z)\preceq\frac{1}{|x-z|^{d+\alpha_1}}\preceq\frac{1}{|x-y|^{d+\alpha_1}}\preceq\rho_0^{y,0}(t,x-y).$$

Combining this with

$$\big(|x-z|^{\beta_1}\wedge 1\big)\big(|z-y|^{\beta_2}\wedge 1\big)\leqslant\big(|z-y|^{\beta_1+\beta_2}\wedge 1\big)+\big(|z-y|^{\beta_2}\wedge 1\big)\big(|x-y|^{\beta_1}\wedge 1\big),\tag{3.7}$$

we arrive at

$$\begin{aligned}J_2 &\preceq (t-s)^{\theta_1}s^{\theta_2}\rho_0^{y,0}(t,x-y)\int_{\mathbb{R}^d}\rho_0^{y,\beta_1+\beta_2}(s,z-y)\,dz\\&\quad+(t-s)^{\theta_1}s^{\theta_2}\rho_0^{y,\beta_1}(t,x-y)\int_{\mathbb{R}^d}\rho_0^{y,\beta_2}(s,z-y)\,dz\\&\preceq (t-s)^{\theta_1}s^{\theta_2+[((\beta_1+\beta_2)/\alpha(y))\wedge 1]-1}\big(1+|\log s|\mathbb{1}_{\{\beta_1+\beta_2=\alpha(y)\}}\big)\rho_0^{y,0}(t,x-y)\\&\quad+(t-s)^{\theta_1}s^{\theta_2+[(\beta_2/\alpha(y))\wedge 1]-1}\big(1+|\log s|\mathbb{1}_{\{\beta_2=\alpha(y)\}}\big)\rho_0^{y,\beta_1}(t,x-y),\end{aligned}\tag{3.8}$$

where the second inequality follows from the first inequality in (2.3).

According to the above estimates for J_1 and J_2, (3.2) holds in this case.

Case (b): $2t^{1/\overline{\alpha}(y;t^{1/\alpha_2})}\leqslant |x-y|\leqslant 2$**.**

Let J_1 and J_2 be defined by (3.3). If $|z-y|\geqslant |x-y|/2$ and $0<s<t<1$, then we have

$$\begin{aligned}\rho_0^{y,0}(s,z-y)&\preceq\begin{cases}|z-y|^{-d-\alpha(y)}, & |z-y|\leqslant 1,\\ |z-y|^{-d-\alpha_1}, & |z-y|>1\end{cases}\\&\preceq\begin{cases}|x-y|^{-d-\alpha(y)}, & |z-y|\leqslant 1,\\ |x-y|^{-d-\alpha_1}, & |z-y|>1\end{cases}\\&\preceq |x-y|^{-d-\alpha(y)}\preceq\rho_0^{y,0}(t,x-y),\end{aligned}\tag{3.9}$$

where the second inequality is due to $|z-y|\geqslant|x-y|/2$, in the third inequality we have used the fact that

$$|x-y|^{-d-\alpha_1}\preceq|x-y|^{-d-\alpha(y)}\quad\text{for all } |x-y|\leqslant 2,$$

and the last inequality follows from

$$|x-y|^{-d-\alpha(y)}\preceq\rho_0^{y,0}(t,x-y)\quad\text{for all } |x-y|\geqslant 2t^{1/\overline{\alpha}(y;t^{1/\alpha_2})}\geqslant 2t^{1/\alpha(y)}.$$

Using (3.9) and following the same argument of (3.6), we know that (3.6) is satisfied too.

If $|z-y|<|x-y|/2$, then $|x-z|\geqslant|x-y|/2$, and so we obtain

$$\begin{aligned}\rho_0^{z,0}(t-s,x-z)&\preceq\begin{cases}|x-z|^{-d-\alpha(z)}, & |x-z|\leqslant 1,\\ |x-z|^{-d-\alpha_1}, & |x-z|>1\end{cases}\\&\preceq|x-y|^{-d-\alpha(z)}\preceq|x-y|^{\alpha(y)-\alpha(z)}\cdot|x-y|^{-d-\alpha(y)}\\&\preceq|x-y|^{-|\alpha(y)-\alpha(z)|}\cdot|x-y|^{-d-\alpha(y)}\\&\preceq|x-y|^{-C|x-y|^{\beta_0}}\cdot|x-y|^{-d-\alpha(y)}\\&\preceq\exp\big(C|\log|x-y||\cdot|x-y|^{\beta_0}\big)\cdot|x-y|^{-d-\alpha(y)}\\&\preceq|x-y|^{-d-\alpha(y)}\preceq\rho_0^{y,0}(t,x-y),\end{aligned}\tag{3.10}$$

where in the second inequality we used the fact that if $|x-z|\geqslant|x-y|/2$ and $|x-y|\leqslant 2$, then $|x-z|^{-d-\alpha_1}\preceq|x-y|^{-d-\alpha_1}\preceq|x-y|^{-d-\alpha(z)}$, the fifth inequality follows from the fact that

$$|\alpha(y)-\alpha(z)|\preceq|z-y|^{\beta_0}\leqslant C|x-y|^{\beta_0}\quad\text{for all } |z-y|\leqslant|x-y|/2,$$

in the seventh inequality we used

$$\sup_{z\in\mathbb{R}^d:|z|\leqslant 2}\exp\big(C|\log|z||\cdot|z|^{\beta_0}\big)<\infty,$$

and the last inequality is due to the same argument of the last one in (3.9).

Using the above inequality for $\rho_0^{z,0}(t-s,x-z)$ and following the same procedure of (3.8), we know that (3.8) still holds.

Therefore, according to the estimates for J_1 and J_2, (3.2) also holds in this case.

Case (c): $|x-y|\leqslant 2t^{1/\overline{\alpha}(y;t^{1/\alpha_2})}$**.**

Note that for any $t\in(0,1]$ and $y\in\mathbb{R}^d$,

$$1\leqslant\frac{t^{1/\overline{\alpha}(y;t^{1/\alpha_2})}}{t^{1/\alpha(y)}}\leqslant t^{-\frac{\overline{\alpha}(y;t^{1/\alpha_2})-\alpha(y)}{\alpha(y)\overline{\alpha}(y;t^{1/\alpha_2})}}\leqslant t^{-\frac{Ct^{\beta_0/\alpha_2}}{\alpha_1^2}}\leqslant\exp\big(C|\log t|\cdot t^{\beta_0/\alpha_2}\big)\leqslant C_1,\tag{3.11}$$

hence $|x-y|\leqslant 2t^{1/\overline{\alpha}(y;t^{1/\alpha_2})}$ implies that $|x-y|\leqslant C_2t^{1/\alpha(y)}$ for some constant $C_2>0$.

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^d}\rho_{\theta_1}^{z,\beta_1}(t-s,x-z)\rho_{\theta_2}^{y,\beta_2}(s,z-y)\,dz\leqslant&\int_{\{|z-y|>t^{1/\alpha(y)}\}}\rho_{\theta_1}^{z,\beta_1}(t-s,x-z)\rho_{\theta_2}^{y,\beta_2}(s,z-y)\,dz\\&+\int_{\{|z-y|\leqslant t^{1/\alpha(y)}\}}\rho_{\theta_1}^{z,\beta_1}(t-s,x-z)\rho_{\theta_2}^{y,\beta_2}(s,z-y)\,dz\\&=:J_1+J_2.\end{aligned}$$

If $|z-y|\geqslant t^{1/\alpha(y)}>s^{1/\alpha(y)}$, then

$$\rho_0^{y,0}(s,z-y)\preceq|z-y|^{-d-\alpha(y)}\preceq t^{-(d+\alpha(y))/\alpha(y)}\preceq\rho_0^{y,0}(t,x-y),$$

where the second inequality follows from $|z-y|\geqslant t^{1/\alpha(y)}$, and the last inequality is due to $t^{-(d+\alpha(y))/\alpha(y)}\preceq\rho_0^{y,0}(t,x-y)$ for all $|x-y|\leqslant C_2t^{1/\alpha(y)}$. Applying the above estimate for $\rho_0^{y,0}(s,z-y)$ and following the same argument of (3.6), we can find that (3.6) still holds for J_1 here.

Now we turn to estimate for J_2, which is divided into the following two subcases:

Subcase (c1): $s>t/2$.

If $|z-y|\leqslant t^{1/\alpha(y)}$, then

$$\rho_0^{y,0}(s,z-y)\preceq s^{-(d/\alpha(y))-1}\preceq t^{-(d/\alpha(y))-1}\preceq\rho_0^{y,0}(t,x-y),$$

where the second inequality follows from $s>t/2$, and in the last inequality we have used the fact that $t^{-(d/\alpha(y))-1}\preceq\rho_0^{y,0}(t,x-y)$ for all $|x-y|\leqslant C_2t^{1/\alpha(y)}$.

Having the above estimate for $\rho_0^{y,0}(s,z-y)$ at hand, we also can follow the same procedure of (3.6) to get that

$$J_2\preceq(t-s)^{\theta_1+((\beta_1+\beta_2)/\alpha(x))-1}s^{\theta_2}\rho_0^{y,0}(t,x-y)+(t-s)^{\theta_1+(\beta_1/\alpha(x))-1}s^{\theta_2}\rho_0^{y,\beta_2}(t,x-y).$$

Subcase (c2): $s\leqslant t/2$.

If $|z-y|\leqslant t^{1/\alpha(y)}$, then we have

$$\begin{aligned}\rho_0^{z,0}(t-s,x-z)&\preceq(t-s)^{-(d/\alpha(z))-1}\preceq t^{-(d/\alpha(z))-1}\preceq t^{-|(d/\alpha(z))-(d/\alpha(y))|}\cdot t^{-(d/\alpha(y))-1}\\&\preceq t^{-Ct^{\beta_0/\alpha_2}}\cdot t^{-(d/\alpha(y))-1}\preceq t^{-(d/\alpha(y))-1}\preceq\rho_0^{y,0}(t,x-y),\end{aligned}$$

where the second inequality is due to $(t-s)\geqslant t/2$, in the fourth inequality we used the fact that $|\alpha(y)-\alpha(z)|\preceq|z-y|^{\beta_0}\preceq t^{\beta_0/\alpha(y)}\preceq t^{\beta_0/\alpha_2}$, the fifth inequality follows from

$$\sup_{t\in(0,1]}t^{-Ct^{\beta_0/\alpha_2}}\preceq\sup_{t\in(0,1]}\exp\left(C|\log t|t^{\beta_0/\alpha_2}\right)<\infty,$$

and the last inequality is again due to the fact that $t^{-(d/\alpha(y))-1}\preceq\rho_0^{y,0}(t,x-y)$ for all $|x-y|\leqslant C_2t^{1/\alpha(y)}$.

Using the above estimate for $\rho_0^{z,0}(t-s,x-z)$ and following the same procedure of (3.8), we can also get (3.8).

Combining both estimates for J_1 and J_2, we know (3.2) holds for **Case (c)**. Therefore, the proof of (3.1) is complete.

(2) We can apply the second inequality in (2.1) in the proofs of (3.6) and (3.8), and follow the same arguments as above to prove (3.2). □

According to Lemma 3.1, we can immediately derive the following statement.

Corollary 3.2. *Let $\mathscr{B}(\gamma,\beta)$ denote the Beta function with respect to γ and β. Then, the following two statements hold.*

(1) For any $\theta_1,\theta_2\in\mathbb{R}$ and $\beta_1,\beta_2\in(0,1)$, there is a constant $c>0$ such that for all $t\in(0,1]$ and $x,y\in\mathbb{R}^d$ with $\beta_1+\beta_2<\alpha_*(x,y):=\alpha(x)\wedge\alpha(y)$, $\theta_1+(\beta_1/\alpha(x))>0$ and $\theta_2+(\beta_2/\alpha(y))>0$,

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\rho_{\theta_1}^{z,\beta_1}(t-s,x-z)\rho_{\theta_2}^{y,\beta_2}(s,z-y)\,dz\,ds\\&\leqslant c\mathscr{B}\big((\beta_1/\alpha_2)+\theta_1,(\beta_2/\alpha_2)+\theta_2\big)\\&\quad\times\Big(\rho^{y,0}_{\theta_1+\theta_2+((\beta_1+\beta_2)/\alpha^*(x,y))}+\rho^{y,\beta_2}_{\theta_1+\theta_2+(\beta_1/\alpha(x))}+\rho^{y,\beta_1}_{\theta_1+\theta_2+(\beta_2/\alpha(y))}\Big)(t,x-y),\end{aligned}\tag{3.12}$$

where $\alpha^*(x,y):=\alpha(x)\vee\alpha(y)$.

(2) For any $\theta_1, \theta_2 \in \mathbb{R}$ and $\beta_1, \beta_2 \in (0,1)$ such that $\beta_1 + \beta_2 < \alpha_2$, $\theta_1 + (\beta_1/\alpha_2) > 0$ and $\theta_2 + (\beta_2/\alpha_2) > 0$, it holds for any $t \in (0,1]$ and $x, y \in \mathbb{R}^d$ that

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d} \rho_{\theta_1}^{z,\beta_1}(t-s, x-z)\rho_{\theta_2}^{y,\beta_2}(s, z-y)\,dz\,ds\\ &\preceq \mathscr{B}\big((\beta_1/\alpha_2)+\theta_1, (\beta_2/\alpha_2)+\theta_2\big)\\ &\quad \times \big(\rho^{y,0}_{\theta_1+\theta_2+((\beta_1+\beta_2)/\alpha_2)} + \rho^{y,\beta_2}_{\theta_1+\theta_2+(\beta_1/\alpha_2)} + \rho^{y,\beta_1}_{\theta_1+\theta_2+(\beta_2/\alpha_2)}\big)(t, x-y).\end{aligned} \tag{3.13}$$

Proof. Note that

$$\int_0^t (t-s)^{\gamma-1}s^{\beta-1}\,ds = t^{\gamma+\beta-1}\mathscr{B}(\gamma,\beta), \quad \gamma>0,\ \beta>0.$$

This, along with (3.1), (3.2) and the decreasing property of $\mathscr{B}(\gamma,\beta)$ with respect to γ and β, yields the desired assertions (3.12) and (3.13). □

At the end of this subsection, we make some remarks.

Remark 3.3. (1) According to the proofs of [20, Lemma 2.1 (ii) and (iii)], we know that for $0 \leqslant s \leqslant t \leqslant 1$, $x, y \in \mathbb{R}^d$ and $\beta_1, \beta_2 \in (0,1)$, $\theta_1, \theta_2 \in \mathbb{R}$, it holds that

$$\begin{aligned}&\int_{\mathbb{R}^d} \rho_{\theta_1}^{y,\beta_1}(t-s, x-z)\rho_{\theta_2}^{y,\beta_2}(s, z-y)\,dz\\ &\preceq \Big[(t-s)^{\theta_1+[((\beta_1+\beta_2)/\alpha(y))\wedge 1]-1}s^{\theta_2}\big(1+|\log(t-s)|\mathbb{1}_{\{\beta_1+\beta_2=\alpha(y)\}}\big)\\ &\qquad + (t-s)^{\theta_1}s^{\theta_2+[((\beta_1+\beta_2)/\alpha(y))\wedge 1]-1}\big(1+|\log s|\mathbb{1}_{\{\beta_1+\beta_2=\alpha(y)\}}\big)\Big]\rho_0^{y,0}(t, x-y)\\ &\qquad + (t-s)^{\theta_1+[(\beta_1/\alpha(y))\wedge 1]-1}s^{\theta_2}\big(1+|\log(t-s)|\mathbb{1}_{\{\beta_1=\alpha(y)\}}\big)\rho_0^{y,\beta_2}(t, x-y)\\ &\qquad + (t-s)^{\theta_1}s^{\theta_2+[(\beta_2/\alpha(y))\wedge 1]-1}\big(1+|\log s|\mathbb{1}_{\{\beta_2=\alpha(y)\}}\big)\rho_0^{y,\beta_1}(t, x-y).\end{aligned} \tag{3.14}$$

If, in addition, $\beta_1 + \beta_2 < \alpha(x)$, $(\beta_1/\alpha(y)) + \theta_1 > 0$ and $(\beta_2/\alpha(y)) + \theta_2 > 0$, then it also holds

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d} \rho_{\theta_1}^{y,\beta_1}(t-s, x-z)\rho_{\theta_2}^{y,\beta_2}(s, z-y)\,dz\,ds\\ &\preceq \mathscr{B}\big((\beta_1/\alpha(y))+\theta_1, (\beta_2/\alpha(y))+\theta_2\big)\\ &\quad \times \big(\rho^{y,0}_{\theta_1+\theta_2+((\beta_1+\beta_2)/\alpha(y))} + \rho^{y,\beta_1}_{\theta_1+\theta_2+(\beta_2/\alpha(y))}(t, x-y) + \rho^{y,\beta_2}_{\theta_1+\theta_2+(\beta_1/\alpha(y))}\big)(t, x-y).\end{aligned} \tag{3.15}$$

Note that in (3.14) and (3.15) the index $\alpha(y)$ is independent of the integrand variable z. However, in the present setting we also need (3.1) and (3.2) as well as the convolution inequalities (3.12) and (3.13), where the index $\alpha(z)$ will depend on the integrand variable z.

(2) For every fixed $\varepsilon > 0$ small enough, $\gamma \in \mathbb{R}$, $\theta \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^d$, define

$$\tilde\rho_{\gamma,\varepsilon}^{y,\theta}(t,x) = t^\gamma(|x|^\theta \wedge 1)\begin{cases}\frac{1}{(t^{1/\alpha(y)}+|x|)^{d+\alpha(y)}}, & |x| \leqslant 1,\\ \frac{1}{|x|^{d+\alpha_1-\varepsilon}}, & |x| > 1.\end{cases} \tag{3.16}$$

By carefully tracking the proofs of Lemma 2.1, Lemma 3.1, Corollary 3.2 and [20, Lemma 2.1 (ii) and (iii)], we know that the inequalities (2.1), (2.3), (3.1), (3.2), (3.12), (3.13), (3.14) and (3.15) are valid with ρ replaced by $\tilde\rho$. (In particular, (3.9) holds true.) For simplicity, in the remainder of this paper we often omit the parameter ε in $\tilde\rho$.

3.2. **Existence, upper bounds and continuity of $q(t,x,y)$.** We will prove the existence and some estimates for the solution $q(t,x,y)$ to the equation (1.27). For this, we first define $q_n(t,x,y)$ inductively by

$$q_n(t,x,y) := \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z)q_{n-1}(s,z,y)\,dz\,ds,\ n \geqslant 1,\ t \in (0,1]. \tag{3.17}$$

Then we can construct $q(t,x,y)$ as follows.

Proposition 3.4. *Let $\beta_0^* \in (0,\beta_0] \cap (0,\alpha_2)$ and $\beta_0^{**} \in (0,\beta_0] \cap (0,\alpha_2/2)$. Then, the following two statements hold.*

(1) If $\kappa(x,z)$ is independent of z, then $q(t,x,y) := \sum_{n=0}^\infty q_n(t,x,y)$ is absolutely convergent on $(0,1] \times \mathbb{R}^d \times \mathbb{R}^d$, it solves equation (1.27) and satisfies that for any $0 < \gamma < \theta < \beta_0^*/\alpha_2$ and $\varepsilon > 0$, there exists a positive constant $c_1 := c_1(\alpha,\kappa,\gamma,\theta,\varepsilon)$ such that for every $(t,x,x',y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$,

$$|q(t,x,y)| \leqslant c_1\left(\rho^{y,0}_{(\beta_0^*/\alpha_2)-\gamma} + \rho^{y,\beta_0^*}_{-\gamma}\right)(t, x-y) \tag{3.18}$$

and if, moreover, $\beta_0^* < \alpha(x)$, then

$$\begin{aligned}|q(t,x,y)-q(t,x',y)| \leqslant & c_1 \left(|x-x'|^{\alpha_1(\beta_0^*-\alpha_2\theta)/\alpha(x)} \wedge 1\right) \\ & \times \Big[\Big(\tilde\rho^{y,0}_{\theta-\gamma+(\beta_0^*/\alpha_2)-(\beta_0^*/\alpha_1)}(t,x-y)+\tilde\rho^{y,0}_{\theta-\gamma+(\beta_0^*/\alpha_2)-(\beta_0^*/\alpha_1)}(t,x'-y)\Big].\end{aligned} \tag{3.19}$$

Suppose, in addition, that there are $y_0 \in \mathbb{R}^d$ and $r_0 \in (0,\infty]$ such that $\alpha(z) = \alpha(y_0)$ for any $z \in B(y_0, r_0)$. Then there is a constant $c_3 := c_3(\alpha,\kappa,r_0) > 0$ such that for every $t \in (0,1]$ and $x \in \mathbb{R}^d$,

$$|q(t,x,y_0)| \leqslant c_3 \left(\rho^{y_0,0}_{\beta_0^*/\alpha_2} + \rho_0^{y_0,\beta_0^*}\right)(t,x-y_0). \tag{3.20}$$

(2) *If $\kappa(x,z)$ depends on z and $(\alpha_2/\alpha_1) - 1 < \beta_0^{**}/\alpha_2$ holds true, then $q(t,x,y) =: \sum_{n=0}^\infty q_n(t,x,y)$ is still absolutely convergent, solves the equation (1.27), and satisfies that for every $\gamma,\theta > 0$ such that $(\alpha_2/\alpha_1) - 1 + \gamma =: \gamma_2 < \theta < \beta_0^{**}/\alpha_2$ and $\varepsilon > 0$, there exists a positive constant $c_2 := c_2(\alpha,\kappa,\gamma,\theta,\varepsilon)$ such that for every $(t,x,x',y) \in (0,1]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$,*

$$|q(t,x,y)| \leqslant c_2 \left(\rho^{y,0}_{(\beta_0^{**}/\alpha_2)-\gamma_2} + \rho^{y,\beta_0^{**}}_{-\gamma_2}\right)(t,x-y) \tag{3.21}$$

and if, moreover, $\beta_0^* < \alpha(x)$, then

$$\begin{aligned}|q(t,x,y)-q(t,x',y)| \leqslant & c_2 \left(|x-x'|^{\alpha_1(\beta_0^*-\alpha_2\theta)/\alpha(x)} \wedge 1\right) \\ & \times \Big[\tilde\rho^{y,0}_{\theta-\gamma_2+(\beta_0^*/\alpha_2)-(\beta_0^*/\alpha_1)}(t,x-y)+\tilde\rho^{y,0}_{\theta-\gamma_2+(\beta_0^*/\alpha_2)-(\beta_0^*/\alpha_1)}(t,x'-y)\Big].\end{aligned} \tag{3.22}$$

If additionally there are some $y_0 \in \mathbb{R}^d$ and $r_0 \in (0,\infty]$ such that $\alpha(z) = \alpha(y_0)$ for all $z \in B(y_0,r_0)$, then there is a constant $c_4 := c_4(\alpha,\kappa,r_0) > 0$ such that for every $t \in (0,1]$ and $x \in \mathbb{R}^d$,

$$|q(t,x,y_0)| \leqslant c_4 \left(\rho^{y_0,0}_{1-(\alpha_2/\alpha_1)+(\beta_0^{**}/\alpha_2)} + \rho^{y_0,\beta_0^{**}}_{1-(\alpha_2/\alpha_1)}\right)(t,x-y_0). \tag{3.23}$$

Proof. Without loss of generality, throughout the proof we will assume that $\beta_0 < \alpha_2$ and $\beta_0^* = \beta_0$; otherwise, we will replace β_0 by β_0^*.

(i) According to (2.16), if $\kappa(x,z)$ is independent of z, then

$$|q_0(t,x,y)| \preceq \rho^{y,\beta_0}_{-\gamma}(t,x-y), \quad t \in (0,1] \text{ and } x,y \in \mathbb{R}^d.$$

Therefore, by Corollary 3.2,

$$\begin{aligned}|q_1(t,x,y)| \preceq & \int_0^t \int \rho^{z,\beta_0}_{-\gamma}(t-s,x-z)\rho^{y,\beta_0}_{-\gamma}(s,z-y)\,dz\,ds \\ \preceq & \mathscr{B}\big((\widehat\beta_0/\alpha_2)-\gamma,(\beta_0/\alpha_2)-\gamma\big)\big(\rho^{y,0}_{2((\widehat\beta_0/\alpha_2)-\gamma)} + \rho^{y,\beta_0}_{-\gamma+((\beta_0/\alpha_2)-\gamma)}\big)(t,x-y),\end{aligned}$$

where $\widehat\beta_0 := \beta_0 \wedge (\alpha_2/2)$.

Suppose now that

$$|q_n(t,x,y)| \leqslant \omega_n\Big(\rho^{y,0}_{(n+1)((\widehat\beta_0/\alpha_2)-\gamma)} + \rho^{y,\beta_0}_{-\gamma+n((\beta_0/\alpha_2)-\gamma)}\Big)(t,x-y), \quad n \geqslant 0,$$

where the constant ω_n is to be determined later. Hence, according to Corollary 3.2 again,

$$\begin{aligned}|q_{n+1}(t,x,y)| \leqslant & C\int_0^t \int \rho^{z,\beta_0}_{-\gamma}(t-s,x-z)\omega_n\Big(\rho^{y,0}_{(n+1)((\widehat\beta_0/\alpha_2)-\gamma)} + \rho^{y,\beta_0}_{-\gamma+n((\beta_0/\alpha_2)-\gamma)}\Big)(s,z-y)\,dz\,ds \\ \leqslant & C\omega_n\mathscr{B}((\beta_0/\alpha_2)-\gamma,(n+1)((\widehat\beta_0/\alpha_2)-\gamma))\Big(\rho^{y,0}_{(n+2)((\widehat\beta_0/\alpha_2)-\gamma)} + \rho^{y,\beta_0}_{-\gamma+(n+1)((\beta_0/\alpha_2)-\gamma)}\Big)(t,x-y) \\ =: & \,\omega_{n+1}\Big(\rho^{y,0}_{(n+2)((\widehat\beta_0/\alpha_2)-\gamma)} + \rho^{y,\beta_0}_{-\gamma+(n+1)((\beta_0/\alpha_2)-\gamma)}\Big)(t,x-y),\end{aligned}$$

where

$$\omega_{n+1} = C\mathscr{B}\big((\beta_0/\alpha_2)-\gamma,(n+1)((\widehat\beta_0/\alpha_2)-\gamma)\big)\omega_n,$$

and C is a constant independent of t,n,x and y. Note that $\omega_0 \leqslant C$ and $\mathscr{B}(\gamma,\beta) = \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}$, where Γ is the standard Gamma function. By iteration procedure we have

$$\begin{aligned}\omega_n \leqslant & C^{n+1}\mathscr{B}\big((\beta_0/\alpha_2)-\gamma,n((\widehat\beta_0/\alpha_2)-\gamma)\big) \\ & \times \mathscr{B}\big((\beta_0/\alpha_2)-\gamma,(n-1)((\widehat\beta_0/\alpha_2)-\gamma)\big)\cdots\mathscr{B}\big((\beta_0/\alpha_2)-\gamma,(\widehat\beta_0/\alpha_2)-\gamma\big) \\ = & \frac{\big[C\Gamma((\beta_0/\alpha_2)-\gamma)\big]^{n+1}}{\Gamma\big((n+1)((\widehat\beta_0/\alpha_2)-\gamma)\big)}.\end{aligned}$$

Therefore,

$$|q_n(t,x,y)| \leqslant \frac{\big[C\Gamma((\beta_0/\alpha_2)-\gamma)\big]^{n+1}}{\Gamma\big((n+1)((\widehat{\beta}_0/\alpha_2)-\gamma)\big)}\Big(\rho^{y,0}_{(n+1)((\widehat{\beta}_0/\alpha_2)-\gamma)}+\rho^{y,\beta_0}_{-\gamma+n((\beta_0/\alpha_2)-\gamma)}\Big)(t,x-y), \tag{3.24}$$

and so combing all estimates together

$$\sum_{n=0}^\infty |q_n(t,x,y)| \leqslant \left(\sum_{n=0}^\infty \frac{\big[C\Gamma((\beta_0/\alpha_2)-\gamma)\big]^{n+1}}{\Gamma\big((n+1)((\widehat{\beta}_0/\alpha_2)-\gamma)\big)}\right)\Big(\rho^{y,0}_{2((\widehat{\beta}_0/\alpha_2)-\gamma)}+\rho^{y,\beta_0}_{-\gamma}\Big)(t,x-y),$$

which means that $q(t,x,y):=\sum_{n=0}^\infty q_n(t,x,y)$ is absolutely convergent, and

$$|q(t,x,y)|\leqslant C_1\Big(\rho^{y,0}_{2((\widehat{\beta}_0/\alpha_2)-\gamma)}+\rho^{y,\beta_0}_{-\gamma}\Big)(t,x-y).$$

Thus, (3.18) is proved by using the fact that $\beta_0\leqslant 2\widehat{\beta}_0$ and changing the constant γ properly. From (3.17) and the fact $q(t,x,y)=\sum_{n=0}^\infty q_n(t,x,y)$, it is easy to see that $q(t,x,y)$ solves the equation (1.27).

(ii) Suppose that there are some $y_0\in\mathbb{R}^d$ and $r_0\in(0,\infty]$ such that $\alpha(z)=\alpha(y_0)$ for all $z\in B(y_0,r_0)$. According to (2.17), it is easy to verify that for every $z\in\mathbb{R}^d$,

$$\begin{aligned}|q_0(s,z,y_0)| &\preceq \big(|z-y_0|^{\beta_0}\wedge 1\big)\cdot\begin{cases}\frac{1}{(s^{1/\alpha(y_0)}+|z-y_0|)^{d+\alpha(y_0)}}, & |z-y_0|\leqslant r_0,\\ \frac{1}{|z-y_0|^{d+\alpha_1}}, & |z-y_0|>r_0\end{cases}\\ &\preceq \rho_0^{y_0,\beta_0}(s,z-y_0).\end{aligned}\tag{3.25}$$

This along with the fact that $|q_0(t-s,x,z)|\preceq\rho^{z,\beta_0}_{-\gamma}(t-s,x-z)$ for every $x,z\in\mathbb{R}^d$ yields that for all $x\in\mathbb{R}^d$,

$$\begin{aligned}|q_1(t,x,y_0)| &\preceq \int_0^t\int_{\mathbb{R}^d}\rho^{z,\beta_0}_{-\gamma}(t-s,x-z)\rho_0^{y_0,\beta_0}(s,z-y_0)\,dz\,ds\\ &\leqslant C\mathscr{B}\big((\widehat{\beta}_0/\alpha_2)-\gamma,(\beta_0/\alpha_2)-\gamma\big)\big(\rho^{y_0,0}_{(\widehat{\beta}_0/\alpha_2)+((\widehat{\beta}_0/\alpha_2)-\gamma)}+\rho^{y_0,\beta_0}_{(\beta_0/\alpha_2)-\gamma}\big)(t,x-y_0).\end{aligned}$$

Following the same arguments in part (i), we find that for every $n\geqslant 0$,

$$|q_n(t,x,y_0)|\leqslant\frac{\big[C\Gamma((\beta_0/\alpha_2)-\gamma)\big]^{n+1}}{\Gamma\big((n+1)((\widehat{\beta}_0/\alpha_2)-\gamma)\big)}\Big(\rho^{y_0,0}_{(\widehat{\beta}_0/\alpha_2)+n((\widehat{\beta}_0/\alpha_2)-\gamma)}+\rho^{y_0,\beta_0}_{n((\beta_0/\alpha_2)-\gamma)}\Big)(t,x-y_0),$$

which implies $q(t,x,y):=\sum_{n=1}^\infty q_n(t,x,y)$ is absolutely convergent and gives us (3.20) immediately.

(iii) It suffices to prove (3.19) for the case that $|x-x'|\leqslant R_1$ holds with some $R_1>0$, since the case that $|x-x'|>1$ follows from (3.18) immediately. For any (fixed) $\varepsilon>0$ small enough, let $\tilde\rho^{y,\beta}_\gamma$ be defined by (3.16). By (2.38), we have

$$\begin{aligned}&|q_0(t,x,y)-q_0(t,x',y)|\\ &\preceq(|x-x'|^{\tilde\theta}\wedge 1)\times\Big[\Big(\tilde\rho^{y,0}_{\theta-\gamma}+\tilde\rho^{y,\beta_0}_{\theta-\gamma-(\beta_0/\alpha(x))}\Big)(t,x-y)+\Big(\tilde\rho^{y,0}_{\theta-\gamma}+\tilde\rho^{y,\beta_0}_{\theta-\gamma-(\beta_0/\alpha(x'))}\Big)(t,x'-y)\Big],\end{aligned}\tag{3.26}$$

where $\tilde\theta:=\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x)$. Since $\beta_0^*=\beta_0<\alpha(x)$, by (1.3) we can find a constant $R_1>0$ small enough such that $\beta_0<\alpha(x)\wedge\alpha(x')$ for any $x'\in\mathbb{R}^d$ with $|x-x'|\leqslant R_1$. Then, according to (1.27), it holds that

$$\begin{aligned}&|q(t,x,y)-q(t,x',y)|\\ &\leqslant|q_0(t,x,y)-q_0(t,x',y)|+\int_0^t\int_{\mathbb{R}^d}|q_0(t-s,x,z)-q_0(t-s,x',z)||q(s,z,y)|\,dz\,ds\\ &\preceq(|x-x'|^{\tilde\theta}\wedge 1)\\ &\quad\times\Big\{\Big[\big(\tilde\rho^{y,0}_{\theta-\gamma}+\tilde\rho^{y,\beta_0}_{\theta-\gamma-(\beta_0/\alpha(x))}\big)(t,x-y)+\Big(\tilde\rho^{y,0}_{\theta-\gamma}+\tilde\rho^{y,\beta_0}_{\theta-\gamma-(\beta_0/\alpha(x'))}\Big)(t,x'-y)\Big]\\ &\qquad+\int_0^t\int_{\mathbb{R}^d}\Big[\big(\tilde\rho^{z,0}_{\theta-\gamma}+\tilde\rho^{z,\beta_0}_{\theta-\gamma-(\beta_0/\alpha(x))}\big)(t-s,x-z)\\ &\qquad\qquad+\big(\tilde\rho^{z,0}_{\theta-\gamma}+\tilde\rho^{z,\beta_0}_{\theta-\gamma-(\beta_0/\alpha(x'))}\big)(t-s,x'-z)\Big]\Big[\big(\tilde\rho^{y,0}_{(\beta_0/\alpha_2)-\gamma}+\tilde\rho^{y,\beta_0}_{-\gamma}\big)(s,z-y)\Big]\,dz\,ds\Big\}\\ &\preceq(|x-x'|^{\tilde\theta}\wedge 1)\cdot\Big[\tilde\rho^{y,0}_{\theta-2\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}(t,x-y)+\tilde\rho^{y,0}_{\theta-2\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}(t,x'-y)\Big].\end{aligned}$$

Here the second inequality follows from (3.18) and (3.26), thanks to the fact that $\beta_0 < \alpha(x)\wedge\alpha(x')$, and the last inequality is due to Remark 3.3(2) (which indicates that the convolution inequality (3.12) holds for $\tilde\rho$) and the fact that $\gamma < \theta$.

(iv) Now we are going to prove the case that $\kappa(x,z)$ depends on z. Due to the assumption $(\alpha_2/\alpha_1) - 1 < \beta_0^{**}/\alpha_2$, we can choose $\gamma > 0$ small enough such that $\gamma + (\alpha_2/\alpha_1) - 1 =: \gamma_2 < \beta_0^{**}/\alpha_2$. Using (2.18) and following the same arguments in part (i) with γ_2 instead of γ, we can obtain that for every $n \geqslant 1$ and $(t,x,y) \in (0,1]\times\mathbb{R}^d\times\mathbb{R}^d$,

$$|q_n(t,x,y)| \leqslant \frac{\big[C\Gamma((\beta_0^{**}/\alpha_2)-\gamma_2)\big]^{n+1}}{\Gamma\big((n+1)((\beta_0^{**}/\alpha_2)-\gamma_2)\big)}\left(\rho^{y,0}_{(n+1)((\beta_0^{**}/\alpha_2)-\gamma_2)} + \rho^{y,\beta_0}_{-\gamma_2+n(\beta_0^{**}/\alpha_2-\gamma_2)}\right)(t,x-y). \tag{3.27}$$

Using this and (2.18), we find immediately that $\sum_{n=0}^\infty |q_n(t,x,y)| < \infty$ and so (3.21) is true.

Note that, according to Proposition 2.9(2), in this case (3.26) holds with γ_2 in place of γ. (Note that here β_0 is not replaced by β_0^{**}.) This along with (3.21) and the same argument in (iii) gives us (3.22).

(v) Suppose that the assumptions in part (2) of the Proposition hold, and that there are some $y_0 \in \mathbb{R}^d$ and $r_0 \in (0,\infty]$ such that $\alpha(z) = \alpha(y_0)$ for all $z \in B(y_0,r_0)$. Then, using (2.19) and repeating the arguments in part (ii) above, we can prove (3.23). □

Furthermore, according to Proposition 2.8, we also have the following estimates for $q(t,x,y)$.

Proposition 3.5. *(1) Suppose that $\kappa(x,z)$ is independent of z. Then there exists a constant $c_5 := c_5(\alpha,\kappa) > 0$ such that for every $t \in (0,1]$ and $x,y \in \mathbb{R}^d$,*

$$|q(t,x,y)| \leqslant \begin{cases} c_5 t^{-1-(d/\alpha(y))}, & |x-y| \leqslant t^{1/\alpha(y)},\\ \frac{c_5}{|x-y|^{d+\alpha_2}}, & t^{1/\alpha(y)} < |x-y| \leqslant 1,\\ \frac{c_5}{|x-y|^{d+\alpha_1}}, & |x-y| > 1.\end{cases} \tag{3.28}$$

(2) Let $\beta_0^{**} \in (0,\beta_0]\cap(0,\alpha_2/2)$. Suppose that $\kappa(x,z)$ depends on z, and $\beta_0^{**}/\alpha_2 > (\alpha_2/\alpha_1) - 1$. Then exists a constant $c_6 := c_6(\alpha,\kappa) > 0$ such that for every $t \in (0,1]$ and $x,y \in \mathbb{R}^d$,

$$|q(t,x,y)| \leqslant \begin{cases} c_6 t^{-1-(d/\alpha(y))}, & |x-y| \leqslant t^{1/\alpha(y)},\\ \frac{c_6 t^{1-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_2}}, & t^{1/\alpha(y)} < |x-y| \leqslant 1,\\ \frac{c_6 t^{1-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_1}}, & |x-y| > 1.\end{cases} \tag{3.29}$$

Proof. Without loss of generality, throughout the proof we still assume that $\beta_0 < \alpha_2/2$ and $\beta_0^{**} = \beta_0$.

We first suppose that $\kappa(x,z)$ is independent of z. As in the proof of Proposition 3.4(1), we know that $q(t,x,y) = \sum_{n=0}^\infty q_n(t,x,y)$ is absolutely convergent. In particular,

$$|q(t,x,y)| \leqslant |q_0(t,x,y)| + \sum_{n=1}^\infty |q_n(t,x,y)|. \tag{3.30}$$

According to (3.24), we have

$$\sum_{n=1}^\infty |q_n(t,x,y)| \leqslant \left(\sum_{n=1}^\infty \frac{\big[C\Gamma((\beta_0/\alpha_2)-\gamma)\big]^{n+1}}{\Gamma\big((n+1)((\beta_0/\alpha_2)-\gamma)\big)}\right)\left(\rho^{y,0}_{3((\beta_0/\alpha_2)-\gamma)} + \rho^{y,\beta_0}_{(\beta_0/\alpha_2)-2\gamma}\right)(t,x-y) \preceq \rho_0^{y,0}(t,x-y),$$

where in the last inequality we choose $\gamma > 0$ small enough such that $\gamma < \beta_0/(2\alpha_2)$. Combining this estimate with (2.34) and (3.30), we obtain (3.28) immediately.

If $\kappa(x,z)$ depends on z and $\beta_0/\alpha_2 > (\alpha_2/\alpha_1) - 1$, then one can use (2.35) and follow the same procedure above to prove (3.29). □

4. Estimates for $p(t,x,y)$

4.1. **Existence and upper bounds for** $p(t,x,y)$**.** By making full use of the estimates for $q(t,x,y)$ in Proposition 3.4, we now can prove that $p(t,x,y)$ is well defined by (1.26).

Proposition 4.1. *(1) If $\kappa(x,z)$ is independent of z, then $p(t,x,y)$ is well defined by (1.26), and for every $\gamma, c_0 > 0$, there exist positive constants $c_1 := c_1(\alpha,\kappa,\gamma,c_0)$ and $R_0 := R_0(\alpha,\kappa,\gamma,c_0)$ such that for any $t \in (0,1]$ and $x,y \in \mathbb{R}^d$,*

$$p(t,x,y) \leqslant \begin{cases} c_1 t^{-d/\alpha(x)}, & |y-x| \leqslant c_0 t^{1/\alpha(x)},\\ \frac{c_1 t^{1-\gamma}}{|x-y|^{d+\alpha(x)}}, & c_0 t^{1/\alpha(x)} \leqslant |y-x| \leqslant R_0,\\ \frac{c_1 t^{1-\gamma}}{|x-y|^{d+\alpha_1}}, & |y-x| > R_0.\end{cases} \tag{4.1}$$

Suppose additionally that there are some $x_0 \in \mathbb{R}^d$ and $r_0 \in (0,\infty]$ such that $\alpha(z)=\alpha(x_0)$ for every $z \in B(x_0,r_0)$. Then we can find a constant $c_2 := c_2(\alpha,\kappa,\gamma,r_0)>0$ such that for every $t\in(0,1]$ and $y\in\mathbb{R}^d$,

$$p(t,x_0,y) \leqslant \begin{cases} \frac{c_2 t}{(t^{1/\alpha(x_0)}+|x_0-y|)^{d+\alpha(x_0)}}, & |x_0-y|\leqslant r_0/2,\\ \frac{c_2 t^{1-\gamma}}{|x_0-y|^{d+\alpha_1}}, & |x_0-y|>r_0/2. \end{cases} \tag{4.2}$$

(2) If $\kappa(x,z)$ depends on z and $(\alpha_2/\alpha_1)-1<\min\{\beta_0/\alpha_2,1/2\}$, then $p(t,x,y)$ is well defined by (1.26), and for every $\gamma, c_0>0$, there exist constants $c_1:=c_1(\alpha,\kappa,\gamma,c_0)$ and $R_0:=R_0(\alpha,\kappa,\gamma,c_0)$ such that for any $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,

$$p(t,x,y)\leqslant \begin{cases} c_1 t^{-d/\alpha(x)}, & |y-x|\leqslant c_0 t^{1/\alpha(x)},\\ \frac{c_1 t^{1-\tilde\gamma}}{|x-y|^{d+\alpha(x)}}, & c_0 t^{1/\alpha(x)}\leqslant |y-x|\leqslant R_0,\\ \frac{c_1 t^{1-\tilde\gamma}}{|x-y|^{d+\alpha_1}}, & |y-x|>R_0, \end{cases} \tag{4.3}$$

where $\tilde\gamma := (\alpha_2/\alpha_1)-1+\gamma$. Suppose additionally there exist some $x_0\in\mathbb{R}^d$ and $r_0\in(0,\infty]$ such that $\alpha(z)=\alpha(x_0)$ for every $z\in B(x_0,r_0)$. Then, for any $c_0>0$, we can find a constant $c_2 := c_2(\alpha,\kappa,\gamma,r_0,c_0)>0$ such that for every $t\in(0,1]$ and $y\in\mathbb{R}^d$,

$$p(t,x_0,y)\leqslant \begin{cases} c_2 t^{-d/\alpha(x_0)}, & |y-x_0|<c_0 t^{1/\alpha(x_0)},\\ \frac{c_2 t^{2-(\alpha_2/\alpha_1)}}{|y-x_0|^{d+\alpha(x_0)}}, & c_0 t^{1/\alpha(x_0)}\leqslant |y-x_0|<r_0/2,\\ \frac{c_2 t^{1-\tilde\gamma}}{|x-y_0|^{d+\alpha_1}}, & |y-x_0|\geqslant r_0/2. \end{cases} \tag{4.4}$$

Proof. Without loss of generality, throughout this proof we will assume that $2\beta_0<\alpha_2$ and $\beta_0^{**}=\beta_0$; otherwise, we will replace β_0 by $\beta_0^{**}\in(0,\beta_0]\cap(0,\alpha_2/2)$. For simplicity, we only verify the case that $c_0=1$.

(i) We first consider the case that $\kappa(x,z)$ is independent of z. Note that

$$p^z(t,x-z)\preceq \rho_1^{z,0}(t,x-z),\quad t\in(0,1], x,z\in\mathbb{R}^d.$$

This, along with (3.18) and (3.13), yields that for every $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$ and any constant $0<\gamma<\beta_0/\alpha_2$,

$$\int_0^t\int_{\mathbb{R}^d} p^z(t-s,x-z)q(s,z,y)\,dz\,ds \preceq \big(\rho_{1-\gamma}^{y,\beta_0}+\rho^{y,0}_{1+(\beta_0/\alpha_2)-\gamma}\big)(t,x-y).$$

Therefore, $p(t,x,y)$ is well defined by (1.26), and

$$p(t,x,y)\preceq \Big(\rho_1^{y,0}+\rho_{1-\gamma}^{y,\beta_0}+\rho^{y,0}_{1+(\beta_0/\alpha_2)-\gamma}\Big)(t,x-y)\preceq \big(\rho_1^{y,0}+\rho_{1-\gamma}^{y,\beta_0}\big)(t,x-y). \tag{4.5}$$

When $|x-y|\leqslant t^{1/\alpha(x)}\leqslant t^{1/\alpha_2}$,

$$\rho_{1-\gamma}^{y,\beta_0}(t,x-y)\preceq t^{-\gamma}|x-y|^{\beta_0}\rho_1^{y,0}(t,x-y)\preceq t^{(\beta_0/\alpha_2)-\gamma}\rho_1^{y,0}(t,x-y).$$

Hence, due to $\gamma<\beta_0/\alpha_2$, we get

$$p(t,x,y)\preceq \rho_1^{y,0}(t,x-y)\preceq t^{-d/\alpha(y)}\preceq t^{-d/\overline{\alpha}(y;t^{1/\alpha_2})}\preceq t^{-d/\alpha(x)}, \tag{4.6}$$

where the third inequality follows from (3.11).

Next, we will verify the other two cases in the upper bound (4.1). Observe that for every $t\in(0,1]$ and $x,y,z\in\mathbb{R}^d$ such that $|x-y|\leqslant 1$ and $|z|\leqslant 1$,

$$\begin{aligned}\frac{1}{(t^{1/\alpha(y)}+|z|)^{d+\alpha(y)}} &= \frac{1}{(t^{1/\alpha(x)}t^{(1/\alpha(y))-(1/\alpha(x))}+|z|)^{d+\alpha(y)}}\preceq \frac{1}{(t^{1/\alpha(x)}t^{|\alpha(x)-\alpha(y)|/\alpha_1^2}+|z|)^{d+\alpha(y)}}\\ &\preceq \frac{1}{(t^{1/\alpha(x)}t^{|\alpha(x)-\alpha(y)|/\alpha_1^2}+|z|t^{|\alpha(x)-\alpha(y)|/\alpha_1^2})^{d+\alpha(y)}}\\ &= t^{-|\alpha(x)-\alpha(y)|(d+\alpha(y))/\alpha_1^2}\cdot\frac{1}{(t^{1/\alpha(x)}+|z|)^{\alpha(y)-\alpha(x)}}\cdot\frac{1}{(t^{1/\alpha(x)}+|z|)^{d+\alpha(x)}}\\ &\preceq t^{-(d+\alpha_2)|\alpha(x)-\alpha(y)|/\alpha_1^2}\cdot t^{-|\alpha(x)-\alpha(y)|/\alpha_1}\cdot\frac{1}{(t^{1/\alpha(x)}+|z|)^{d+\alpha(x)}}\preceq\frac{t^{-C_1R_0^{\beta_0}}}{(t^{1/\alpha(x)}+|z|)^{d+\alpha(x)}},\end{aligned}$$

where the second inequality we used $t^{|\alpha(x)-\alpha(y)|/\alpha_1^2}\leqslant 1$, and the third inequality follows from the fact that for all $t\in(0,1]$ and $x,y,z\in\mathbb{R}^d$ with $|z|\leqslant 1$,

$$\frac{1}{(t^{1/\alpha(x)}+|z|)^{\alpha(y)-\alpha(x)}}\preceq t^{-|\alpha(y)-\alpha(x)|/\alpha(x)}.$$

Thus, choosing $R_0 := R_0(\alpha,\gamma)$ small enough such that $C_1 R_0^{\beta_0} < \gamma$ and using the definition of $\rho_0^{x,0}$, we can get that for every $t \in (0,1]$ and $x, y, z \in \mathbb{R}^d$ with $|x-y| \leqslant R_0$ and $|z| \leqslant 1$,

$$\rho_0^{y,0}(t,z) \preceq t^{-\gamma} \rho_0^{x,0}(t,z). \tag{4.7}$$

By the definition of (1.10), we know immediately that (4.7) still holds for every $|x-y| \leqslant R_0$ and $|z| > 1$. This is, (4.7) holds for all $x, y, z \in \mathbb{R}^d$ with $|x-y| \leqslant R_0$.

Hence, combining (4.5), (4.6) with (4.7) and changing the constant γ properly, we find that

$$p(t,x,y) \preceq \left(\rho_1^{y,0} + \rho_{1-\gamma}^{y,\beta_0}\right)(t, x-y) \preceq \begin{cases} t^{-d/\alpha(x)}, & |y-x| \leqslant t^{1/\alpha(x)}, \\ \frac{t^{1-\gamma}}{|x-y|^{d+\alpha(x)}}, & t^{1/\alpha(x)} \leqslant |y-x| \leqslant R_0, \\ \frac{t^{1-\gamma}}{|x-y|^{d+\alpha_1}}, & |y-x| > R_0. \end{cases}$$

(ii) If there are some $x_0 \in \mathbb{R}^d$ and $r_0 \in (0,\infty]$ such that $\alpha(z) = \alpha(x_0)$ for all $z \in B(x_0, r_0)$, then for every $y \in B(x_0, r_0/2)$ and $z \in B(y, r_0/2)$, $\alpha(z) = \alpha(y) = \alpha(x_0)$. Hence, by (3.20), we have

$$|q(t,x,y)| \preceq \left(\rho_{\beta_0/\alpha_2}^{y,0} + \rho_0^{y,\beta_0}\right)(t, x-y) \quad \text{for all } x \in \mathbb{R}^d \text{ and } y \in B\big(x_0, r_0/2\big).$$

Based on the inequality above, the computation in part (i) is valid with $\gamma = 0$ for $x = x_0$ and $y \in B(x_0, r_0/2)$, which proves (4.2) for the case that $y \in B(x_0, r_0/2)$. The upper bound for the case $|y - x_0| > r_0/2$ is just the same as that of (4.1).

(iii) If $\kappa(x,z)$ depends on z and $(\alpha_2/\alpha_1) - 1 < \beta_0/\alpha_2$, then, according to (3.21), we know that the computation in part (i) holds with γ replaced by $\tilde{\gamma}$. Thus, following the argument in part (i), we can obtain (4.3). Similarly as in (ii), (4.4) could be verified by using (3.23). □

Remark 4.2. The estimate (4.2) indicates that if $\alpha(x)$ is a constant $\alpha \in (0,2)$ locally, then we can get a upper bound for $p(t,x,y)$ which is also locally comparable with that for the heat kernel of rotationally symmetric α-stable process. In particular, when $\alpha(x) \equiv \alpha$ for all $x \in \mathbb{R}^d$, (4.4) coincides with the upper bound given in [20, Therem 1.1].

Besides Proposition 4.1, we also can obtain the following upper bound for $p(t,x,y)$, which is based on Proposition 3.5.

Proposition 4.3. *(1) If $\kappa(x,z)$ is independent of z, then for any $c_0 > 0$, there exists a positive constant $c_1 := c_1(\alpha,\kappa,c_0)$ such that for every $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$p(t,x,y) \leqslant \begin{cases} c_1 t^{-d/\alpha(x)}, & |x-y| \leqslant c_0 t^{1/\alpha(x)}, \\ \frac{c_1 t}{|x-y|^{d+\alpha_2}}, & c_0 t^{1/\alpha(x)} < |x-y| \leqslant 1, \\ \frac{c_1 t}{|x-y|^{d+\alpha_1}}, & |x-y| > 1. \end{cases} \tag{4.8}$$

(2) Let $\beta_0^{**} \in (0,\beta_0] \cap (0,\alpha_2/2)$. Suppose that $\kappa(x,z)$ depends on z and $\beta_0^{**}/\alpha_2 > (\alpha_2/\alpha_1) - 1$. Then, for any $c_0 > 0$, there exists a positive constant $c_2 := c_2(\alpha,\kappa,c_0)$ such that for every $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$p(t,x,y) \leqslant \begin{cases} c_2 t^{-d/\alpha(x)}, & |x-y| \leqslant c_0 t^{1/\alpha(x)}, \\ \frac{c_2 t^{2-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_2}}, & c_0 t^{1/\alpha(x)} < |x-y| \leqslant 1, \\ \frac{c_2 t^{2-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_1}}, & |x-y| > 1. \end{cases} \tag{4.9}$$

Proof. Throughout the proof, we assume that $\beta_0 < \alpha_2/2$ and that $\beta_0 = \beta_0^{**}$. We only verify the case that $c_0 = 1$.

(1) We first suppose that $\kappa(x,z)$ is independent of z.

Case (a): $|x-y| \leqslant t^{1/\alpha(x)}$.

According to (4.1), we can easily see that (4.8) holds.

Case (b): $|x-y| > 2$.

We have

$$\begin{aligned} \left| \int_{\mathbb{R}^d} p^z(t-s, x-z) q(s,z,y)\, dz \right| &\leqslant \int_{\{|z-y| > |x-y|/2\}} p^z(t-s, x-z) |q(s,z,y)|\, dz \\ &\quad + \int_{\{|z-y| \leqslant |x-y|/2\}} p^z(t-s, x-z) |q(s,z,y)|\, dz \\ &=: J_1 + J_2. \end{aligned} \tag{4.10}$$

When $|z-y|\geqslant |x-y|/2\geqslant 1$, it follows from (3.28) that for any $s\in(0,1]$ and $y,z\in\mathbb{R}^d$,

$$|q(s,z,y)|\preceq \frac{1}{|z-y|^{d+\alpha_1}}\preceq \frac{1}{|x-y|^{d+\alpha_1}}.$$

Therefore, by (2.21) and (2.1),

$$J_1\preceq \frac{1}{|x-y|^{d+\alpha_1}}\int_{\mathbb{R}^d}\rho_1^{z,0}(t-s,x-z)\,dz\preceq \frac{1}{|x-y|^{d+\alpha_1}}.$$

Meanwhile, it is not difficult to see that condition $|z-y|\leqslant |x-y|/2$ implies $|z-x|\geqslant |x-y|/2\geqslant 1$, and so

$$p^z(t-s,x-z)\preceq \frac{t-s}{|x-z|^{d+\alpha_1}}\preceq \frac{t}{|x-z|^{d+\alpha_1}}.$$

This, along with (3.18) and (2.3), yields that

$$J_2\preceq \frac{t}{|x-y|^{d+\alpha_1}}\int_{\mathbb{R}^d}\left(\rho^{y,0}_{(\beta_0/\alpha_2)-\gamma}+\rho^{y,\beta_0}_{-\gamma}\right)(s,z-y)\,dz\preceq \frac{t}{|x-y|^{d+\alpha_1}}\cdot s^{-1+(\beta_0/\alpha_2)-\gamma}.$$

Combining all the estimates above together and choosing $0<\gamma<\beta_0/\alpha_2$, we find that

$$\left|\int_0^t\int_{\mathbb{R}^d}p^z(t-s,x-z)q(s,z,y)\,dz\,ds\right|\preceq\int_0^t(J_1+J_2)\,ds\preceq \frac{t}{|x-y|^{d+\alpha_1}}\cdot\left(1+t^{(\beta_0/\alpha_2)-\gamma}\right)\preceq \frac{t}{|x-y|^{d+\alpha_1}}.$$

Then, the desired assertion (4.8) immediately follows from the estimate above and (1.26).

Case (c): $t^{1/\alpha(x)}\leqslant |x-y|\leqslant 2$.

We still define J_1 and J_2 by those in (4.10). If $|z-y|\geqslant |x-y|/2$, then, by (3.28), we have

$$\begin{aligned}|q(s,z,y)|&\preceq\begin{cases}s^{-d/\alpha(y)}, & |z-y|\leqslant s^{1/\alpha(y)},\\ |z-y|^{-d-\alpha_2}, & s^{1/\alpha(y)}\leqslant |z-y|\leqslant 1,\\ |z-y|^{-d-\alpha_1}, & |z-y|>1\end{cases}\\ &\preceq\begin{cases}|x-y|^{-d-\alpha(y)}, & |z-y|\leqslant s^{1/\alpha(y)},\\ |x-y|^{-d-\alpha_2}, & s^{1/\alpha(y)}\leqslant |z-y|\leqslant 1,\\ |x-y|^{-d-\alpha_1}, & |z-y|>1\end{cases}\\ &\preceq |x-y|^{-d-\alpha_2},\end{aligned}$$

where in the second inequality we have used that fact that if $|z-y|\leqslant s^{1/\alpha(y)}$ and $|x-y|\preceq|z-y|$, then

$$s^{-d/\alpha(y)}\preceq |z-y|^{-d-\alpha(y)}\preceq |x-y|^{-d-\alpha(y)},$$

the last inequality follows from the fact that $|x-y|^{-d-\alpha_1}\preceq |x-y|^{-d-\alpha_2}$, thanks to $|x-y|\leqslant 2$. Hence,

$$J_1\preceq \frac{1}{|x-y|^{d+\alpha_2}}\int_{\mathbb{R}^d}\rho_1^{z,0}(t-s,x-z)\,dz\preceq \frac{1}{|x-y|^{d+\alpha_2}}.$$

At the same time, if $|z-y|\leqslant |x-y|/2$, then

$$p^z(t-s,x-z)\preceq \frac{t-s}{|x-z|^{d+\alpha(z)}}\preceq \frac{t}{|x-y|^{d+\alpha(z)}}\preceq \frac{t}{|x-y|^{d+\alpha_2}},$$

where the last inequality follows from $|x-y|^{-d-\alpha(z)}\preceq |x-y|^{-d-\alpha_2}$ since $|x-y|\leqslant 2$. Combining this estimate with (3.18), we arrive at

$$J_2\preceq \frac{t}{|x-y|^{d+\alpha_2}}\int_{\mathbb{R}^d}\left(\rho^{y,0}_{(\beta_0/\alpha_2)-\gamma}+\rho^{y,\beta_0}_{-\gamma}\right)(s,z-y)\,dz\preceq \frac{t}{|x-y|^{d+\alpha_2}}\cdot s^{-1+(\beta_0/\alpha_2)-\gamma}.$$

Using all the estimates above and choosing $0<\gamma<\beta_0/\alpha_2$, we get

$$\left|\int_0^t\int_{\mathbb{R}^d}p^z(t-s,x-z)q(s,z,y)\,dz\,ds\right|\preceq\int_0^t(J_1+J_2)\,ds\preceq \frac{t}{|x-y|^{d+\alpha_2}}\cdot\left(1+t^{(\beta_0/\alpha_2)-\gamma}\right)\preceq \frac{t}{|x-y|^{d+\alpha_2}}.$$

This, along with (1.26) immediately yields (4.8).

(2) If $\kappa(x,z)$ depends on z and $\beta_0/\alpha_2>(\alpha_2/\alpha_1)-1$, then, applying (3.29) and following the same arguments as above, we can prove (3.19). The details are omitted here. □

4.2. **Hölder regularity and gradient estimates of** $p(t,x,y)$**.** In this part, we consider the Hölder regularity and gradient estimates of $p(t,\cdot,y)$.

Lemma 4.4. *There exists a constant* $c_1 := c_1(\alpha,\kappa) > 0$ *such that for all* $x, x', y \in \mathbb{R}^d$ *and* $t \in (0,1]$,

$$\left|\nabla p^y(t,x) - \nabla p^y(t,x')\right| \leqslant c_1 \left[\left(t^{-1/\alpha(y)}|x-x'|\right) \wedge 1\right] \left(\rho^{y,0}_{1-(1/\alpha(y))}(t,x) + \rho^{y,0}_{1-(1/\alpha(y))}(t,x')\right). \tag{4.11}$$

Proof. We write $\mathcal{L}^y = \mathcal{L}^y_{\kappa_1/2} + \mathcal{L}^y_{\widehat{\kappa}}$ with

$$\mathcal{L}^y_{\kappa_1/2} f(x) = \frac{1}{2}\int_{\mathbb{R}^d} \delta_f(x;z) \frac{\kappa_1}{2} \frac{1}{|z|^{d+\alpha(y)}}\,dz, \quad f \in C_c^2(\mathbb{R}^d),$$

$$\mathcal{L}^y_{\widehat{\kappa}} f(x) = \frac{1}{2}\int_{\mathbb{R}^d} \delta_f(x;z) \frac{\widehat{\kappa}(y,z)}{|z|^{d+\alpha(y)}}\,dz, \quad f \in C_c^2(\mathbb{R}^d),$$

where $\kappa_1 > 0$ is the constant in (1.4), $\widehat{\kappa}(y,z) := \kappa(y,z) - \kappa_1/2$ and

$$\delta_f(x;z) = f(x+z) + f(x-z) - 2f(x).$$

Then, we have

$$p^y(t,x) = \int_{\mathbb{R}^d} p^y_{\kappa_1/2}(t,x-z) p^y_{\widehat{\kappa}}(t,z)\,dz, \quad x \in \mathbb{R}^d, t \in (0,1], \tag{4.12}$$

where $p^y_{\kappa_1/2}$ and $p^y_{\widehat{\kappa}}$ denote the fundamental solutions (i.e. heat kernels) associated with the operators $\mathcal{L}^y_{\kappa_1/2}$ and $\mathcal{L}^y_{\widehat{\kappa}}$, respectively.

We first show that (4.11) holds for $p^y_{\kappa_1/2}$. Indeed, for every $x, x' \in \mathbb{R}^d$ such that $|x-x'| \leqslant 1$,

$$\begin{aligned}
\left|\nabla p^y_{\kappa_1/2}(1,x) - \nabla p^y_{\kappa_1/2}(1,x')\right| &= \left|\int_0^1 \frac{d}{d\theta}\left(\nabla p^y_{\kappa_1/2}\left(1, x+\theta(x'-x)\right)\right) d\theta\right| \\
&\leqslant |x-x'| \cdot \int_0^1 \left|\nabla^2 p^y_{\kappa_1/2}\left(1, x+\theta(x'-x)\right)\right| d\theta \\
&\preceq |x-x'| \cdot \int_0^1 \left(1 + \left|x+\theta(x'-x)\right|\right)^{-d-\alpha(y)-2} d\theta \\
&\preceq |x-x'| \cdot (1+|x|)^{-d-\alpha(y)-2},
\end{aligned}$$

where in the third inequality we have used (2.20), and the last inequality is due to the fact that for all $\theta \in [0,1]$ and $x, x' \in \mathbb{R}^d$ with $|x-x'| \leqslant 1$

$$\left(1 + \left|x+\theta(x'-x)\right|\right)^{-1} \preceq (1+|x|)^{-1}.$$

Also by (2.20), we have that for every $x, x' \in \mathbb{R}^d$ with $|x-x'| > 1$,

$$\begin{aligned}
\left|\nabla p^y_{\kappa_1/2}(1,x) - \nabla p^y_{\kappa_1/2}(1,x')\right| &\leqslant \left|\nabla p^y_{\kappa_1/2}(1,x)\right| + \left|\nabla p^y_{\kappa_1/2}(1,x')\right| \\
&\preceq (1+|x|)^{-d-\alpha(y)-1} + (1+|x'|)^{-d-\alpha(y)-1}.
\end{aligned}$$

Combining both estimates above yields that for all $x, x' \in \mathbb{R}^d$,

$$\left|\nabla p^y_{\kappa_1/2}(1,x) - \nabla p^y_{\kappa_1/2}(1,x')\right| \preceq \left(|x-x'| \wedge 1\right)\left((1+|x|)^{-d-\alpha(y)} + (1+|x'|)^{-d-\alpha(y)}\right). \tag{4.13}$$

Since the Markov process $(X_t^{y,\kappa_1/2})_{t\geqslant 0}$ associated with $\mathcal{L}^y_{\kappa_1/2}$ is a constant time-change of standard rotationally invariant $\alpha(y)$-stable process, by the scaling property, for all $t > 0$ and $x \in \mathbb{R}^d$,

$$p^y_{\kappa_1/2}(t,x) = t^{-d/\alpha(y)} p^y_{\kappa_1/2}(1, t^{-1/\alpha(y)}x).$$

This along with (4.13) yields that for all $x, x' \in \mathbb{R}^d$ and $t \in (0,1]$,

$$\left|\nabla p^y_{\kappa_1/2}(t,x) - \nabla p^y_{\kappa_1/2}(t,x')\right| \preceq \left[\left(t^{-1/\alpha(y)}|x-x'|\right) \wedge 1\right] \left(\rho^{y,0}_{1-(1/\alpha(y))}(t,x) + \rho^{y,0}_{1-(1/\alpha(y))}(t,x')\right). \tag{4.14}$$

Therefore, for every $x, x' \in \mathbb{R}^d$ and $t \in (0,1]$,

$$\begin{aligned}
&\left|\nabla p^y(t,x) - \nabla p^y(t,x')\right| \\
&= \left|\int_{\mathbb{R}^d} \left(\nabla p^y_{\kappa_1/2}(t,x-z) - \nabla p^y_{\kappa_1/2}(t,x'-z)\right) \cdot p^y_{\widehat{\kappa}}(t,z)\,dz\right| \\
&\leqslant \int_{\mathbb{R}^d} \left|\nabla p^y_{\kappa_1/2}(t,x-z) - \nabla p^y_{\kappa_1/2}(t,x'-z)\right| \cdot p^y_{\widehat{\kappa}}(t,z)\,dz
\end{aligned}$$

$$\begin{aligned}
&\preceq \left[\left(t^{-1/\alpha(y)}|x-x'|\right)\wedge 1\right]\cdot\left(\int_{\mathbb{R}^d}\rho^{y,0}_{1-(1/\alpha(y))}(t,x-z)\rho^{y,0}_1(t,z)\,dz+\int_{\mathbb{R}^d}\rho^{y,0}_{1-(1/\alpha(y))}(t,x'-z)\rho^{y,0}_1(t,z)\,dz\right)\\
&\preceq \left[\left(t^{-1/\alpha(y)}|x-x'|\right)\wedge 1\right]\cdot\left(\rho^{y,0}_{1-(1/\alpha(y))}(t,x)+\rho^{y,0}_{1-(1/\alpha(y))}(t,x')\right),
\end{aligned}$$

where the equality above is due to (4.12), the second inequality follows from (4.14) and (2.21), and in the last inequality we have used (3.14). By now we have finished the proof. □

Lemma 4.5. *For every $0<\varepsilon,\theta<\alpha_1$ and $0<\gamma<(\theta/\alpha_2)\wedge((1-(\theta/\alpha_1))/2)$, there exist $R_1:=R_1(\alpha,\kappa,\gamma,\theta)\in(0,1]$ and $c_1:=c_1(\alpha,\kappa,\gamma,\theta,\varepsilon)>0$ such that for all $t\in(0,1]$ and $x,y,z,w\in\mathbb{R}^d$ with $|x-y|\leqslant R_1$,*

$$|p^x(t,z)-p^y(t,z)|\leqslant c_1|x-y|^{\beta_0}\tilde\rho^{x,0}_{1-2\gamma-(\theta/\alpha_1)}(t,z),\tag{4.15}$$

$$|\nabla p^x(t,z)-\nabla p^y(t,z)|\leqslant c_1|x-y|^{\beta_0}\tilde\rho^{x,0}_{1-(1/\alpha(x))-2\gamma-(\theta/\alpha_1)}(t,z)\tag{4.16}$$

and

$$|\mathcal{L}^w p^x(t,z)-\mathcal{L}^w p^y(t,z)|\leqslant c_1|x-y|^{\beta_0}\tilde\rho^{x,0}_{-2\gamma-(\theta/\alpha_1)}(t,z),\tag{4.17}$$

where $\tilde\rho$ is defined by (3.16).

Proof. (i) Note that for all $t\in(0,1]$ and $x,y,z\in\mathbb{R}^d$,

$$\begin{aligned}
p^x(t,z)-p^y(t,z)&=\int_0^t\frac{d}{ds}\left(\int_{\mathbb{R}^d}p^x(s,w)p^y(t-s,z-w)\,dw\right)ds\\
&=\int_0^t\left(\int_{\mathbb{R}^d}\left(\mathcal{L}^x p^x(s,w)\,p^y(t-s,z-w)-p^x(s,w)\,\mathcal{L}^y p^y(t-s,z-w)\right)dw\right)ds\\
&=\int_0^{t/2}\int_{\mathbb{R}^d}(\mathcal{L}^x-\mathcal{L}^y)p^x(s,w)\,p^y(t-s,z-w)\,dw\,ds\\
&\quad+\int_{t/2}^t\int_{\mathbb{R}^d}p^x(s,w)\big(\mathcal{L}^x-\mathcal{L}^y\big)p^y(t-s,z-w)\,dw\,ds\\
&=\int_0^{t/2}\left(\int_{\mathbb{R}^d}(\mathcal{L}^x-\mathcal{L}^y)p^x(s,w)\big(p^y(t-s,z-w)-p^y(t-s,z)\big)\,dw\right)ds\\
&\quad+\int_{t/2}^t\left(\int_{\mathbb{R}^d}\big(p^x(s,w)-p^x(s,z)\big)\big(\mathcal{L}^x-\mathcal{L}^y\big)p^y(t-s,z-w)\,dw\right)ds\\
&=:J_1+J_2.
\end{aligned}\tag{4.18}$$

Here, the first equality is due to the following estimate

$$\int_0^t\left|\frac{d}{ds}\left(\int_{\mathbb{R}^d}p^x(s,w)p^y(t-s,z-w)\,dw\right)\right|ds<\infty,$$

which can be verified by the proofs of (4.22) and (4.23) below. By estimates for $p^x(\cdot,\cdot)$ and $p^y(\cdot,\cdot)$ in Lemma 2.6, we can change the order of derivatives and integrals in the second equality. The third and the fourth equalities above follow from the following facts respectively

$$\int_{\mathbb{R}^d}\mathcal{L}^x f(z_1)g(z_1)\,dz_1=\int_{\mathbb{R}^d}f(z_1)\mathcal{L}^x g(z_1)\,dz_1,\quad f,g\in C_c^\infty(\mathbb{R}^d),$$

$$\int_{\mathbb{R}^d}\mathcal{L}^y f(z_1)g(z_1)\,dz_1=\int_{\mathbb{R}^d}f(z_1)\mathcal{L}^y g(z_1)\,dz_1,\quad f,g\in C_c^\infty(\mathbb{R}^d)$$

and

$$\int_{\mathbb{R}^d}\mathcal{L}^x f(z_1)\,dz_1=\int_{\mathbb{R}^d}\mathcal{L}^y f(z_1)\,dz_1=0,\quad f\in C_c^\infty(\mathbb{R}^d).$$

Note that, also due to estimates for $p^x(\cdot,\cdot)$ and $p^y(\cdot,\cdot)$, the equalities above are still true for $p^x(t,\cdot)$ and $p^y(t,\cdot)$.

By Lemma 2.7, for any $\gamma>0$ there is a constant $R_0:=R_0(\alpha,\kappa,\gamma)\in(0,1)$ such that for all $t\in(0,1]$ and $x,y,w\in\mathbb{R}^d$ with $|x-y|\leqslant R_0$,

$$\left|(\mathcal{L}^x-\mathcal{L}^y)p^x(t,w)\right|\preceq(|x-y|^{\beta_0}\wedge 1)\cdot\tilde\rho^{x,0}_{-\gamma}(t,w).\tag{4.19}$$

On the other hand, for any $\theta \in (0,1)$ and $\gamma \in (0,1)$, there exists a constant $R_1 := R_1(\alpha, \kappa, \gamma) \in (0, R_0)$ such that for all $x, y, z \in \mathbb{R}^d$ with $|x-y| \leqslant R_1$, $0 < s \leqslant t \leqslant 1$,

$$\begin{aligned}
\big|p^y(t-s, z-w) - p^y(t-s,z)\big| &\preceq \big[\big((t-s)^{-1/\alpha(y)}|w|\big) \wedge 1\big] \cdot \big[\rho_1^{y,0}(t-s, z-w) + \rho_1^{y,0}(t-s,z)\big] \\
&\preceq \big[\big((t-s)^{-1/\alpha(y)}|w|\big)^\theta \wedge 1\big] \cdot \big[\rho_1^{y,0}(t-s, z-w) + \rho_1^{y,0}(t-s,z)\big] \\
&\preceq \big[\big((t-s)^{-1/\alpha_1}|w|\big)^\theta \wedge 1\big] \cdot \big[\rho_{1-\gamma}^{x,0}(t-s, z-w) + \rho_{1-\gamma}^{x,0}(t-s,z)\big] \\
&\preceq \big(|w|^\theta \wedge 1\big) \cdot \big[\rho_{1-\gamma-(\theta/\alpha_1)}^{x,0}(t-s, z-w) + \rho_{1-\gamma-(\theta/\alpha_1)}^{x,0}(t-s,z)\big],
\end{aligned} \tag{4.20}$$

where the first inequality follows from (2.23) and in the third inequality we have used (4.7).

Next, we choose $\theta \in (0, 1\wedge\alpha_1)$ and $0 < \gamma < (\theta/\alpha_2) \wedge ((1-(\theta/\alpha_1))/2)$. Then, according to (4.19) and (4.20), for any $t \in (0,1]$ and $x, y, z \in \mathbb{R}^d$ with $|x-y| \leqslant R_1$,

$$\begin{aligned}
J_1 &\preceq |x-y|^{\beta_0} \cdot \Big(\int_0^{t/2} \int_{\mathbb{R}^d} \tilde\rho_{-\gamma}^{x,\theta}(s,w) \tilde\rho_{1-\gamma-(\theta/\alpha_1)}^{x,0}(t-s, z-w)\, dw\, ds \\
&\quad + \int_0^{t/2} \int_{\mathbb{R}^d} \tilde\rho_{-\gamma}^{x,\theta}(s,w) \tilde\rho_{1-\gamma-(\theta/\alpha_1)}^{x,0}(t-s, z)\, dw\, ds \Big) \\
&=: |x-y|^{\beta_0} \cdot \big(J_{11} + J_{12}\big).
\end{aligned}$$

As mentioned in Remark 3.3(2), (3.15) holds for $\tilde\rho$, from which we can obtain that

$$J_{11} \preceq \tilde\rho_{1-2\gamma-(\theta/\alpha_1)}^{x,0}(t,z).$$

At the same time, observe that for every $0 < s < t/2$,

$$\tilde\rho_{1-\gamma-(\theta/\alpha_1)}^{x,0}(t-s,z) \preceq \tilde\rho_{1-\gamma-(\theta/\alpha_1)}^{x,0}(t,z),$$

then we have

$$\begin{aligned}
J_{12} &\preceq \tilde\rho_{1-\gamma-(\theta/\alpha_1)}^{x,0}(t,z) \int_0^{t/2} \int_{\mathbb{R}^d} \tilde\rho_{-\gamma}^{x,\theta}(s,w)\, dw\, ds \\
&\preceq \tilde\rho_{1-\gamma-(\theta/\alpha_1)}^{x,0}(t,z) \int_0^{t/2} s^{-1-\gamma+(\theta/\alpha_2)}\, ds \preceq \tilde\rho_{1-2\gamma-(\theta/\alpha_1)}^{x,0}(t,z),
\end{aligned} \tag{4.21}$$

where in the second inequality we used the fact that (2.3) is true for $\tilde\rho$, see again Remark 3.3(2). By both estimates for J_{11} and J_{12}, we get that for any $t \in (0,1]$ and $x, y, z \in \mathbb{R}^d$ with $|x-y| \leqslant R_1$,

$$J_1 \preceq |x-y|^{\beta_0} \cdot \tilde\rho_{1-2\gamma-(\theta/\alpha_1)}^{x,0}(t,z). \tag{4.22}$$

For J_2, we need to handle the singularity near $s = t$. As the same way as before, we can obtain that for all $t \in (0,1]$ and $x, y, z, w \in \mathbb{R}^d$ with $|x-y| \leqslant R_1$,

$$\big|\big(\mathcal{L}^x - \mathcal{L}^y\big) p^y(t-s, z-w)\big| \preceq \big(|x-y|^{\beta_0} \wedge 1\big) \cdot \tilde\rho_{-\gamma}^{y,0}(t-s, z-w) \preceq \big(|x-y|^{\beta_0} \wedge 1\big) \cdot \tilde\rho_{-2\gamma}^{x,0}(t-s, z-w)$$

and

$$\big|p^x(s,w) - p^x(s,z)\big| \preceq \big(|z-w|^\theta \wedge 1\big) \cdot \big(\rho_{1-(\theta/\alpha_1)}^{x,0}(s,z) + \rho_{1-(\theta/\alpha_1)}^{x,0}(s,w)\big).$$

Using both estimates above and following the same argument as that for J_1, we can get that for any $t \in (0,1]$ and $x, y, z \in \mathbb{R}^d$ with $|x-y| \leqslant R_1$,

$$J_2 \preceq |x-y|^{\beta_0} \tilde\rho_{1-2\gamma-(\theta/\alpha_1)}^{x,0}(t,z). \tag{4.23}$$

Then, (4.15) is proved.

(ii) Following the argument of (4.18), we can verify that for all $t \in (0,1]$ and $x, y, z \in \mathbb{R}^d$,

$$\begin{aligned}
&\nabla p^x(t,z) - \nabla p^y(t,z) \\
&= \int_0^t \frac{d}{ds} \Big(\int_{\mathbb{R}^d} \nabla p^x(s,w) p^y(t-s, z-w)\, dw \Big)\, ds \\
&= \int_0^t \Big(\int_{\mathbb{R}^d} \Big(\mathcal{L}^x p^x(s,w) \nabla p^y(t-s, z-w) - \nabla p^x(s,w)\, \mathcal{L}^y p^y(t-s, z-w) \Big)\, dw \Big)\, ds \\
&= \int_0^{t/2} \int_{\mathbb{R}^d} \big(\mathcal{L}^x - \mathcal{L}^y\big) p^x(s,w)\, \nabla p^y(t-s, z-w)\, dw\, ds \\
&\quad + \int_{t/2}^t \int_{\mathbb{R}^d} \nabla p^x(s,w) \big(\mathcal{L}^x - \mathcal{L}^y\big) p^y(t-s, z-w)\, dw\, ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{t/2} \Big(\int_{\mathbb{R}^d} (\mathcal{L}^x - \mathcal{L}^y) p^x(s,w) \big(\nabla p^y(t-s, z-w) - \nabla p^y(t-s,z) \big) \, dw \Big) \, ds \\
&\quad + \int_{t/2}^t \Big(\int_{\mathbb{R}^d} \big(\nabla p^x(s,w) - \nabla p^x(s,z) \big) \big(\mathcal{L}^x - \mathcal{L}^y \big) p^y(t-s, z-w) \, dw \Big) \, ds \\
&=: I_1 + I_2,
\end{aligned}
$$

where in the first and the second equalities we used the integration by part formula and in the second equality we also used the fact that $\nabla \mathcal{L}^x p^y(t,w) = \mathcal{L}^x \nabla p^y(t,w)$.

Observe that

$$
t^{-d/\alpha(y)} \leqslant t^{-d/\alpha(x)} t^{-\left| \frac{d}{\alpha(y)} - \frac{d}{\alpha(x)} \right|} \leqslant t^{-d/\alpha(x)} t^{-\frac{|\alpha(x)-\alpha(z)|}{\alpha_1^2}} \leqslant t^{-d/\alpha(x)} t^{-C|x-z|^{\beta_0}}.
$$

Then, according to (4.11) and the proof of (4.20), for any $\theta \in (0,1)$, $\gamma \in (0,1)$, there exists a constant $R_1 := R_1(\alpha, \kappa, \gamma) \in (0, R_0)$ such that for all $0 < t \leqslant 1$ and $x, y \in \mathbb{R}^d$ with $|x-y| \leqslant R_1$,

$$
\begin{aligned}
&|\nabla p^y(t, z-w) - \nabla p^y(t,z)| \\
&\preceq \big(|w|^\theta \wedge 1 \big) \cdot \big[\rho^{x,0}_{1-(1/\alpha(x))-\gamma-(\theta/\alpha_1)}(t, z-w) + \rho^{x,0}_{1-(1/\alpha(x))-\gamma-(\theta/\alpha_1)}(t,z) \big].
\end{aligned}
\tag{4.24}
$$

Choosing $\theta \in (0, 1\wedge \alpha_1)$ and $0 < \gamma < (\theta/\alpha_2) \wedge ((1-(\theta/\alpha_1))/2)$, and using (4.19) and (4.24), we arrive at that for any $t \in (0,1]$ and $x,y,z \in \mathbb{R}^d$ with $|x-y| \leqslant R_1$,

$$
\begin{aligned}
I_1 &\preceq |x-y|^{\beta_0} \cdot \Big(\int_0^{t/2} \int_{\mathbb{R}^d} \tilde{\rho}^{x,\theta}_{-\gamma}(s,w) \tilde{\rho}^{x,0}_{1-(1/\alpha(x))-\gamma-(\theta/\alpha_1)}(t-s, z-w) \, dw \, ds \\
&\qquad + \int_0^{t/2} \int_{\mathbb{R}^d} \tilde{\rho}^{x,\theta}_{-\gamma}(s,w) \tilde{\rho}^{x,0}_{1-(1/\alpha(x))-\gamma-(\theta/\alpha_1)}(t-s, z) \, dw \, ds \Big) \\
&=: |x-y|^{\beta_0} \cdot \big(I_{11} + I_{12} \big).
\end{aligned}
$$

Noticing that, by Remark 3.3(2), (3.14) still holds for $\tilde{\rho}$, we have

$$
\begin{aligned}
I_{11} &\preceq \tilde{\rho}^{x,0}_0(t,z) \int_0^{t/2} \Big(s^{-\gamma} (t-s)^{-(1/\alpha(x))-\gamma-(\theta/\alpha_1)} + s^{-1+\theta/\alpha_2-\gamma} (t-s)^{1-(1/\alpha(x))-\gamma-(\theta/\alpha_1)} \Big) \, ds \\
&\preceq \tilde{\rho}^{x,0}_0(t,z) \cdot t^{1-(1/\alpha(x))-2\gamma-(\theta/\alpha_1)} \preceq \tilde{\rho}^{x,0}_{1-(1/\alpha(x))-2\gamma-(\theta/\alpha_1)}(t,z).
\end{aligned}
$$

On the other hand, following the argument of (4.21), we can obtain that

$$
I_{12} \preceq \tilde{\rho}^{x,0}_{1-(1/\alpha(x))-\gamma-(\theta/\alpha_1)}(t,z) \cdot \int_0^{t/2} \int_{\mathbb{R}^d} \tilde{\rho}^{x,\theta}_{-\gamma}(s,w) \, dw \, ds \preceq \tilde{\rho}^{x,0}_{1-(1/\alpha(x))-2\gamma-(\theta/\alpha_1)}(t,z).
$$

By the same argument as that for I_1, we can also obtain that for every $t \in (0,1]$ and $x,y,z \in \mathbb{R}^d$ with $|x-y| \leqslant R_1$,

$$
I_2 \preceq |x-y|^{\beta_0} \tilde{\rho}^{x,0}_{1-(1/\alpha(x))-2\gamma-(\theta/\alpha_1)}(t,z).
$$

Combining all estimates together, we have shown (4.16).

(iii) Following the same procedure as these of (4.15) and (4.16), we can also verify (4.17). □

Proposition 4.6. *(1) Suppose that $\kappa(x,z)$ is independent of z. Then, for every $\gamma > 0$ small enough, there exist positive constants $c_1 := c_1(\alpha,\kappa,\gamma)$ and $R_1 := R_1(\alpha,\kappa,\gamma)$ such that for all $t \in (0,1]$ and $x, x', y \in \mathbb{R}^d$ with $|x-x'| \leqslant R_1$*

$$
\big| p(t,x,y) - p(t,x',y) \big| \leqslant c_1 |x-x'|^{(\alpha(x)-\gamma)_+ \wedge 1} \big(\rho^{y,0}_{\gamma/(2\alpha_2)}(t, x-y) + \rho^{y,0}_{\gamma/(2\alpha_2)}(t, x'-y) \big).
\tag{4.25}
$$

(2) Let $\beta_0^{**} \in (0,\beta_0] \cap (0, \alpha_2/2)$. If $\kappa(x,z)$ depends on z and $(\alpha_2/\alpha_1) - 1 < \beta_0^{**}/\alpha_2$, then for every $\gamma > 0$, there exist positive constants $c_1 := c_1(\alpha,\kappa,\gamma)$ and $R_1 := R_1(\alpha,\kappa,\gamma)$ such that for all $t \in (0,1]$ and $x, x', y \in \mathbb{R}^d$ with $|x-x'| \leqslant R_1$

$$
\big| p(t,x,y) - p(t,x',y) \big| \leqslant c_1 |x-x'|^{(\alpha_1-\gamma)_+ \wedge 1} \big(\rho^{y,0}_{\gamma_1}(t, x-y) + \rho^{y,0}_{\gamma_1}(t, x'-y) \big),
\tag{4.26}
$$

where $\gamma_1 := 1 - (\alpha_2/\alpha_1) + (\gamma/(2\alpha_2))$.

Proof. For simplicity, we assume that $\beta_0 < \alpha_2/2$ and $\beta_0^{**} = \beta_0$ as before.

(1) We first suppose that $\kappa(x,z)$ is independent of z. According to (2.23), for any $\gamma>0$,

$$\begin{aligned}&\big|p^z(t-s,x-z)-p^z(t-s,x'-z)\big|\\&\preceq\big[\big((t-s)^{-1/\alpha(z)}|x-x'|\big)\wedge 1\big]\cdot\big[\rho_1^{z,0}(t-s,x-z)+\rho_1^{z,0}(t-s,x'-z)\big]\\&\preceq\big[\big((t-s)^{-1/\alpha(z)}|x-x'|\big)^{(\alpha(x)-\gamma)_+\wedge 1}\wedge 1\big]\big[\rho_1^{z,0}(t-s,x-z)+\rho_1^{z,0}(t-s,x'-z)\big]\\&\preceq|x-x'|^{(\alpha(x)-\gamma)_+\wedge 1}(t-s)^{-(\alpha(x)-\gamma)_+/\alpha(z)}\times\big[\rho_1^{z,0}(t-s,x-z)+\rho_1^{z,0}(t-s,x'-z)\big].\end{aligned}\tag{4.27}$$

On the one hand, observing that

$$\frac{\alpha(x)-\gamma}{\alpha(z)}\leqslant 1+\frac{|\alpha(x)-\alpha(z)|}{\alpha_1}-\frac{\gamma}{\alpha_2}\leqslant 1+\frac{C|x-z|^{\beta_0}}{\alpha_1}-\frac{\gamma}{\alpha_2},$$

we can find a constant $R_2:=R_2(\alpha,\gamma)>0$ such that for every $x,x',z\in\mathbb{R}^d$ satisfying $|x-x'|\leqslant R_2$ and $|x-z|\leqslant 2R_2$ (which imply that $|x'-z|\leqslant 3R_2$), it holds

$$\frac{\alpha(x)-\gamma}{\alpha(z)}\leqslant 1-\frac{2\gamma}{3\alpha_2},$$

which in turn yields that for every $0<s\leqslant t\leqslant 1$ and $x,x',z\in\mathbb{R}^d$ with $|x-x'|\leqslant R_2$ and $|x-z|\leqslant 2R_2$

$$\big|p^z(t-s,x-z)-p^z(t-s,x'-z)\big|\preceq|x-x'|^{(\alpha(x)-\gamma)_+\wedge 1}\Big(\rho^{z,0}_{2\gamma/(3\alpha_2)}(t-s,x-z)+\rho^{z,0}_{2\gamma/(3\alpha_2)}(t-s,x'-z)\Big).$$

On the other hand, when $|x-x'|\leqslant R_2$ and $|z-x|>2R_2$ (which imply that $|z-x'|>R_2$), we obtain from (2.20) and the mean value theorem that

$$\begin{aligned}|p^z(t-s,x-z)-p^z(t-s,x'-z)|&\leqslant|x-x'|\cdot|\nabla p^z(t-s,x-z+\tilde\theta_{x,x',z}(x'-x))|\\&\preceq|x-x'|\cdot\frac{t-s}{((t-s)^{1/\alpha(z)}+|x-z|)^{d+\alpha(z)+1}}\\&\preceq|x-x'|\cdot\frac{t-s}{|x-z|^{d+\alpha_1}}\preceq|x-x'|\cdot\rho_1^{z,0}(t-s,x-z).\end{aligned}$$

where in the first inequality $|\tilde\theta_{x,x'z}|\leqslant 1$ is a constant (which may depend on x, x' and z), in the second inequality we have used the fact that

$$|x-z+\tilde\theta_{x,x',z}(x'-x)|\geqslant|x-z|-|x'-x|\geqslant|x-z|/2,$$

and the fourth inequality follows from $|z-x|>R_2>C(t-s)^{1/\alpha(z)}$.

As a result, we obtain that for all $x,x',z\in\mathbb{R}^d$ with $|x-x'|\leqslant R_2$,

$$\big|p^z(t,x-z)-p^z(t,x'-z)\big|\preceq|x-x'|^{(\alpha(x)-\gamma)_+\wedge 1}\Big(\rho^{z,0}_{2\gamma/(3\alpha_2)}(t,x-z)+\rho^{z,0}_{2\gamma/(3\alpha_2)}(t,x'-z)\Big).\tag{4.28}$$

Then, using (3.18) and (3.13) and changing the constant γ properly, we arrive at for every $|x-x'|\leqslant R_2$,

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}|p^z(t-s,x-z)-p^z(t-s,x'-z)||q(s,z,y)|\,dz\,ds\\&\preceq|x-x'|^{(\alpha(x)-\gamma)_+\wedge 1}\cdot\Big[\Big(\rho^{y,0}_{(\beta_0/\alpha_2)+(\gamma/(2\alpha_2))}+\rho^{y,\beta_0}_{\gamma/(2\alpha_2)}\Big)(t,x-y)+\Big(\rho^{y,0}_{(\beta_0/\alpha_2)+(\gamma/(2\alpha_2))}+\rho^{y,\beta_0}_{\gamma/(2\alpha_2)}\Big)(t,x'-y)\Big].\end{aligned}$$

Combining all the estimates together with (1.26) and using again (4.28), we can prove (4.25) immediately.

(2) Now we assume that $\kappa(x,z)$ depends on z and $(\alpha_2/\alpha_1)-1<\beta_0/\alpha_2$. Replacing $\alpha(x)$ by $\alpha(z)$ in (4.27), we find that for every $x,x'\in\mathbb{R}^d$,

$$\begin{aligned}&\big|p^z(t-s,x-z)-p^z(t-s,x'-z)\big|\\&\preceq(|x-x'|^{(\alpha_1-\gamma)_+\wedge 1}\wedge 1)(t-s)^{-(\alpha(z)-\gamma)_+/\alpha(z)}\big(\rho_1^{z,0}(t-s,x-z)+\rho_1^{z,0}(t-s,x'-z)\big)\\&\preceq(|x-x'|^{(\alpha_1-\gamma)_+\wedge 1}\wedge 1)\big[\rho^{z,0}_{\gamma/\alpha_2}(t-s,x-z)+\rho^{z,0}_{\gamma/\alpha_2}(t-s,x'-z)\big].\end{aligned}$$

Combining this with (1.26), (3.21) and (3.13), and following the same arguments as above, we arrive at for all $x,x'\in\mathbb{R}^d$ with $|x-x'|\leqslant R_2$,

$$\begin{aligned}\big|p(t,x,y)-p(t,x',y)\big|&\preceq|x-x'|^{(\alpha_1-\gamma)_+\wedge 1}\\&\quad\times\Big[\big(\rho^{y,0}_{\gamma/\alpha_2}(t,x-y)+\rho^{y,0}_{\gamma/\alpha_2}(t,x'-y)\big)\\&\qquad+\int_0^t\int_{\mathbb{R}^d}\big(\rho^{z,0}_{\gamma/\alpha_2}(t-s,x-z)+\rho^{z,0}_{\gamma/\alpha_2}(t-s,x'-z)\big)\\&\qquad\times\Big(\rho^{y,0}_{(\beta_0/\alpha_2)-[(\alpha_2/\alpha_1)-1+\gamma/(2\alpha_2)]}+\rho^{y,\beta_0}_{-[(\alpha_2/\alpha_1)-1+\gamma/(2\alpha_2)]}\Big)(s,z-y)\,dz\,ds\Big]\end{aligned}$$

$$\preceq |x-x'|^{(\alpha_1-\gamma)_+\wedge 1}\big[\big(\rho^{y,0}_{\beta_0/\alpha_2+\gamma_1}+\rho^{y,\beta_0}_{\gamma_1}\big)(t,x-y)+\big(\rho^{y,0}_{\beta_0/\alpha_2+\gamma_1}+\rho^{y,\beta_0}_{\gamma_1}\big)(t,x'-y)\big],$$

where $\gamma_1 := 1-\alpha_2/\alpha_1+\gamma/(2\alpha_2)$. By now we have verified (4.26). The proof is complete. □

Furthermore, we have the following gradient estimates for $p(t,x,y)$.

Proposition 4.7. *(1) Suppose that $\kappa(x,z)$ is independent of z. Let $\beta_0^* := \beta_0\wedge\alpha_2$. If $\tilde\beta_0(x_0) := (\alpha_1\beta_0/\alpha(x_0))\wedge\alpha_1 > 1-\alpha(x_0)$ for some $x_0\in\mathbb{R}^d$, then for every fixed $t\in(0,1]$ and $y\in\mathbb{R}^d$, $p(t,\cdot,y)$ is differentiable at $x=x_0$. Moreover, for every $\gamma>0$,*

$$|\nabla p(t,\cdot,y)(x_0)|\leqslant c_1\rho^{y,0}_{1-(1/\alpha(x_0))+(\beta_0^*/\alpha_2)-(\beta_0^*/\alpha_1)-\gamma}(t,x_0-y) \tag{4.29}$$

holds for some $c_1:=c_1(\alpha,\kappa,\gamma,x_0)>0$.

(2) Suppose that $\kappa(x,z)$ depends on z and $(\alpha_2/\alpha_1)-1<\beta_0^{**}/\alpha_2$, where $\beta_0^{**}:=\beta_0\wedge(\alpha_2/2)$. If $\tilde\beta_0(x_0)-(\alpha_1\alpha_2/\alpha(x_0))((\alpha_2/\alpha_1)-1)>1-\alpha_1$ for some $x_0\in\mathbb{R}^d$ with $\tilde\beta_0(x_0)$ defined in (i), then for every fixed $t\in(0,1]$ and $y\in\mathbb{R}^d$, $p(t,\cdot,y)$ is differentiable at $x_0\in\mathbb{R}^d$. Furthermore, for every $\gamma>0$, there exists a constant $c_2:=c_2(\alpha,\kappa,\gamma,x_0)>0$ such that

$$|\nabla p(t,\cdot,y)(x_0)|\leqslant c_2\rho^{y,0}_{1-(1/\alpha_1)+(\beta_0^*/\alpha_2)-(\beta_0^*/\alpha_1)-\gamma_2}(t,x_0-y), \tag{4.30}$$

where $\gamma_2:=(\alpha_2/\alpha_1)-1+\gamma$.

Proof. (1) We first assume that $\kappa(x,z)$ is independent of z. For simplicity, we assume that $\beta_0<\alpha(x_0)$, and so $\beta_0^*=\beta_0$ and $\tilde\beta_0(x_0)=\alpha_1\beta_0/\alpha(x_0)$. We will show that we can take the gradient with respect to variable x in the equation (1.26). Note that

$$\begin{aligned}
&\int_0^t\left|\int_{\mathbb{R}^d}\nabla p^z(t-s,x_0-z)q(s,z,y)\,dz\right|ds\\
&\leqslant\int_0^{t/2}\left|\int_{\mathbb{R}^d}\nabla p^z(t-s,x_0-z)q(s,z,y)\,dz\right|ds\\
&\quad+\int_{t/2}^t\left|\int_{\mathbb{R}^d}\nabla p^z(t-s,x_0-z)\left(q(s,z,y)-q(s,x_0,y)\right)dz\right|ds\\
&\quad+\int_{t/2}^t\left|\int_{\mathbb{R}^d}\left(\nabla p^z(t-s,x_0-z)-\nabla p^{x_0}(t-s,x_0-z)\right)q(s,x_0,y)\,dz\right|ds\\
&=:\int_0^{t/2}\left|\int_{\mathbb{R}^d}J_1(s,z)\,dz\right|ds+\int_{t/2}^t\left|\int_{\mathbb{R}^d}J_2(s,z)\,dz\right|ds+\int_{t/2}^t\left|\int_{\mathbb{R}^d}J_3(s,z)\,dz\right|ds,
\end{aligned}$$

where in the inequality above we used the fact that for any $s\in(t/2,t)$,

$$\int_{\mathbb{R}^d}\nabla p^{x_0}(t-s,x_0-z)q(s,x_0,y)\,dz=q(s,x_0,y)\int_{\mathbb{R}^d}\nabla p^{x_0}(t-s,x_0-z)\,dz=0.$$

According to (2.20),

$$|\nabla p^z(t,x-z)|\preceq\frac{t}{(t^{1/\alpha(z)}+|x-z|)^{d+\alpha(z)+1}}. \tag{4.31}$$

Observing that

$$\left|\frac{1}{\alpha(z)}-\frac{1}{\alpha(x)}\right|\leqslant\frac{|\alpha(x)-\alpha(z)|}{\alpha_1^2}\leqslant C|x-z|^{\beta_0},$$

we find by (4.31) that for every (small enough) $\gamma>0$, there is a constant $R_1:=R_1(\alpha,\kappa,\gamma)>0$ such that

$$\begin{aligned}
|\nabla p^z(t,x-z)|&\preceq\begin{cases}\rho^{z,0}_{1-(1/\alpha(z))}(t,x-z), & |x-z|\leqslant R_1,\\ \frac{t}{|x-z|^{d+\alpha_1+1}}, & |x-z|>R_1\end{cases}\\
&\preceq\rho^{z,0}_{1-(1/\alpha(x))-\gamma}(t,x-z).
\end{aligned} \tag{4.32}$$

Thus, choosing $\gamma>0$ small enough such that $\gamma<\beta_0/\alpha_2$, and using (3.1), (3.18) and (4.32), we can find that

$$\begin{aligned}
&\int_0^{t/2}\int_{\mathbb{R}^d}|J_1(s,z)|\,dz\,ds\\
&\preceq\int_0^{t/2}\int_{\mathbb{R}^d}\rho^{z,0}_{1-(1/\alpha(x_0))-\gamma}(t-s,x_0-z)\big(\rho^{y,0}_{(\beta_0/\alpha_2)-\gamma}(s,z-y)+\rho^{y,\beta_0}_{-\gamma}(s,z-y)\big)\,dz\,ds
\end{aligned}$$

$$\preceq \rho_0^{y,0}(t,x_0-y)\Big(\int_0^{t/2}(t-s)^{-(1/\alpha(x_0))-\gamma}s^{(\beta_0/\alpha_2)-\gamma}+(t-s)^{1-(1/\alpha(x_0))-\gamma}s^{-1+(\beta_0/\alpha_2)-\gamma}\,ds\Big)$$
$$\preceq \rho_{1-(1/\alpha(x_0))-2\gamma}^{y,0}(t,x-y).$$

On the other hand, according to (3.19), (4.32) and the fact that $\beta_0<\alpha(x_0)$, we arrive at

$$\begin{aligned}\int_{t/2}^t\int_{\mathbb{R}^d}|J_2(s,z)|\,dz\,ds &\preceq \int_{t/2}^t\int_{\mathbb{R}^d}\Big(\rho_{1-(1/\alpha(x_0))-\gamma}^{z,\widehat\beta_0}(t-s,x_0-z)\tilde\rho_{\theta-\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}^{y,0}(s,z-y)\\ &\qquad+\rho_{1-(1/\alpha(x_0))-\gamma}^{z,\widehat\beta_0}(t-s,x_0-z)\tilde\rho_{\theta-\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}^{y,0}(s,x_0-y)\Big)\,dz\,ds\\ &=: \int_{t/2}^t\int_{\mathbb{R}^d}(J_{21}(s,z)+J_{22}(s,z))\,dz\,ds,\end{aligned}$$

where $\widehat\beta_0=\widehat\beta_0(\theta):=\alpha_1(\beta_0-\alpha_2\theta)/\alpha(x_0)$.

Since $\tilde\beta_0(x_0)>1-\alpha(x_0)$, we can choose $\theta,\gamma>0$ small enough such that $1+(\widehat\beta_0/\alpha(x_0))-(1/\alpha(x_0))-\gamma>0$, which along with (3.1) yields

$$\begin{aligned}\int_{t/2}^t\int_{\mathbb{R}^d}J_{21}(s,z)\,dz\,ds &\preceq \tilde\rho_0^{y,0}(t,x_0-y)\cdot\Big(\int_{t/2}^t(t-s)^{1-(1/\alpha(x_0))-\gamma}s^{-1+\theta-\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}\\ &\qquad+(t-s)^{(\widehat\beta_0/\alpha(x_0))-(1/\alpha(x_0))-\gamma}s^{\theta-\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}\,ds\Big)\\ &\preceq \tilde\rho_{1-(1/\alpha(x_0))+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)-2\gamma}^{y,0}(t,x_0-y).\end{aligned}$$

On the other hand, noting that for $t/2<s\leqslant t$, it holds that

$$\tilde\rho_{\theta-\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}^{y,\beta_0}(s,x_0-y)\preceq\tilde\rho_{-\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}^{y,\beta_0}(t,x_0-y).\tag{4.33}$$

Since $\tilde\beta_0>1-\alpha(x_0)$, by (2.1) we have

$$\begin{aligned}\int_{t/2}^t\int_{\mathbb{R}^d}J_{22}(s,z)\,dz\,ds &\preceq \tilde\rho_{-\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}^{y,\beta_0}(t,x_0-y)\int_{t/2}^t\int_{\mathbb{R}^d}\rho_{1-(1/\alpha(x_0))-\gamma}^{z,\widehat\beta_0}(t-s,x_0-z)\,dz\,ds\\ &\preceq \tilde\rho_{1+(\widehat\beta_0/\alpha(x_0))-(1/\alpha(x_0))+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)-2\gamma}^{y,0}(t,x_0-y).\end{aligned}$$

According to (4.16) for the case that $|x_0-z|\leqslant R_1$ and (4.32) for the case that $|x_0-z|>R_1$), we arrive at that (by changing the constant γ properly) for any $t>0$ and $x_0,z\in\mathbb{R}^d$,

$$|\nabla p^{x_0}(t,x_0-z)-\nabla p^z(t,x_0-z)|\preceq\tilde\rho_{1-1/\alpha(x_0)-\gamma}^{z,\beta_0}(t,x_0-z).$$

This, along with (3.18) yields

$$\begin{aligned}\int_{t/2}^t\int_{\mathbb{R}^d}|J_3(s,z)|\,dz\,ds &\preceq \tilde\rho_{-\gamma}^{y,0}(t,x_0-y)\int_{t/2}^t\int_{\mathbb{R}^d}\rho_{1-(1/\alpha(x_0))-\gamma}^{z,\beta_0}(t-s,x_0-z)\,dz\,ds\\ &\preceq \tilde\rho_{1+(\beta_0/\alpha(x_0))-1/\alpha(x_0)-2\gamma}^{y,0}(t,x_0-y),\end{aligned}$$

where in the last inequality we used the fact that $\beta_0\geqslant\tilde\beta_0(x_0)>1-\alpha(x_0)$.

Combining all estimates together, we obtain that

$$\int_0^t\Big|\int_{\mathbb{R}^d}\nabla p^z(t-s,x_0-z)q(s,z,y)\,dz\Big|\,ds\preceq\tilde\rho_{1-(1/\alpha(x_0))+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)-2\gamma}^{y,0}(t,x_0-y).$$

According to the estimate above and the dominated convergence theorem, we know that $\nabla p(t,\cdot)(x_0)$ exists, and (4.29) holds by changing θ and γ properly.

(2) Suppose that $\kappa(x,z)$ depends on z and $(\alpha_2/\alpha_1)-1<\beta_0^{**}/\alpha_2$. If for some $x_0\in\mathbb{R}^d$, $\tilde\beta_0(x_0)-(\alpha_1\alpha_2/\alpha(x_0))\big((\alpha_2/\alpha_1)-1\big)>1-\alpha_1$, then, by (4.16) and (4.31),

$$|\nabla p^z(t,x-z)|\preceq t^{-1/\alpha(z)}\rho_1^{z,0}(t,x-z)\preceq\rho_{1-(1/\alpha_1)}^{z,0}(t,x-z)$$

and

$$|\nabla p^x(t,x-z)-\nabla p^z(t,x-z)|\preceq\tilde\rho_{1-(1/\alpha_1)-\gamma}^{z,\beta_0}(t,x-z).$$

In the following, we choose θ,γ small enough such that

$$\widehat\beta_0:=\alpha_1\big(\beta_0-\alpha_2\theta\big)/\alpha(x_0)>-1+(1/\alpha_1)+\gamma.$$

Using estimates and following the same argument in part (1), we can obtain that for every fixed $t \in (0,1]$ and $y \in \mathbb{R}^d$, $p(t,\cdot,y)$ is differentiable at $x_0 \in \mathbb{R}^d$ and (4.30) holds. □

Remark 4.8. Propositions 4.6 and 4.7 show that, the regularity of $p(t,\cdot,y)$ at $x = x_0$ depends on the index $\alpha(x_0)$.

Proposition 4.9. *Suppose that $\kappa(x,z)$ is independent of z, or that $\kappa(x,z)$ depends on z and $(\alpha_2/\alpha_1) - 1 < \beta_0^{**}/\alpha_2$, where $\beta_0^{**} := \beta_0 \wedge (\alpha_2/2)$. Then, $p : (0,1] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is continuous.*

Proof. We only consider the case that $\kappa(x,z)$ is independent of z. Then other case can be verified similarly.

According to (1.26), (3.18) and (4.1), we know immediately the continuity of p with respect to time variable. It remains to show the continuity of p with respect to space variables.

For any x, y, z_1 and $z_2 \in \mathbb{R}^d$,

$$\begin{aligned}&|p^{y+z_2}(t,x+z_1-y-z_2) - p^y(t,x-y)|\\ &\leqslant |p^{y+z_2}(t,x+z_1-y-z_2) - p^{y+z_2}(t,x-y)| + |p^{y+z_2}(t,x-y) - p^y(t,x-y)|.\end{aligned}$$

According to (2.23) and (4.15),

$$\lim_{|z_1|,|z_2|\to 0} |p^{y+z_2}(t,x+z_1-y-z_2) - p^{y+z_2}(t,x-y)| = 0$$

and

$$\lim_{|z_2|\to 0} |p^{y+z_2}(t,x-y) - p^y(t,x-y)| = 0,$$

respectively. Hence, we can show that $(x,y) \mapsto p^y(t,x-y)$ is continuous. On the other hand, (2.24), (2.38) and (4.17) imply that $(x,y) \mapsto q_0(t,x,y)$ is continuous. By the iteration estimates in Proposition 3.4 and the dominated convergence theorem, we also can verify that $(x,y) \mapsto q(t,x,y)$ is continuous. Due to the expression (1.26) and again the dominated convergence theorem, we know that $(x,y) \mapsto p(t,x,y)$ is continuous. The proof is finished. □

5. Existence and Uniqueness of the Solution to (1.5)

5.1. **Existence.** The purpose of this subsection is to prove rigorously that $p(t,x,y)$ defined by (1.26) satisfies (1.5). First, as a direct consequence of Lemma 4.5, we have the following statement.

Lemma 5.1. *Let $\beta_0^* \in (0,\beta_0] \cap (0,\alpha_2)$. Then, for any $0 < \theta < \beta_0^*\alpha_1/\alpha_2$ and $0 < \gamma < (\theta/\alpha_2) \wedge ((\beta_0^*/\alpha_2 - \theta/\alpha_1)/2)$, $t \in (0,1]$ and $x \in \mathbb{R}^d$,*

$$\left| \int_{\mathbb{R}^d} p^y(t,x-y)\,dy - 1 \right| \preceq t + t^{(\beta_0^*/\alpha_2)-2\gamma-(\theta/\alpha_1)}. \tag{5.1}$$

In particular,

$$\limsup_{t\downarrow 0}\, \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^y(t,x-y)\,dy - 1 \right| = 0. \tag{5.2}$$

Proof. Throughout the proof, we will assume that $\beta_0 < \alpha_2$ and $\beta_0^* = \beta_0$ for simplicity. Noting that $\int_{\mathbb{R}^d} p^x(t,x-y)\,dy = 1$ for all $x \in \mathbb{R}^d$ and $t \in (0,1]$, we have

$$\begin{aligned}\left| \int_{\mathbb{R}^d} p^y(t,x-y)\,dy - 1 \right| &= \left| \int_{\mathbb{R}^d} p^y(t,x-y)\,dy - \int_{\mathbb{R}^d} p^x(t,x-y)\,dy \right|\\ &\leqslant \int_{\mathbb{R}^d} |p^x(t,x-y) - p^y(t,x-y)|\,dy\\ &\leqslant \int_{\{|x-y|\leqslant R_1\}} |p^x(t,x-y) - p^y(t,x-y)|\,dy\\ &\quad + \int_{\{|x-y|> R_1\}} |p^x(t,x-y) - p^y(t,x-y)|\,dy\\ &=: J_1 + J_2,\end{aligned}$$

where R_1 is the constant in Lemma 4.5.

On the one hand, (4.15) yields that

$$J_1 \preceq \int_{\mathbb{R}^d} \tilde{\rho}^{x,\beta_0}_{1-2\gamma-(\theta/\alpha_1)}(t,x-y)\,dy \preceq t^{(\beta_0/\alpha_2)-2\gamma-(\theta/\alpha_1)},$$

where the last inequality follows from the fact that (2.3) holds for $\tilde{\rho}$.

On the other hand,

$$J_2 \preceq \int_{\{|x-y|>R_1\}} \big(\rho_1^{x,0}(t,x-y)+\rho_1^{y,0}(t,x-y)\big)\,dy$$
$$\preceq \int_{\{|x-y|>R_1\}} \left(\frac{t}{|x-y|^{d+\alpha(x)}}+\frac{t}{|x-y|^{d+\alpha(y)}}\right)dy \preceq t\int_{\{|x-y|>R_1\}} \frac{1}{|x-y|^{d+\alpha_1}}\,dy \preceq t.$$

Combining with both estimates for J_1 and J_2, we prove (5.1).

Choosing $\gamma,\theta>0$ small enough such that $2\gamma+(\theta/\alpha_1)<\beta_0/\alpha_2$, we immediately get (5.2). □

Lemma 5.2. *For every $0<\varepsilon,\theta<\alpha_1$ and $0<\gamma<(\theta/\alpha_2)\wedge((1-(\theta/\alpha_1))/2)$, there are constants $R_1:=R_1(\alpha,\kappa,\gamma,\theta)\in(0,1)$ and $c_1:=c_1(\alpha,\kappa,\gamma,\theta,\varepsilon)>0$ such that for every $t\in(0,1]$ and $x,y\in\mathbb{R}^d$ with $|x-y|\leqslant R_1$,*

$$\int_{\mathbb{R}^d} \big|\delta_{p^x}(t,x-y;z)-\delta_{p^y}(t,x-y;z)\big|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz \leqslant c_1\tilde\rho^{x,\beta_0}_{-2\gamma-(\theta/\alpha_1)}(t,x-y), \tag{5.3}$$

where $\tilde\rho$ is defined by (3.16).

Proof. By using (4.18), we can verify that

$$\begin{aligned}
&\delta_{p^x}(t,x-y;z)-\delta_{p^y}(t,x-y;z)\\
&=\int_0^{t/2}\left(\int_{\mathbb{R}^d}(\mathcal{L}^x-\mathcal{L}^y)p^x(s,w)\big(\delta_{p^y}(t-s,x-y-w;z)-\delta_{p^y}(t-s,x-y;z)\big)\,dw\right)ds\\
&\quad+\int_{t/2}^t\left(\int_{\mathbb{R}^d}(\mathcal{L}^x-\mathcal{L}^y)p^y(t-s,x-y-w)\big(\delta_{p^x}(s,w;z)-\delta_{p^x}(s,x-y;z)\big)\,dw\right)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\big|\delta_{p^x}(t,x-y;z)-\delta_{p^y}(t,x-y;z)\big|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz\\
&\leqslant \int_0^{t/2}\left[\int_{\mathbb{R}^d}\big|(\mathcal{L}^x-\mathcal{L}^y)p^x(s,w)\big|\left(\int_{\mathbb{R}^d}\big|\delta_{p^y}(t-s,x-y-w;z)-\delta_{p^y}(t-s,x-y;z)\big|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz\right)dw\right]ds\\
&+\int_{t/2}^t\left[\int_{\mathbb{R}^d}\big|(\mathcal{L}^x-\mathcal{L}^y)p^y(t-s,x-y-w)\big|\left(\int_{\mathbb{R}^d}\big|\delta_{p^x}(s,w;z)-\delta_{p^x}(s,x-y;z)\big|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz\right)dw\right]ds\\
&=: J_1+J_2.
\end{aligned}$$

According to (2.24) and (4.7), for every $\theta\in(0,1)$ and $\gamma>0$, there exists a constant $R_1:=R_1(\alpha,\kappa,\gamma)>0$ such that for every $0<s<t<1$ and $x,y\in\mathbb{R}^d$ with $|x-y|\leqslant R_1$,

$$\begin{aligned}
&\big|\delta_{p^y}(t-s,x-y-w;z)-\delta_{p^y}(t-s,x-y;z)\big|\\
&\preceq\Big[\Big((t-s)^{-1/\alpha(y)}|w|\Big)^\theta\wedge 1\Big]\cdot\Big[\Big((t-s)^{-2/\alpha(y)}|z|^2\Big)\wedge 1\Big]\\
&\quad\times\big[\rho_1^{y,0}(t-s,x-y)+\rho_1^{y,0}(t-s,x-y-w)+\rho_1^{y,0}(t-s,x-y\pm z)+\rho_1^{y,0}(t-s,x-y-w\pm z)\big]\\
&\preceq\Big(|w|^\theta\wedge 1\Big)\cdot\Big[\Big((t-s)^{-2/\alpha(x)}|z|^2\Big)\wedge 1\Big]\cdot\big[\rho^{x,0}_{1-\gamma-(\theta/\alpha_1)}(t-s,x-y)+\rho^{x,0}_{1-\gamma-(\theta/\alpha_1)}(t-s,x-y-w)\\
&\qquad+\rho^{x,0}_{1-\gamma-(\theta/\alpha_1)}(t-s,x-y\pm z)+\rho^{x,0}_{1-\gamma-(\theta/\alpha_1)}(t-s,x-y-w\pm z)\big].
\end{aligned}\tag{5.4}$$

Furthermore, following the argument of Lemma 2.3, we can derive that for every $t\in(0,1]$ and $x,y\in\mathbb{R}^d$

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Big[\Big(t^{-2/\alpha(x)}|z|^2\Big)\wedge 1\Big]\cdot\rho_1^{x,0}(t,y\pm z)\cdot|z|^{-d-\alpha(x)}\,dz\preceq\rho_0^{x,0}(t,y),\\
&\int_{\mathbb{R}^d}\Big[\Big(t^{-2/\alpha(x)}|z|^2\Big)\wedge 1\Big]\cdot\rho_1^{x,0}(t,y)\cdot|z|^{-d-\alpha(x)}\,dz\preceq\rho_0^{x,0}(t,y).
\end{aligned}\tag{5.5}$$

Combining (4.19) with (5.4) and (5.5), we find that for all $t\in(0,1]$ and $x,y\in\mathbb{R}^d$ with $|x-y|\leqslant R_1$,

$$\begin{aligned}
J_1&\preceq|x-y|^{\beta_0}\cdot\left[\int_0^{t/2}\int_{\mathbb{R}^d}\tilde\rho^{x,\theta}_{-\gamma}(s,w)\cdot\tilde\rho^{x,0}_{-\gamma-(\theta/\alpha_1)}(t-s,x-y-w)\,dw\,ds\right.\\
&\qquad\left.+\int_0^{t/2}\int_{\mathbb{R}^d}\tilde\rho^{x,\theta}_{-\gamma}(s,w)\cdot\tilde\rho^{x,0}_{-\gamma-(\theta/\alpha_1)}(t-s,x-y)\,dw\,ds\right]\\
&=:|x-y|^{\beta_0}\cdot(J_{11}+J_{12}).
\end{aligned}$$

Since (3.14) holds true for $\tilde\rho$ as mentioned in Remark 3.3(2), for every $0<\theta<\alpha_1$ and $0<\gamma<(\theta/\alpha_2)\wedge((1-(\theta/\alpha_1))/2)$, we have

$$\begin{aligned} J_{11} &\preceq \tilde\rho_0^{x,0}(t,x-y)\int_0^{t/2}\big(s^{-1-\gamma+(\theta/\alpha_2)}(t-s)^{-\gamma-(\theta/\alpha_1)}+s^{-\gamma}(t-s)^{-1-\gamma-(\theta/\alpha_1)}\big)\,ds\\ &\preceq \tilde\rho_0^{x,0}(t,x-y)\bigg(t^{-\gamma-(\theta/\alpha_1)}\int_0^{t/2}s^{-1-\gamma+(\theta/\alpha_2)}\,ds+t^{-1-\gamma-(\theta/\alpha_1)}\int_0^{t/2}s^{-\gamma}\,ds\bigg)\\ &\preceq \tilde\rho^{x,0}_{-2\gamma-(\theta/\alpha_1)}(t,x-y). \end{aligned} \tag{5.6}$$

On the other hand, note that for every $0<s<t/2$,

$$\tilde\rho^{x,0}_{-\gamma-(\theta/\alpha_1)}(t-s,x-y)\preceq \tilde\rho^{x,0}_{-\gamma-(\theta/\alpha_1)}(t,x-y).$$

Then, it holds that

$$J_{12}\preceq \tilde\rho^{x,0}_{-\gamma-(\theta/\alpha_1)}(t,x-y)\cdot\int_0^{t/2}\int_{\mathbb{R}^d}\tilde\rho^{x,\theta}_{-\gamma}(s,w)\,dw\,ds\preceq \tilde\rho^{x,0}_{-2\gamma-(\theta/\alpha_1)}(t,x-y),$$

where (2.3) was used in the last inequality. Hence, we find that for every $t\in(0,1]$ and $x,y\in\mathbb{R}^d$ with $|x-y|\leqslant R_1$,

$$J_1\preceq \tilde\rho^{x,\beta_0}_{-2\gamma-(\theta/\alpha_1)}(t,x-y).$$

By estimates for the terms

$$\big|(\mathcal{L}^x-\mathcal{L}^y)p^y(t-s,x-y-w)\big|$$

and

$$\big|\delta_{p^x}(s,w;z)-\delta_{p^x}(s,x-y;z)\big|,$$

we can deal with the singularity as s near t by the same arguments above and obtain that for all $t\in(0,1]$ and $x,y\in\mathbb{R}^d$ with $|x-y|\leqslant R_1$,

$$J_2\preceq \tilde\rho^{x,\beta_0}_{-2\gamma-(\theta/\alpha_1)}(t,x-y).$$

Combining both of the estimates for J_1 and J_2, we finally obtain that (5.3) holds true. □

The following result is a consequence of Lemma 5.2.

Lemma 5.3. *Let $\beta_0^*\in(0,\beta_0]\cap(0,\alpha_2)$. Then, we have the following two statements.*

(1) If $\kappa(x,z)$ is independent of z, then for every positive constants $\gamma,\theta\in(0,1)$ such that $\gamma<\theta/\alpha_2$ and $2\gamma+(\theta/\alpha_1)<\beta_0^*/\alpha_2$, there exists a constant $c_1:=c_1(\alpha,\kappa,\gamma,\theta)>0$ such that for all $t\in(0,1]$ and $x\in\mathbb{R}^d$,

$$\Big|\int_{\mathbb{R}^d}\mathcal{L}^xp^y(t,x-y)\,dy\Big|\leqslant c_1t^{-1+(\beta_0^*/\alpha_2)-2\gamma-(\theta/\alpha_1)}. \tag{5.7}$$

(2) If $\kappa(x,z)$ depends on z, then for every positive constants $\gamma,\theta\in(0,1)$ such that $\gamma<\theta/\alpha_2$ and $2\gamma+(\theta/\alpha_1)<\beta_0^*/\alpha_2$, the estimate (5.7) will be replaced by

$$\Big|\int_{\mathbb{R}^d}\mathcal{L}^xp^y(t,x-y)\,dy\Big|\leqslant c_1\big(t^{-1+(\beta_0^*/\alpha_2)-2\gamma-(\theta/\alpha_1)}+t^{1-(\alpha_2/\alpha_1)}\big). \tag{5.8}$$

Proof. We assume that $\beta_0<\alpha_2$ and $\beta_0^*=\beta_0$ for simplicity. Observe that

$$\begin{aligned}\int_{\mathbb{R}^d}\mathcal{L}^xp^y(t,x-y)\,dy&=\int_{\mathbb{R}^d}\big(\mathcal{L}^xp^y(t,x-y)-\mathcal{L}^xp^x(t,x-y)\big)\,dy\\ &=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(\delta_{p^y}(t,x-y;z)-\delta_{p^x}(t,x-y;z)\big)\cdot\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\,dz\,dy.\end{aligned}$$

Therefore, for all $t\in(0,1]$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}\mathcal{L}^xp^y(t,x-y)\,dy\Big|&\preceq\int_{\{|y-x|\leqslant R_1\}}\int_{\mathbb{R}^d}|\delta_{p^y}(t,x-y;z)-\delta_{p^x}(t,x-y;z)|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz\,dy\\ &\quad+\int_{\{|y-x|>R_1\}}\int_{\mathbb{R}^d}|\delta_{p^y}(t,x-y;z)-\delta_{p^x}(t,x-y;z)|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz\,dy\\ &=:J_1+J_2,\end{aligned}$$

where R_1 is the constant in Lemma 5.2. According to (5.3), we have for all $t\in(0,1]$ and $x\in\mathbb{R}^d$,

$$J_1\preceq\int_{\{|y-x|\leqslant R_1\}}\tilde\rho^{x,\beta_0}_{-2\gamma-(\theta/\alpha_1)}(t,x-y)\,dy\preceq t^{-1+(\beta_0^*/\alpha_2)-2\gamma-(\theta/\alpha_1)},$$

where we used the fact that (2.3) holds for $\tilde{\rho}$.

(1) Suppose that $\kappa(x,z)$ is independent of z. Then, according to the argument for J_1 in (2.30), for every $t\in(0,1]$ and $x,y\in\mathbb{R}^d$ with $|x-y|>R_1$,

$$\begin{aligned}\int_{\mathbb{R}^d}|\delta_{p^y}(t,x-y;z)|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz&\preceq\frac{1}{|x-y|^{d+\alpha_1}},\\ \int_{\mathbb{R}^d}|\delta_{p^x}(t,x-y;z)|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz&\preceq\frac{1}{|x-y|^{d+\alpha_1}},\end{aligned}\tag{5.9}$$

which imply that for all $t\in(0,1]$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}J_2&\preceq\int_{\{|y-x|>R_2\}}\int_{\mathbb{R}^d}|\delta_{p^y}(t,x-y;z)|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz\,dy+\int_{\{|y-x|>R_2\}}\int_{\mathbb{R}^d}|\delta_{p^x}(t,x-y;z)|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz\,dy\\&\leqslant C_1.\end{aligned}$$

Combining both the estimates for J_1 and J_2, we prove (5.7).

(2) When $\kappa(x,z)$ depends on z, by the argument of (2.33), the first inequality in (5.9) will be changed into

$$\int_{\mathbb{R}^d}|\delta_{p^y}(t,x-y;z)|\cdot\frac{1}{|z|^{d+\alpha(x)}}\,dz\preceq\frac{t^{1-(\alpha_2/\alpha_1)}}{|x-y|^{d+\alpha_1}}.$$

Using this inequality and following the same line as above, we will obtain (5.8). □

The statement below is the main result in this subsection.

Proposition 5.4. *The following two statements hold.*

(1) If $\kappa(x,z)$ is independent of z, then $p(t,x,y)$ defined by (1.26) *satisfies the equation* (1.5) *pointwise.*

(2) If $\kappa(x,z)$ depends on z and $(\alpha_2/\alpha_1)-1<\min\{\beta_0/\alpha_2,1/2\}$, then $p(t,x,y)$ also satisfies the equation (1.5) *pointwise.*

Proof. For simplicity, we assume that $\beta_0<\alpha_2/2$ and $\beta_0^*=\beta_0$.

(1) We first assume that $\kappa(x,z)$ is independent of z. The proof is split into four parts.

(i) For every $0<s<t<1$ and $x,y\in\mathbb{R}^d$, define

$$\phi(t,s,x,y):=\int_{\mathbb{R}^d}p^z(t-s,x-z)q(s,z,y)\,dz,$$

where $q(t,x,y)$ is constructed in Proposition 3.4. By (1.26), it holds that

$$p(t,x,y)=p^y(t,x-y)+\int_0^t\phi(t,s,x,y)\,ds.\tag{5.10}$$

Note that for every $t\in(0,1]$, $x,y\in\mathbb{R}^d$ and $\varepsilon>0$,

$$\begin{aligned}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\phi(t+\varepsilon,s,x,y)\,ds-q(t,x,y)&=\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}p^z(t+\varepsilon-s,x-z)q(s,z,y)\,dz\,ds-q(t,x,y)\\&=\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}p^z(t+\varepsilon-s,x-z)\big(q(s,z,y)-q(s,x,y)\big)\,dz\,ds\\&\quad+\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\left(\int_{\mathbb{R}^d}p^z(t+\varepsilon-s,x-z)\,dz\right)\cdot\big(q(s,x,y)-q(t,x,y)\big)\,ds\\&\quad+q(t,x,y)\cdot\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\left(\int_{\mathbb{R}^d}p^z(t+\varepsilon-s,x-z)\,dz-1\right)ds\\&=:J_1(\varepsilon)+J_2(\varepsilon)+J_3(\varepsilon).\end{aligned}$$

First, by (3.19), for every $t>0$, $x,y\in\mathbb{R}^d$ and $\sigma>0$, there exists a constant $\varepsilon_1:=\varepsilon_1(t,x,y,\sigma)>0$ such that for all $z\in\mathbb{R}^d$ with $|z-x|\leqslant\varepsilon_1$,

$$|q(s,z,y)-q(s,x,y)|\leqslant\sigma\tag{5.11}$$

holds for all $t<s<t+\varepsilon$ and $0<\varepsilon<1$. Let

$$\begin{aligned}|J_1(\varepsilon)|&\preceq\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\{|z-x|\leqslant\varepsilon_1\}}p^z(t+\varepsilon-s,x-z)\big|q(s,z,y)-q(s,x,y)\big|\,dz\,ds\\&\quad+\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\{|z-x|>\varepsilon_1\}}p^z(t+\varepsilon-s,x-z)\big|q(s,z,y)-q(s,x,y)\big|\,dz\,ds\\&=:J_{11}(\varepsilon)+J_{12}(\varepsilon).\end{aligned}$$

According to (5.11) and (2.1), we know immediately that

$$J_{11}(\varepsilon) \preceq \sigma \cdot \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d} \rho_1^{z,0}(t+\varepsilon-s, x-z)\,dz\,ds \preceq \frac{\sigma}{\varepsilon}\cdot\int_t^{t+\varepsilon} ds \leqslant C_1\sigma,$$

where $C_1>0$ is a constant independent of ε and σ. At the same time, it holds that for any $\gamma<\beta_0/\alpha_2$,

$$\begin{aligned}
|J_{12}(\varepsilon)| &\preceq \frac{1}{\varepsilon}\cdot\left(\int_t^{t+\varepsilon}(t+\varepsilon-s)\int_{\{|z-x|>\varepsilon_1\}}\frac{|q(s,z,y)|+|q(s,x,y)|}{|x-z|^{d+\alpha_1}}\,dz\,ds\right)\\
&\preceq \varepsilon_1^{-d-\alpha_1}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\big(\rho^{y,0}_{(\beta_0/\alpha_2)-\gamma}+\rho^{y,\beta_0}_{-\gamma}\big)(s,z-y)\,dz\,ds\\
&\quad+\int_t^{t+\varepsilon}\int_{\{|z-x|>\varepsilon_1\}}\frac{\big(\rho^{y,0}_{(\beta_0/\alpha_2)-\gamma}+\rho^{y,\beta_0}_{-\gamma}\big)(s,x-y)}{|z-x|^{d+\alpha_1}}\,dz\,ds\\
&\preceq \varepsilon_1^{-d-\alpha_1}\left(\int_t^{t+\varepsilon}s^{-1+(\beta_0/\alpha_2)-\gamma}\,ds+\varepsilon_1^d t^{-1-(d/\alpha_1)-\gamma}\int_t^{t+\varepsilon}ds\right)\\
&\leqslant C(\varepsilon_1)t^{-1-(d/\alpha_1)-\gamma}\varepsilon,
\end{aligned}$$

where in the first inequality we used the fact that $p^z(t,x)\preceq\frac{t}{|x|^{d+\alpha_1}}$ for every $t\in(0,1]$ and $x\in\mathbb{R}^d$ with $|x|>\varepsilon_1$, the second inequality follows from (3.18), and in the third inequality we used (2.3) and the fact that $\rho^{y,0}_{-\gamma}(s,x)\preceq t^{-1-(d/\alpha_1)-\gamma}$ for all $t<s<t+\varepsilon$ and $x\in\mathbb{R}^d$.

Second, note that for every fixed $x,y\in\mathbb{R}^d$, $q_0(\cdot,x,y)$ is continuous in $(0,1]$. Then, by (1.27), (3.18) and the dominated convergence theorem, we know that $q(\cdot,x,y)$ is continuous in $(0,1]$. Thus, for every $t\in(0,1]$, $x,y\in\mathbb{R}^d$ and $\sigma>0$, there exists a constant $\varepsilon_2:=\varepsilon_2(t,x,y,\sigma)>0$ such that for all $|s-t|<\varepsilon_2$,

$$|q(s,x,y)-q(t,x,y)|\leqslant\sigma,$$

from which we have that when $\varepsilon<\varepsilon_2$,

$$|J_2(\varepsilon)|\preceq\frac{\sigma}{\varepsilon}\cdot\left(\int_t^{t+\varepsilon}\left|\int_{\mathbb{R}^d}p^z(t+\varepsilon-s,x-z)\,dz\right|ds\right)\leqslant C_2\sigma,$$

where in the last inequality we have used Lemma 5.1, and $C_2>0$ is a constant independent of ε and σ.

Third, according to Lemma 5.1, we arrive at

$$\begin{aligned}
\lim_{\varepsilon\downarrow0}|J_3(\varepsilon)| &\leqslant |q(t,x,y)|\cdot\left(\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\cdot\int_t^{t+\varepsilon}\left|\int_{\mathbb{R}^d}p^z(t+\varepsilon-s,x-z)\,dz-1\right|ds\right)\\
&\leqslant |q(t,x,y)|\cdot\left(\lim_{\varepsilon\downarrow0}\sup_{s\in(0,\varepsilon)}\left|\int_{\mathbb{R}^d}p^z(s,x-z)\,dz-1\right|\right)=0.
\end{aligned}$$

Combining all the estimates together, we arrive at for every $\sigma>0$,

$$\lim_{\varepsilon\downarrow0}\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\phi(t+\varepsilon,s,x,y)\,ds-q(t,x,y)\right|\leqslant C_3\sigma,$$

where C_3 is independent of ε and σ. Since σ is arbitrary, we finally obtain that

$$\lim_{\varepsilon\downarrow0}\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\phi(t+\varepsilon,s,x,y)\,ds-q(t,x,y)\right|=0. \tag{5.12}$$

(ii) By the proof of (5.5) (also see [20, Theorem 2.4]), it holds that

$$|\mathcal{L}^z p^z(t-s,x-z)|\preceq\rho_0^{z,0}(t-s,x-z), \tag{5.13}$$

and so we can verify that for every $0<s<t<1$ and $x,y\in\mathbb{R}^d$,

$$\int_{\mathbb{R}^d}|\mathcal{L}^z p^z(t-s,x-z)||q(s,z,y)|\,dz<\infty.$$

Thus, by the dominated convergence theorem, for every $0 < s < t$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}\partial_t \phi(t,s,x,y) &= \int_{\mathbb{R}^d} \mathcal{L}^z p^z(t-s,x-z) q(s,z,y)\,dz \\ &= \int_{\mathbb{R}^d} \mathcal{L}^z p^z(t-s,x-z)(q(s,z,y)-q(s,x,y))\,dz \\ &\quad + q(s,x,y)\int_{\mathbb{R}^d} (\mathcal{L}^z - \mathcal{L}^x) p^z(t-s,x-z)\,dz \\ &\quad + q(s,x,y)\int_{\mathbb{R}^d} \mathcal{L}^x p^z(t-s,x-z)\,dz \\ &=: \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3.\end{aligned} \tag{5.14}$$

Combining (5.13) with (3.19) yields that for any $0 < \gamma < \theta < \beta_0/\alpha_2$,

$$\begin{aligned}|\tilde{J}_1| &\preceq \int_{\mathbb{R}^d} \tilde{\rho}_0^{z,\tilde{\theta}}(t-s,x-z)\cdot \tilde{\rho}^{y,0}_{\theta-\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}(s,z-y)\,dz \\ &\quad + \int_{\mathbb{R}^d} \tilde{\rho}_0^{z,\tilde{\theta}}(t-s,x-z)\cdot \tilde{\rho}^{y,0}_{\theta-\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)}(s,x-y)\,dz \\ &=: \tilde{J}_{11} + \tilde{J}_{12},\end{aligned}$$

where $\tilde{\theta} := \alpha_1(\beta_0 - \alpha_2\theta)/\alpha_2$. According to (3.2) (which holds for $\tilde{\rho}$), we obtain that for every $t/2 \leqslant s \leqslant t$,

$$\tilde{J}_{11} \preceq \tilde{\rho}_0^{y,0}(t,x-y)\cdot\left[(t-s)^{-1+(\tilde{\theta}/\alpha_2)} s^{\theta-\gamma+(\beta_0/\alpha_2)-(\beta_0/\alpha_1)-1}\right] \preceq t^{-\gamma-(\beta_0/\alpha_1)-(d/\alpha_1)-2}(t-s)^{-1+(\tilde{\theta}/\alpha_2)},$$

where in the last inequality we have used the fact that $\tilde{\rho}_0^{y,0}(t,x-y) \preceq t^{-1-d/\alpha_1}$ and $s^{-1} \preceq t^{-1}$ for $t/2 \leqslant s \leqslant t$. Note that $\tilde{\rho}^{y,\beta_0}_{\theta-\gamma-(\beta_0/\alpha_1)}(s,x-y) \preceq t^{-\gamma-(\beta_0/\alpha_1)-(d/\alpha_1)-1}$ for every $t/2 \leqslant s \leqslant t$, and (2.1) holds for $\tilde{\rho}$, then we have

$$\tilde{J}_{12} \preceq t^{-\gamma-(\beta_0/\alpha_1)-(d/\alpha_1)-1}\int_{\mathbb{R}^d} \tilde{\rho}_0^{z,\tilde{\theta}}(t-s,x-z)\,dz \preceq t^{-\gamma-(\beta_0/\alpha_1)-(d/\alpha_1)-1}(t-s)^{-1+(\tilde{\theta}/\alpha_2)}.$$

By (2.16) and (3.18), we arrive at that for any $0 < \gamma < \theta < \beta_0/\alpha_2$ and every $t/2 \leqslant s \leqslant t$,

$$|(\mathcal{L}^z - \mathcal{L}^x)p^z(t-s,x-z)| \preceq |q_0(t-s,x,z)| \preceq \rho^{z,\beta_0}_{-\gamma}(t-s,x-z)$$

and

$$|q(s,x,y)| \preceq \left(\rho^{y,0}_{(\beta_0/\alpha_2)-\gamma} + \rho^{y,\beta_0}_{-\gamma}\right)(s,x-y) \preceq t^{-\gamma-(d/\alpha_1)-1}.$$

Hence, combining both estimates above with (2.1), we obtain

$$|\tilde{J}_2| \preceq t^{-\gamma-(d/\alpha_1)-1}\int_{\mathbb{R}^d} \rho^{z,\beta_0}_{-\gamma}(t-s,x-z)\,dz \preceq t^{-\gamma-(d/\alpha_1)-1}(t-s)^{-1+(\beta_0/\alpha_2)-\gamma}.$$

Furthermore, (3.18) and (5.7) yield that for every $t/2 \leqslant s \leqslant t$ and constants $\gamma, \theta \in (0,1)$ such that $\gamma < \theta/\alpha_2$ and $2\gamma + (\theta/\alpha_1) < \beta_0/\alpha_2$,

$$|\tilde{J}_3| \preceq t^{-\gamma-(d/\alpha_1)-1}(t-s)^{-1+(\beta_0/\alpha_2)-2\gamma-(\theta/\alpha_1)}.$$

On the other hand, when $0 < s \leqslant t/2$, it follows from (5.13), (3.18) and (3.2) that

$$\begin{aligned}\left|\partial_t\phi(t,s,x,y)\right| &\preceq \int_{\mathbb{R}^d} \left|\mathcal{L}^z p^z(t-s,x-z)\right|\left|q(s,z,y)\right|dz \\ &\preceq \int_{\mathbb{R}^d} \rho_0^{z,0}(t-s,x-z)\cdot\left(\rho^{y,0}_{(\beta_0/\alpha_2)-\gamma} + \rho^{y,\beta_0}_{-\gamma}\right)(s,z-y)\,dz \\ &\preceq \rho_0^{y,0}(t,x-y)\cdot\left[(t-s)^{-1}s^{(\beta_0/\alpha_2)-\gamma} + (t-s)^{-1+(\beta_0/\alpha_2)}s^{-\gamma} + s^{-1+(\beta_0/\alpha_2)-\gamma}\right] \\ &\quad + \rho_0^{y,\beta_0}(t,x-y)\cdot(t-s)^{-1}s^{-\gamma} \\ &\preceq t^{-2-(d/\alpha_1)}s^{-1+(\beta_0/\alpha_2)-\gamma},\end{aligned}$$

where in the last inequality we have used the facts that $\rho_0^{y,0}(t,x-y) \preceq t^{-1-(d/\alpha_1)}$ and $(t-s)^{-1} \preceq t^{-1}$ for every $0 < s \leqslant t/2$. Therefore, choosing $\gamma, \theta \in (0,1)$ such that $\gamma < \theta/\alpha_2$ and $2\gamma + (\theta/\alpha_1) < \beta_0/\alpha_2$, and combining all the estimates above, we know that (5.14) is well defined, and that for every $t \in (0,1)$, $\sup_{t_0 \in (t, t+\varepsilon_1)} |\partial_t \phi(t_0,s,x,y)| \leqslant \eta(t,s)$, where η is a non-negative measurable function such that $\int_0^t \eta(t,s)\,ds <$

∞ and $\varepsilon_1 > 0$ is a constant small enough (which may depend on t). Now, according to the dominated convergence theorem, for every $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \cdot \left(\int_0^t \big(\phi(t+\varepsilon, s, x, y) - \phi(t, s, x, y)\big)\, ds \right) = \int_0^t \partial_t \phi(t, s, x, y)\, ds. \tag{5.15}$$

(iii) We obtain by (5.12) and (5.15) that for every $t \in (0,1)$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \left| \frac{1}{\varepsilon} \cdot \Big(\int_0^{t+\varepsilon} \phi(t+\varepsilon, s, x, y)\, ds - \int_0^t \phi(t, s, x, y)\, ds \Big) - q(t, x, y) - \int_0^t \partial_t \phi(t, s, x, y)\, ds \right| \\
&\leqslant \lim_{\varepsilon \downarrow 0} \left| \frac{1}{\varepsilon} \cdot \Big(\int_t^{t+\varepsilon} \phi(t+\varepsilon, s, x, y)\, ds \Big) - q(t, x, y) \right| \\
&\quad + \lim_{\varepsilon \downarrow 0} \left| \frac{1}{\varepsilon} \cdot \Big(\int_0^t (\phi(t+\varepsilon, s, x, y) - \phi(t, s, x, y))\, ds \Big) - \int_0^t \partial_t \phi(t, s, x, y)\, ds \right| \\
&= 0.
\end{aligned}$$

By the same way, it is not difficult to verify

$$\lim_{\varepsilon \uparrow 0} \left| \frac{1}{\varepsilon} \cdot \Big(\int_0^{t+\varepsilon} \phi(t+\varepsilon, s, x, y)\, ds - \int_0^t \phi(t, s, x, y)\, ds \Big) - q(t, x, y) - \int_0^t \partial_t \phi(t, s, x, y)\, ds \right| = 0.$$

Hence, we have for any $t_0 \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\partial_t \left(\int_0^{\cdot} \phi(\cdot, s, x, y)\, ds \right)(t_0) = q(t_0, x, y) + \int_0^{t_0} \partial_t \phi(t_0, s, x, y)\, ds.$$

Combining all estimates above with (5.10), (1.27) and (5.14), we have for every $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
\frac{\partial p(t, x, y)}{\partial t} &= \mathcal{L}^y p^y(t, x-y) + q(t, x, y) + \int_0^t \partial_t \phi(t, s, x, y)\, ds \\
&= \mathcal{L}^y p^y(t, x-y) + (\mathcal{L}^x - \mathcal{L}^y) p^y(t, x-y) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} (\mathcal{L}^x - \mathcal{L}^z) p^z(t-s, x-z) \cdot q(s, z, y)\, dz\, ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^z p^z(t-s, x-z) q(s, z, y)\, dz\, ds \\
&= \mathcal{L}^x p^y(t, x-y) + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^x p^z(t-s, x-z) q(s, z, y)\, dz\, ds.
\end{aligned} \tag{5.16}$$

Furthermore, by the same arguments for estimates of $\tilde{J}_1$, $\tilde{J}_2$ and $\tilde{J}_3$ above, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} |\mathcal{L}^x p^z(t-s, x-z) q(s, z, y)|\, dz\, ds \\
&\preceq \int_0^t \int_{\mathbb{R}^d} \big|\mathcal{L}^z p^z(t-s, x-z) q(s, z, y)\big|\, dz\, ds + \int_0^t \int_{\mathbb{R}^d} |(\mathcal{L}^x - \mathcal{L}^z) p^z(t-s, x-z) q(s, z, y)|\, dz\, ds \\
&< \infty
\end{aligned} \tag{5.17}$$

and so

$$\int_0^t \mathcal{L}^x(\phi(t, s, \cdot, y))(x)\, ds = \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^x p^z(t-s, x-z) q(s, z, y)\, dz\, ds,$$

which in turn implies that (5.16) is well defined.

(iv) For any $\varepsilon, \sigma > 0$ small enough,

$$\begin{aligned}
&\int_{\{|z|>\sigma\}} \delta_{\int_0^t \phi(t,s,\cdot,y)\, ds}(x; z) \cdot \frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\, dz \\
&= \int_\varepsilon^{t-\varepsilon} \int_{\{|z|>\sigma\}} \delta_{\phi(t,s,\cdot,y)}(x; z) \cdot \frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\, dz\, ds + \int_{\{(0,\varepsilon) \cup (t-\varepsilon, t)\}} \int_{\{|z|>\sigma\}} \delta_{\phi(t,s,\cdot,y)}(x; z) \cdot \frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\, dz\, ds \\
&=: \widehat{J}_1(\varepsilon, \sigma) + \widehat{J}_2(\varepsilon, \sigma).
\end{aligned}$$

Note that for the integral with respect to time variable in $\widehat{J}_1(\varepsilon,\sigma)$, there is not any singularity. According to the dominated convergence theorem, we have

$$\lim_{\sigma\downarrow 0} J_1(\varepsilon,\sigma)=\int_\varepsilon^{t-\varepsilon}\mathcal{L}^x(\phi(t,s,\cdot,y))(x)\,ds=\int_\varepsilon^{t-\varepsilon}\int_{\mathbb{R}^d}\mathcal{L}^x p^z(t-s,x-z)q(s,z,y)\,dz\,ds,$$

Meanwhile, according to the proof of (5.17) and Fatou's lemma we arrive at

$$\limsup_{\sigma\downarrow 0}|\widehat{J}_2(\varepsilon,\sigma)|\leqslant\int_{\{(0,\varepsilon)\cup(t-\varepsilon,t)\}}|\mathcal{L}^x(\phi(t,s,\cdot,y)(x)|\,ds.$$

Hence, combining all the estimates above with (1.25) yields

$$\begin{aligned}&\Big|\mathcal{L}^x\Big(\int_0^t\phi(t,s,\cdot,y)\,ds\Big)(x)-\int_0^t\mathcal{L}^x(\phi(t,s,\cdot,y))(x)\,ds\Big|\\&\leqslant\limsup_{\sigma\downarrow 0}\Big|\int_{\{|z|>\sigma\}}\delta_{\int_0^t\phi(t,s,\cdot,y)\,ds}(x;z)\cdot\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\,dz-\int_0^t\mathcal{L}^x(\phi(t,s,\cdot,y))(x)\,ds\Big|\\&\leqslant 2\int_{\{(0,\varepsilon)\cup(t-\varepsilon,t)\}}|\mathcal{L}^x(\phi(t,s,\cdot,y))(x)|\,ds.\end{aligned}$$

Then, letting $\varepsilon\to 0$, we know that for every $t\in(0,1)$ and $x,y\in\mathbb{R}^d$,

$$\mathcal{L}^x\Big(\int_0^t\phi(t,s,\cdot,y)\,ds\Big)(x)=\int_0^t\mathcal{L}^x(\phi(t,s,\cdot,y))(x)\,ds=\int_0^t\int_{\mathbb{R}^d}\mathcal{L}^x p^z(t-s,x-z)q(s,z,y)\,dz\,ds,$$

which along with (5.16) yields (1.5) immediately.

(2) Suppose that $\kappa(x,z)$ depends on z. Then, using (2.18), (3.21) and (3.22), and following the same arguments above, we can also show (1.5) holds true. The details are omitted here. □

5.2. **Maximum principle and uniqueness.** Adopting the approach of [20, Theorem 4.1], we will prove the following maximum principle for non-local parabolic PDEs associated with the operator $\mathcal{L}$, which is crucial for the uniqueness of solution to the corresponding Cauchy problem.

Theorem 5.5. *Let $u\in C_b([0,1]\times\mathbb{R}^d)$ be the solution of the following equation*

$$\partial_t u(t,x)=\mathcal{L}u(t,x),\quad (t,x)\in(0,1]\times\mathbb{R}^d. \tag{5.18}$$

Suppose

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\big|u(t,x)-u(0,x)\big|=0, \tag{5.19}$$

and for every $x\in\mathbb{R}^d$, $t\mapsto\mathcal{L}u(t,x)$ is continuous in $t\in(0,1]$. Assume that there exists a function $\theta(x)\in(0,1)$ such that $\inf_{x\in\mathbb{R}^d}\big(\theta(x)+1-\alpha(x)\big)>0$ and for every $\varepsilon\in(0,1)$,

$$\sup_{t\in(\varepsilon,1)}\big|u(t,x)-u(t,x')\big|\leqslant c_1(\varepsilon)|x-x'|^{\theta(x)},\quad x,x'\in\mathbb{R}^d. \tag{5.20}$$

Then for every $t\in(0,1]$,

$$\sup_{x\in\mathbb{R}^d}u(t,x)\leqslant\sup_{x\in\mathbb{R}^d}u(0,x).$$

Proof. Throughout the proof, the constant C denotes a positive constant that is independent of R and x whose exact value may change from line to line. Since (5.19) holds, it suffices to prove that for any $\varepsilon\in(0,1)$ and $t\in(\varepsilon,1]$,

$$\sup_{x\in\mathbb{R}^d}u(t,x)\leqslant\sup_{x\in\mathbb{R}^d}u(\varepsilon,x). \tag{5.21}$$

For every $R>1$, we can choose a smooth cut-off function $l_R:\mathbb{R}^d\to\mathbb{R}$ such that

$$l_R(x)=\begin{cases}1, & |x|\leqslant R,\\ \in[0,1], & R<|x|<2R,\\ 0, & |x|\geqslant 2R,\end{cases}$$

and

$$|\nabla l_R(x)|^2+|\nabla^2 l_R(x)|\leqslant\frac{C}{R^2},\quad x\in\mathbb{R}^d. \tag{5.22}$$

For every $R,\delta>0$ and $\varepsilon\in(0,1)$, define

$$u_R^\delta(t,x)=u(t,x)l_R(x)-(t-\varepsilon)\delta,\quad x\in\mathbb{R}^d,t\in(\varepsilon,1).$$

Then, it follows from the fact $\partial_t u(t,x) = \mathcal{L}u(t,x)$ that

$$\partial_t u_R^\delta(t,x) = \mathcal{L}u_R^\delta(t,x) + g_R^\delta(t,x), \tag{5.23}$$

where $g_R^\delta(t,x) := \mathcal{L}u(t,x)\cdot l_R(x) - \mathcal{L}(u\cdot l_R)(t,x) - \delta$.

Observe that

$$\begin{aligned}
&\mathcal{L}(u\cdot l_R)(t,x) - \mathcal{L}u(t,x)\cdot l_R(x) - \mathcal{L}l_R(x)\cdot u(t,x)\\
&= \int_{\mathbb{R}^d} (u(t,x+z)-u(t,x))\cdot(l_R(x+z)-l_R(x))\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\,dz\\
&\preceq \int_{\{|z|\leqslant 1\}} (u(t,x+z)-u(t,x))\cdot(l_R(x+z)-l_R(x))\frac{1}{|z|^{d+\alpha(x)}}\,dz\\
&\quad + \int_{\{|z|>1\}} (u(t,x+z)-u(t,x))\cdot(l_R(x+z)-l_R(x))\frac{1}{|z|^{d+\alpha(x)}}\,dz\\
&=: J_1 + J_2.
\end{aligned}$$

On the one hand, by (5.20) and (5.22),

$$J_1 \leqslant \frac{C(\varepsilon)}{R}\int_{\{|z|\leqslant 1\}} \frac{|z|^{1+\theta(x)}}{|z|^{d+\alpha(x)}}\,dz \leqslant \frac{C(\varepsilon)}{R},$$

where in the last inequality we used $\inf_{x\in\mathbb{R}^d}\big(\theta(x)+1-\alpha(x)\big)>0$. On the other hand, for every $0<\gamma<\alpha_1\wedge 1$,

$$J_2 \leqslant 2\|u\|_\infty \|l_R\|_\infty^{1-\gamma}\int_{\{|z|>1\}} \frac{|l_R(x+z)-l_R(x)|^\gamma}{|z|^{d+\alpha(x)}}\,dz \leqslant \frac{C\|u\|_\infty}{R^\gamma}\int_{\{|z|>1\}} |z|^{-d-\alpha(x)+\gamma}\,dz \leqslant C\|u\|_\infty R^{-\gamma}.$$

According to the argument above, it is easy to verify that

$$|\mathcal{L}l_R(x)| \leqslant CR^{-\gamma}.$$

Combining all the estimates above yields

$$g_R^\delta(t,x) \leqslant C(\varepsilon,u)R^{-\gamma} - \delta. \tag{5.24}$$

Now we are going to verify that for every fixed R large enough,

$$\sup_{x\in\mathbb{R}^d} u_R^\delta(t,x) \leqslant \sup_{x\in\mathbb{R}^d} u_R^\delta(\varepsilon,x), \quad t\in(\varepsilon,1]. \tag{5.25}$$

Suppose (5.25) does not hold. Then for every large enough R, there exists $(t_0,x_0)\in(\varepsilon,1]\times\mathbb{R}^d$ (which may depend on R, ε and $\delta>0$) such that

$$\sup_{(t,x)\in(\varepsilon,1)\times\mathbb{R}^d} u_R^\delta(t,x) = u_R^\delta(t_0,x_0). \tag{5.26}$$

Note that the existence of (t_0,x_0) follows from the fact that $u_R^\delta(t,x)=0$ for every $t\in(\varepsilon,1]$ and $|x|>2R$. Therefore, by (5.23), we have for every $h\in(0,t_0-\varepsilon)$,

$$0 \leqslant \frac{u_R^\delta(t_0,x_0)-u_R^\delta(t_0-h,x_0)}{h} = \frac{1}{h}\int_{t_0-h}^{t_0} \mathcal{L}u_R^\delta(s,x_0)\,ds + \frac{1}{h}\int_{t_0-h}^{t_0} g_R^\delta(s,x_0)\,ds.$$

Letting $h\downarrow 0$, we arrive at

$$0 \leqslant \mathcal{L}u_R^\delta(t_0,x_0) + g_R^\delta(t_0,x_0), \tag{5.27}$$

thanks to the assumption that the function $t\mapsto \mathcal{L}u(t,x)$ is continuous on $(0,1]$. Furthermore, from (5.26) it is easy to see

$$\mathcal{L}u_R^\delta(t_0,x_0) = \int_{\mathbb{R}^d} \big(u_R^\delta(t_0,x_0+z)+u_R^\delta(t_0,x_0-z)-2u_R^\delta(t_0,x_0)\big)\frac{\kappa(x_0,z)}{|z|^{d+\alpha(x_0)}}\,dz \leqslant 0.$$

Combining this with (5.24), we get that for every $R > \big(\frac{2C(\varepsilon,u)}{\delta}\big)^{1/\gamma}$,

$$\mathcal{L}u_R^\delta(t_0,x_0) + g_R^\delta(t_0,x_0) \leqslant -\delta/2,$$

which contradicts with (5.27). Hence, the assumption above fails and so (5.25) holds. Letting $R\to\infty$ in (5.25), we obtain (5.21) immediately. $\square$

Now, we are in a position to prove Theorems 1.1 and 1.3.

Proofs of Theorems 1.1 *and* 1.3. (i) We first assume that $\kappa(x,z)$ is independent of z. We will prove that $p(t,x,y)$ constructed by (1.26) satisfies all the assertions.

According to Proposition 5.4, (1.5) holds. By Propositions 4.1 and 4.3, (1.6) and (1.7) hold true. Propositions 4.6 and 4.7 imply (1.9) and (1.11), respectively. By Proposition 4.9, we know that $p:(0,1]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}_+$ is continuous. The lower bounds (1.8) will be shown in Corollary 5.10, which is a consequence of Propositions 5.6 and 5.9 in the next subsection.

(ii) According to the proof of Proposition 5.4, we can obtain that for every $\varepsilon\in(0,1)$, there exist a constant $C(\varepsilon)>0$ and a measurable function $\eta:\mathbb{R}^d\to\mathbb{R}_+$ such that $\int_{\mathbb{R}^d}\eta(y)\,dy<\infty$,

$$\sup_{t\in(\varepsilon,1)}\left|\frac{\partial p(t,x,y)}{\partial t}\right|\leqslant C(\varepsilon)\eta(x-y)$$

and

$$\sup_{t\in(\varepsilon,1),\sigma>0}\left|\int_{\{|z|>\sigma\}}\big(p(t,x+z,y)+p(t,x-z,y)-2p(t,x,y)\big)\frac{\kappa(x,z)}{|z|^{d+\alpha(x)}}\,dz\right|\leqslant C(\varepsilon)\eta(x-y).$$

Thus, by the dominated convergence theorem, it is easy to verify that $\mathcal{L}u_f(t,\cdot)(x)$ exists for each $t>0$ and $x\in\mathbb{R}^d$, $t\mapsto\mathcal{L}u_f(t,\cdot)(x)$ is continuous, and that (1.15) holds.

(iii) We have by (1.26) that

$$\begin{aligned}u_f(t,x)-f(x)&=\int_{\mathbb{R}^d}p^y(t,x-y)\big(f(y)-f(x)\big)\,dy+f(x)\left(\int_{\mathbb{R}^d}p^y(t,x-y)\,dy-1\right)\\&\quad+\int_{\mathbb{R}^d}f(y)\int_0^t\int_{\mathbb{R}^d}p^z(t-s,x-z)q(s,z,y)\,dz\,ds\,dy\\&=:J_1+J_2+J_3.\end{aligned}$$

For any $f\in C_{b,u}(\mathbb{R}^d)$ and for every $\varepsilon>0$, there exists a constant $\delta:=\delta(\varepsilon)>0$ such that $|f(x)-f(y)|<\varepsilon$ for all $x,y\in\mathbb{R}^d$ with $|x-y|\leqslant\delta$. Thus,

$$\begin{aligned}|J_1|&\leqslant\int_{\{|y-x|\leqslant\delta\}}p^y(t,x-y)|f(y)-f(x)|\,dy+\int_{\{|y-x|>\delta\}}p^y(t,x-y)|f(y)-f(x)|\,dy\\&\leqslant\varepsilon\int_{\mathbb{R}^d}\rho_1^{y,0}(t,x-y)\,dy+C(\varepsilon)\|f\|_\infty\int_{\{|y-x|>\delta\}}\frac{t}{|x-y|^{d+\alpha_1}}\,dy\leqslant C\varepsilon+C(\varepsilon,\delta)t,\end{aligned}$$

where in the last inequality we have used (2.1). According to (5.2), it holds that $\lim_{t\downarrow0}\sup_{x\in\mathbb{R}^d}|J_2|=0$. Furthermore, by (3.18) and (3.13), we obtain that for $\gamma<\beta_0^*/\alpha_2$ with $\beta^*\in(0,\beta]\cap(0,\alpha_2/2)$,

$$\begin{aligned}|J_3|&\preceq\|f\|_\infty\left[\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\rho_1^{z,0}(t-s,x-z)\Big(\rho^{y,0}_{(\beta_0^*/\alpha_2)-\gamma}(s,z-y)+\rho^{y,\beta_0^*}_{-\gamma}(s,z-y)\Big)\,dz\,ds\,dy\right]\\&\preceq\|f\|_\infty\int_{\mathbb{R}^d}\Big(\rho^{y,0}_{1+(\beta_0^*/\alpha_2)-\gamma}+\rho^{y,\beta_0^*}_{1-\gamma}\Big)(t,x-y)\,dy\preceq t^{(\beta_0^*/\alpha_2)-\gamma}\|f\|_\infty.\end{aligned}$$

Combining all the estimates above together, we arrive at

$$\lim_{t\downarrow0}\sup_{x\in\mathbb{R}^d}\big|u_f(t,x)-f(x)\big|\leqslant\varepsilon.$$

Since ε is arbitrary, we know that (1.16) holds.

(iv) Denote by $C_b^\varepsilon(\mathbb{R}^d)$ the set of bounded Hölder continuous functions, and by $C_b^{2,\varepsilon}(\mathbb{R}^d)$ the set of bounded twice differentiable functions whose second derivatives are uniformly Hölder continuous. We first suppose that $f\in C_b^{2,\varepsilon}(\mathbb{R}^d)$. Let

$$\tilde u_f(t,x):=f(x)+\int_0^t\int_{\mathbb{R}^d}p(s,x,y)\mathcal{L}f(y)\,dy\,ds=f(x)+\int_0^t u_{\mathcal{L}f}(s,x)\,ds.$$

Since $\mathcal{L}f\in C_b^\varepsilon(\mathbb{R}^d)$, it is easy to see that

$$\frac{\partial\tilde u_f(t,x)}{\partial t}=\int_{\mathbb{R}^d}p(t,x,y)\mathcal{L}f(y)\,dy=u_{\mathcal{L}f}(t,x),\quad t\in(0,1]$$

and

$$\lim_{t\downarrow0}\sup_{x\in\mathbb{R}^d}|\tilde u_f(t,x)-f(x)|=0.$$

On the other hand, due to $\mathcal{L}f \in C_b^{\varepsilon}(\mathbb{R}^d)$ again, it follows from (1.15) that $\mathcal{L}u_{\mathcal{L}f}(t,x) = \frac{\partial u_{\mathcal{L}f}(t,x)}{\partial t}$. Furthermore, following the proof of Proposition 5.4, we will get

$$\begin{aligned}\int_0^t \Big| \int_{\mathbb{R}^d} \mathcal{L}^x p^z(t-s, x-z)\mathcal{L}f(z)\,dz\Big|\,ds \leqslant &\int_0^t \Big| \int_{\mathbb{R}^d} \mathcal{L}^z p^z(t-s, x-z)\mathcal{L}f(z)\,dz\Big|\,ds\\ &+\int_0^t \Big| \int_{\mathbb{R}^d} (\mathcal{L}^x-\mathcal{L}^z) p^z(t-s, x-z)\mathcal{L}f(z)\,dz\Big|\,ds<\infty\end{aligned}$$

and so

$$\int_0^t \mathcal{L}u_{\mathcal{L}f}(s,x)\,ds$$

is well defined. Hence,

$$\begin{aligned}\mathcal{L}\tilde{u}_f(t,x) &= \mathcal{L}f(x) + \int_0^t \mathcal{L}u_{\mathcal{L}f}(s,x)\,ds = \mathcal{L}f(x) + \int_0^t \frac{\partial u_{\mathcal{L}f}(s,x)}{\partial s}\,ds\\ &= \mathcal{L}f(x) + u_{\mathcal{L}f}(t,x) - \mathcal{L}f(x) = u_{\mathcal{L}f}(t,x),\end{aligned}$$

where in the third equality we used $\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}|u_{\mathcal{L}f}(t,x)-\mathcal{L}f(x)|=0$, thanks to (1.16). Therefore, both u_f and $\tilde{u}_f$ are solutions of the following PDE

$$\begin{cases}\frac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x),\\ \lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}|u(t,x)-f(x)|=0.\end{cases} \tag{5.28}$$

Let $w_f(t,x) := u_f(t,x) - \tilde{u}_f(t,x)$. Then, (5.18) and (5.19) hold for w_f with $w_f(0,x)\equiv 0$. At the same time, it is easy to verify from (1.26) that the function $t\mapsto \mathcal{L}w_f(t,x)$ is continuous on $(0,1]$, and that (1.9) implies (5.20) holds for w_f. Thus, by Theorem 5.5, we have

$$w_f(t,x)\leqslant w_f(0,x)=0,\quad (t,x)\in(0,1]\times\mathbb{R}^d.$$

Furthermore, applying the argument above to $-w_f$, we finally get that

$$w_f(t,x)\equiv 0,\quad (t,x)\in(0,1]\times\mathbb{R}^d.$$

Therefore, $u_f(t,x)=\tilde{u}_f(t,x)$ for any $t\in(0,1]$ and $x\in\mathbb{R}^d$, which further implies that

$$\int_{\mathbb{R}^d}\mathcal{L}p(t,\cdot,y)(x)f(y)\,dy = \frac{\partial u_f(t,x)}{\partial t} = \frac{\partial \tilde{u}_f(t,x)}{\partial t} = \int_{\mathbb{R}^d} p(t,x,y)\mathcal{L}f(y)\,dy.$$

Thus, (1.17) holds. Observe that

$$u_f(t,x)=\tilde{u}_f(t,x)=f(x)+\int_0^t u_{\mathcal{L}f}(s,x)\,ds.$$

According to (1.16), we arrive at (1.18) immediately. By the standard approximation procedure, we know that (1.17) and (1.18) still hold for every $f\in C_{b,u}^2(\mathbb{R}^d)$.

(v) Let $u(t,x) =: \int_{\mathbb{R}^d} p(t,x,y)\,dy$. Then, according to (1.15) and (1.16), we know that u satisfies the following equation

$$\begin{cases}\frac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x),\\ \lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\big|u(t,x)-1\big|=0.\end{cases} \tag{5.29}$$

At the same time, $v(t,x)\equiv 1$ satisfies the equation (5.29) above. Note that (5.20) and the time continuity condition hold for both $u(t,x)$ and $v(t,x)$. Then, using the same argument as in (iv) and applying Theorem 5.5, we obtain

$$\int_{\mathbb{R}^d} p(t,x,y)\,dy = u(t,x) = 1,\quad (t,x)\in(0,1]\times\mathbb{R}^d,$$

which is (1.13).

For every fixed $s\in(0,1)$ and $y\in\mathbb{R}^d$, we define

$$u_{s,y}(t,x) := \int_{\mathbb{R}^d} p(t,x,z)p(s,z,y)\,dz.$$

Again by (1.15) and (1.16), the following equation holds for $u_{s,y}$:

$$\begin{cases}\frac{\partial u_{s,y}(t,x)}{\partial t} = \mathcal{L}u_{s,y}(t,x),\\ \lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}|u_{s,y}(t,x)-p(s,x,y)|=0.\end{cases} \tag{5.30}$$

On the other hand, it is easy to verify that $v_{s,y}(t,x) := p(t+s,x,y)$ satisfies (5.30), and (5.20) and the time continuity condition hold for both $u_{s,y}(t,x)$ and $v_{s,y}(t,x)$. Following the procedure above and applying Theorem 5.5, we arrive at

$$\int_{\mathbb{R}^d} p(t,x,z)p(s,z,y)\,dz = u_{s,y}(t,x) = v_{s,y}(t,x) = p(t+s,x,y),$$

which is (1.12).

(vi) Suppose that $\widehat{p}(t,x,y)$ is another jointly continuous function on $(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$ that is bounded for each $t>0$ and satisfies (1.9) and (1.14)-(1.16). For every $f\in C_{b,u}(\mathbb{R}^d)$, let

$$\widehat{u}_f(t,x) := \int_{\mathbb{R}^d} \widehat{p}(t,x,y)f(y)\,dy \quad \text{and} \quad \widehat{w}_f(t,x) := u_f(t,x)-\widehat{u}_f(t,x).$$

Then both u_f and $\widehat{u}_f$ satisfy (1.15). By the same argument above, we have $\widehat{w}_f(t,x)=0$ for each $(t,x)\in(0,1]\times\mathbb{R}^d$. This implies that for every $f\in C_{b,u}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy = \int_{\mathbb{R}^d} \widehat{p}(t,x,y)f(y)\,dy.$$

Consequently, $p(t,x,y)=\widehat{p}(t,x,y)$ for a.e. $y\in\mathbb{R}^d$ and hence for every $y\in\mathbb{R}^d$.

(vii) Suppose that $\kappa(x,z)$ depends on z. The desired upper and lower estimates and the regularity can be proved in a similar way as before. On the other hand, the condition $(\alpha_2/\alpha_1)-1<\beta_*/\alpha_2<1/2$ implies that $\alpha_2-\alpha_1<\alpha_1/2<1$, which ensures that (5.20) holds with $u_f(t,x) := \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy$ for every $f\in C_{b,u}(\mathbb{R}^d)$. Therefore, following the arguments in steps (i)-(vi), we can verify that (1.12)-(1.18) hold in such case. □

Sketch of the Proof for Remark 1.5(1). In Theorem 1.1 we assume that $\kappa(x,z)$ is independent of z. According to its proof, the reason why we need such condition is only due to that this implies the gradient estimate (2.20). Thus, for the upper bound estimates for $|\nabla p^y(t,x)|$ and $|\nabla^2 p^y(t,x)|$, the time singularity factor $t^{-1/\alpha(y)}$ will not appear when $|x|$ is large and t is small. This point is crucial for estimates (2.16) and (2.38), which yield Theorem 1.1.

Now, we turn to these two assumptions in Remark 1.5(1). According to [37, Theorem 1.5] and [7, Corollary 7 and Theorem 21], the following gradient estimate

$$|\nabla^k p^y(t,x)| \preceq \frac{t}{|x|^{d+\alpha(y)+k}}, \quad k=1,2$$

holds for all $t\in(0,1]$ and $x\in\mathbb{R}^d$ with $|x|$ is large, which also ensures that the required estimates (2.16) and (2.38) hold true. □

5.3. **Lower bound estimates.** In this subsection, we will establish lower bound estimates for $p(t,x,y)$. The idea of the arguments below is inspired by that in [20, Subsection 4.4]. Throughout this part, we will always suppose that either of two conditions below is satisfied:

(1) $\kappa(x,z)$ is independent of z.
(2) $\kappa(x,z)$ depends on z, and $(\alpha_2/\alpha_1)-1<\beta_0^{**}/\alpha_2$, where $\beta_0^{**}\in(0,\beta_0]\cap(0,\alpha_2/2)$.

Proposition 5.6. (On diagonal lower bounds) *There exists a positive constant $c_1 := c_1(\alpha,\kappa)$ such that for all $t\in(0,1]$ and $x,y\in\mathbb{R}^d$ with $|x-y|\leqslant 5(t^{1/\alpha(x)}\vee t^{1/\alpha(y)})$,*

$$p(t,x,y)\geqslant c_1 t^{-d/\alpha(x)}. \tag{5.31}$$

Proof. For simplicity, we only prove the case that $\kappa(x,z)$ depends on z and $(\alpha_2/\alpha_1)-1<\beta_0^{**}/\alpha_2$, since the other case can be tackled similarly and easily.

First, according to (3.11), for any $t\in(0,1]$ and $x,y\in\mathbb{R}^d$ with $|x-y|\leqslant 5(t^{1/\alpha(x)}\vee t^{1/\alpha(y)})\leqslant 5t^{1/\alpha_2}$,

$$t^{1/\alpha(x)} \asymp t^{1/\alpha(y)}. \tag{5.32}$$

It is well known that for any $t>0$ and $x,y\in\mathbb{R}^d$,

$$p^y(t,x-y)\asymp \frac{t}{(t^{1/\alpha(y)}+|x-y|)^{d+\alpha(y)}}.$$

Thus, for all $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,

$$p^y(t,x-y)\succeq t^{-d/\alpha(y)} \quad \text{when } |x-y|\leqslant 5(t^{1/\alpha(x)}\vee t^{1/\alpha(y)}). \tag{5.33}$$

Second, (3.21) and (3.13) yield that for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$ with $|x-y| \leqslant 5t^{1/\alpha_2}$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} p^z(t-s, x-z)|q(s,z,y)|\,dz\,ds \\
&\preceq \int_0^t \int_{\mathbb{R}^d} \rho_1^{z,0}(t-s, x-z) \cdot \Big(\rho^{y,0}_{(\beta_0^{**}/\alpha_2)-\gamma_2} + \rho^{y,\beta_0^{**}}_{-\gamma_2}\Big)(s,z,y)\,dz\,ds \\
&\preceq \rho^{y,\beta_0^{**}}_{1-\gamma_2}(t, x-y) + \rho^{y,0}_{1+(\beta_0^{**}/\alpha_2)-\gamma_2}(t, x-y) \\
&\preceq |x-y|^{\beta_0^{**}} t^{-d/\alpha(y)-\gamma_2} + t^{-(d/\alpha(y))+(\beta_0^{**}/\alpha_2)-\gamma_2} \preceq t^{-(d/\alpha(y))+(\beta_0^{**}/\alpha_2)-\gamma_2},
\end{aligned} \tag{5.34}$$

where in the first inequality $\gamma_2 := (\alpha_2/\alpha_1) - 1 + \gamma$ and the fourth inequality follows from the fact that $|x-y| \leqslant 5t^{1/\alpha_2}$.

The assumption $(\alpha_2/\alpha_1) - 1 < \beta_0^{**}/\alpha_2$ ensures that we can choose $\gamma > 0$ small enough such that $\gamma_2 < \beta_0^{**}/\alpha_2$. Combining (5.34) with (5.33), (1.26) and (5.32), we arrive at that there is a constant $t_0 \in (0,1]$ such that for all $t \in (0, t_0]$ and $x, y \in \mathbb{R}^d$ with $|x-y| \leqslant 5(t^{1/\alpha(x)} \vee t^{1/\alpha(y)})$,

$$p(t,x,y) \succeq t^{-d/\alpha(y)} \asymp t^{-d/\alpha(x)}. \tag{5.35}$$

Note that, according to the argument above, (5.35) still holds for all $t \in (0, t_0]$ and $x, y \in \mathbb{R}^d$ with $|x-y| \preceq (t^{1/\alpha(x)} \vee t^{1/\alpha(y)})$. Furthermore, due to (5.32), for any $y, z \in \mathbb{R}^d$ with $|z-y| \leqslant 5t^{1/\alpha(y)}$, we have $|x-z| \leqslant |z-y| + |x-y| \preceq t^{1/\alpha(x)}$ for any $x \in \mathbb{R}^d$ with $|x-y| \preceq (t^{1/\alpha(x)} \vee t^{1/\alpha(y)})$, and so

$$p(t,x,z) \succeq t^{-d/\alpha(z)} \asymp t^{-d/\alpha(x)}, \quad t \in (0, t_0],$$

thanks to (5.32) again. Therefore, according to the Chapman-Kolmogorov equation (1.12) and (5.35), for every $t \in [t_0, 2t_0]$ and $x, y \in \mathbb{R}^d$ with $|x-y| \leqslant 5t^{1/\alpha(x)}$,

$$\begin{aligned}
p(t,x,y) &= \int_{\mathbb{R}^d} p(t_0,x,z)p(t-t_0,z,y)\,dz \succeq (t-t_0)^{-d/\alpha(y)} \int_{\{|z-y| \leqslant 5(t-t_0)^{1/\alpha(y)}\}} p(t_0,x,z)\,dz \\
&\succeq (t-t_0)^{-d/\alpha(y)} t_0^{-d/\alpha(x)} (t-t_0)^{d/\alpha(y)} \succeq t^{-d/\alpha(x)}.
\end{aligned}$$

Iterating the arguments above $[1/t_0] + 1$ times, we can obtain (5.31). □

To consider off-diagonal lower bounds for the heat kernel $p(t,x,y)$, we will make use of a strong Markov process, in particular the corresponding Lévy system, associated with the operator $\mathcal{L}$. Note that, from the Chapman-Kolmogorov equation (1.12) and the properties (1.15) and (1.17), it is standard to prove the following result. Since the proof is almost the same as [20, Theorem 4.5], we omit it here.

Proposition 5.7. (1) *There is a strong Markov process* $X := \big((X_t)_{t \geqslant 0}; (\mathbb{P}_x)_{x \in \mathbb{R}^d}\big)$ *such that for every* $f \in C_b^2(\mathbb{R}^d)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\,ds, \quad t \in (0,1] \tag{5.36}$$

is a martingale with respect to the natural filtration $\mathscr{F}_t := \sigma\{X_s, 0 \leqslant s \leqslant t\}$ *under probability measure* $\mathbb{P}_x$ *for all* $x \in \mathbb{R}^d$. *Moreover,* X *has the strong Feller property.*

(2) *For every non-negative measurable function* $g : (0,1] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ *vanishing on* $\{(s,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ *and any stopping time* T, *we have*

$$\mathbb{E}_x\left(\sum_{s \leqslant T \wedge 1} g(s, X_{s-}, X_s)\right) = \mathbb{E}_x\bigg(\int_0^{T\wedge 1} \int_{\mathbb{R}^d} g(s, X_s, y) J(X_s, y)\,dy\,ds\bigg), \tag{5.37}$$

where

$$J(x,y) := \frac{\kappa(x, y-x)}{|y-x|^{d+\alpha(x)}}, \quad x, y \in \mathbb{R}^d.$$

For any subset $D \subseteq \mathbb{R}^d$, define

$$\sigma_D := \inf\{t \geqslant 0 : X_t \in D\}, \qquad \tau_D := \inf\{t \geqslant 0 : X_t \notin D\}.$$

Lemma 5.8. *There exist constants* $R_1, A_0 \in (0,1)$ *such that for every* $r \in (0, R_1)$,

$$\mathbb{P}_x\big(\tau_{B(x, A_0 r)} \leqslant r^{\alpha(x)}\big) \leqslant 1/2, \quad x \in \mathbb{R}^d. \tag{5.38}$$

Proof. Choose a function $f \in C_b^2(\mathbb{R}^d)$ such that $f(0) = 0$ and $f(x) = 1$ for every $|x| \geqslant 1$. For each $r > 0$ and $x \in \mathbb{R}^d$, define $f_{r,x}(y) := f\big(\frac{y-x}{r}\big)$. Then, by (5.36), for every $0 < r < 1$, $A > 0$ and $x \in \mathbb{R}^d$,

$$\mathbb{P}_x\big(\tau_{B(x,Ar)} \leqslant r^{\alpha(x)}\big) \leqslant \mathbb{E}_x f_{Ar,x}\big(X_{\tau_{B(x,Ar)}\wedge r^{\alpha(x)}}\big) = \mathbb{E}_x\Big(\int_0^{\tau_{B(x,Ar)}\wedge r^{\alpha(x)}} \mathcal{L} f_{Ar,x}(X_s)\, ds\Big). \tag{5.39}$$

Observe that for every $y \in B(x, Ar)$ and $\lambda > 1$,

$$\begin{aligned}
|\mathcal{L} f_{Ar,x}(y)| &\preceq \int_{\{|z|\leqslant \lambda r\}} |\delta_{f_{Ar,x}}(y;z)| \cdot \frac{dz}{|z|^{d+\alpha(y)}} + \int_{\{|z|>\lambda r\}} |\delta_{f_{Ar,x}}(y;z)| \cdot \frac{dz}{|z|^{d+\alpha(y)}}\\
&\preceq \frac{\|\nabla^2 f\|_\infty}{(Ar)^2} \int_{\{|z|\leqslant \lambda r\}} |z|^{2-d-\alpha(y)}\, dz + \|f\|_\infty \int_{\{|z|>\lambda r\}} |z|^{-d-\alpha(y)}\, dz\\
&\preceq \Big(\frac{\|\nabla^2 f\|_\infty \lambda^{2-\alpha(y)}}{A^2} + \frac{\|f\|_\infty}{\lambda^{\alpha(y)}}\Big) \cdot r^{-\alpha(y)}.
\end{aligned}$$

Hence, first taking λ large enough and then A large enough, we can find a constant $A_0 > 0$ such that

$$|\mathcal{L} f_{A_0 r,x}(y)| \leqslant \frac{1}{4} r^{-\alpha(y)} \quad \text{for all } y \in B(x, A_0 r).$$

Since for every $y \in B(x, A_0 r)$ and $r \in (0,1)$,

$$r^{-\alpha(y)} = r^{-\alpha(x)} r^{-(\alpha(y)-\alpha(x))} \leqslant r^{-|\alpha(x)-\alpha(y)|} r^{-\alpha(x)} \leqslant r^{-C_1 (A_0 r)^{\beta_0}} r^{-\alpha(x)} \leqslant \exp(C_2 |\log r| r^{\beta_0}) r^{-\alpha(x)},$$

there exists a constant $r_0 > 0$ small enough such that $r^{-\alpha(y)} \leqslant 2 r^{-\alpha(x)}$ for all $r \in (0, r_0)$ and $y \in B(x, A_0 r)$. Hence, we have for every $r \in (0, r_0)$ and $y \in B(x, A_0 r)$,

$$|\mathcal{L} f_{A_0 r,x}(y)| \leqslant \frac{1}{2} r^{-\alpha(x)}.$$

Therefore, putting this estimate into (5.39), we obtain (5.38). □

We now show the following off-diagonal lower bound estimates for $p(t, x, y)$.

Proposition 5.9. (Off-diagonal lower bound estimates) *There exists a constant $c_1 := c_1(\alpha, \kappa) > 0$ such that for every $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$ with $|x - y| > 5 \max\{t^{1/\alpha(x)}, t^{1/\alpha(y)}\}$*

$$p(t, x, y) \geqslant \frac{c_1 t}{|x - y|^{d+\alpha(x)}}. \tag{5.40}$$

Proof. For any $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$, it holds that

$$\begin{aligned}
&\mathbb{P}_x\big(X_{\lambda t} \in B(y, t^{1/\alpha(y)})\big)\\
&\geqslant \mathbb{P}_x\Big(\sigma_{B(y,t^{1/\alpha(y)}/2)} \leqslant \lambda t; \sup_{s\in(\sigma_{B(y,t^{1/\alpha(y)}/2)},\lambda t)} \big|X_s - X_{\sigma_{B(y,t^{1/\alpha(y)}/2)}}\big| \leqslant t^{1/\alpha(y)}/2\Big)\\
&\geqslant \mathbb{P}_x\Big(\sigma_{B(y,t^{1/\alpha(y)}/2)} \leqslant \lambda t; \mathbb{P}_{X_{\sigma_{B(y,t^{1/\alpha(y)}/2)}}}\big(\sup_{0\leqslant s\leqslant \lambda t} |X_s - X_0| \leqslant t^{1/\alpha(y)}/2\big)\Big)\\
&\geqslant \mathbb{P}_x\big(\sigma_{B(y,t^{1/\alpha(y)}/2)} \leqslant \lambda t\big) \cdot \inf_{z\in B(y,t^{1/\alpha(y)}/2)} \mathbb{P}_z\big(\tau_{B(z,t^{1/\alpha(y)}/2)} > \lambda t\big),
\end{aligned} \tag{5.41}$$

where in the second inequality we have used the strong Markov property.

Following the proof of (3.11), we know that $t^{1/\alpha(y)} \geqslant C_1 t^{1/\alpha(z)}$ for every $z \in B(y, t^{1/\alpha(y)}/2)$. Hence, there exists a constant $0 < \lambda_0 < 1$ such that for all $0 < \lambda < \lambda_0$ and $z \in B(y, t^{1/\alpha(y)}/2)$,

$$\mathbb{P}_z\big(\tau_{B(z,t^{1/\alpha(y)}/2)} > \lambda t\big) \geqslant \mathbb{P}_z\big(\tau_{B(z,C_1 t^{1/\alpha(z)}/2)} > \lambda t\big) \geqslant 1/2, \tag{5.42}$$

where the last inequality follows from (5.38). On the other hand, by the Lévy system (5.37), for $\lambda > 0$ small enough and for any $x, y \in \mathbb{R}^d$ with $|x - y| > 5 \max\{t^{1/\alpha(x)}, t^{1/\alpha(y)}\}$,

$$\begin{aligned}
\mathbb{P}_x\big(\sigma_{B(y,t^{1/\alpha(y)}/2)} \leqslant \lambda t\big) &\geqslant \mathbb{P}_x\big(X_{\lambda t\wedge \tau_{B(x,t^{1/\alpha(x)})}} \in B(y, t^{1/\alpha(y)}/2)\big)\\
&= \mathbb{E}_x\Big(\int_0^{\lambda t\wedge\tau_{B(x,t^{1/\alpha(x)})}} \int_{B(y,t^{1/\alpha(y)}/2)} \frac{du}{|X_s - u|^{d+\alpha(X_s)}}\, ds\Big)\\
&\succeq \lambda t^{1+(d/\alpha(y))} \cdot \inf_{z\in B(x,t^{1/\alpha(x)})} \frac{1}{|x - y|^{d+\alpha(z)}} \cdot \mathbb{P}_x\big(\tau_{B(x,t^{1/\alpha(x)})} \geqslant \lambda t\big)\\
&\succeq \frac{t^{1+(d/\alpha(y))}}{|x - y|^{d+\alpha(x)}},
\end{aligned} \tag{5.43}$$

where in the first inequality we used the fact that $|x-y| \geqslant 5\max\{t^{1/\alpha(x)}, t^{1/\alpha(y)}\}$ implies $B(x, t^{1/\alpha(x)}) \cap B(y, t^{1/\alpha(y)}/2) = \emptyset$, the second inequality follows from the fact that $|z-u| \preceq |x-y|$ for any $z \in B\big(x, t^{1/\alpha(x)}\big)$ and $u \in B\big(y, t^{1/\alpha(y)}/2\big)$, and in the last inequality we used (5.38) and the fact that for any $x, y, z \in \mathbb{R}^d$ with $|x-y| \geqslant 5t^{1/\alpha(x)}$ and $z \in B(x, t^{1/\alpha(x)})$ and for every $0 < t \leqslant 1$,

$$|x-y|^{-d-\alpha(z)} \succeq |x-y|^{-d-\alpha(x)+C_2|x-z|^{\beta_0}} \succeq t^{C_2 t^{\beta_0/\alpha_1}}|x-y|^{-d-\alpha(x)} \succeq |x-y|^{-d-\alpha(x)}.$$

According to (5.41), (5.42) and (5.43), we obtain that for any $|x-y| > 5t^{1/\alpha(x)}$ and λ small enough,

$$\mathbb{P}_x\big(X_{\lambda t} \in B(y, t^{1/\alpha(y)})\big) \succeq \frac{t^{1+(d/\alpha(y))}}{|x-y|^{d+\alpha(x)}}. \tag{5.44}$$

Hence, we arrive at

$$\begin{aligned}
p(t,x,y) &= \int_{\mathbb{R}^d} p(\lambda t, x, z) p((1-\lambda)t, z, y)\, dz \\
&\geqslant \int_{B(y, t^{1/\alpha(y)})} p(\lambda t, x, z) p((1-\lambda)t, z, y)\, dz \\
&\geqslant \inf_{|z-y| \leqslant t^{1/\alpha(y)}} p((1-\lambda)t, z, y) \int_{B(y, t^{1/\alpha(y)})} p(\lambda t, x, z)\, dz \\
&\succeq \inf_{|z-y| \leqslant C_3 t^{1/\alpha(z)}} p((1-\lambda)t, z, y) \int_{B(y, t^{1/\alpha(y)})} p(\lambda t, x, z)\, dz \\
&\succeq t^{-d/\alpha(y)} \mathbb{P}_x\big(X_{\lambda t} \in B(y, t^{1/\alpha(y)})\big) \succeq \frac{t}{|x-y|^{d+\alpha(x)}},
\end{aligned}$$

where the third inequality follows from the fact that $t^{1/\alpha(y)} \asymp t^{1/\alpha(z)}$ for every $|y-z| \leqslant t^{1/\alpha(y)}$, in the forth inequality we used (5.31), and in the last inequality we used (5.44). Therefore, (5.40) has been proved. □

According to Propositions 5.6 and 5.9, we immediately get the following

Corollary 5.10. *There exists a constant* $c_0 := c_0(\alpha, \kappa) > 0$ *such that for every* $t \in (0, 1]$ *and* $x, y \in \mathbb{R}^d$,

$$p(t,x,y) \geqslant \frac{c_0 t}{\big(t^{1/\alpha(x)} + |y-x|\big)^{d+\alpha(x)}}.$$

Acknowledgements. The authors thank the referee for helpful comments. The research of Xin Chen is supported by National Natural Science Foundation of China (No. 11501361 and No. 11871338) and "Yang Fan Project" of Science and Technology Commission of Shanghai Municipality (No. 15YF1405900). The research of Zhen-Qing Chen is partially supported by Simons Foundation grant 520542 and a Victor Klee Faculty Fellowship at UW. The research of Jian Wang is supported by the National Natural Science Foundation of China (Nos. 11522106 and 11831014), the Fok Ying Tung Education Foundation (No. 151002), the Alexander von Humboldt foundation, the Program for Probability and Statistics: Theory and Application (No. IRTL1704) and the Program for Innovative Research Team in Science and Technology in Fujian Province University (IRTSTFJ).

References

[1] Bass, R.F.: Uniqueness in law for pure jump type Markov processes, *Probab. Theory Relat. Fields* **79** (1988), 271–287.

[2] Bass, R.F.: Regularity results for stable-like operators, *J. Funct. Anal.* **257** (2009), 2693–2722.

[3] Bass, R.F. and Kassmann, M.: Hölder continuity of harmonic functions with respect to operators of variable order, *Comm. Partial Differential Equations* **30** (2005), 1249–1259.

[4] Bass, R.F. and Kassmann, M.: Harnack inequalities for non-local operators of variable order, *Trans. Amer. Math. Soc.* **357** (2005), 837–850.

[5] Bass, R.F. and Tang, H.L.: The martingale problem for a class of stable-like processes, *Stochastic Process. Appl.* **119** (2009), 1144–1167.

[6] Böttcher, B., Schilling, R.L. and Wang, J.: *Lévy-Type Processes: Construction, Approximation and Sample Path Properties*, Lecture Notes in Mathematics, vol. **2099**, Lévy Matters III, Springer, Berlin 2014.

[7] Bogdan, K., Grzywny, T. and Ryznar, T.: Density and tails of unimodal convolution semigroups, *J. Funct. Anal.* **266** (2014), 3543–3571,

[8] Bogdan, K. and Jakubowski, T.: Estimates of heat kernel of fractional Laplacian perturbed by gradient operators, *Comm. Math. Phys.* **271** (2007), 179–198.

[9] Bogdan, K., Sztonyk, P. and Knopova, V.: Heat kernel of anisotropic nonlocal operators, arXiv:1704.03705

[10] Caffarelli, L. and Silvestre, L.: Regularity theory for fully nonlinear integro-differential equations, *Commun. Pure Appl. Math.* **62** (2009), 597–638.

[11] Caffarelli, L. and Silvestre, L.: The Evans-Krylov theorem for non-local fully nonlinear equations, *Ann. of Math.* **174** (2011), 1163–1187.

[12] Caffarelli, L. and Vasseur, A.: Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, *Ann. of Math.* **171** (2010), 1903–1930.

[13] Chen, Z.-Q. and Dou, X.: Drift perturbation of subordinate Brownian motions with Gaussian component, *Sci. China Math.* **59** (2016), 239–260.

[14] Chen, Z.-Q. and Hu, E.: Heat kernel estimates for $\Delta+\Delta^{\alpha/2}$ under gradient perturbation, *Stochastic Processes Appl.* **125** (2015), 2603–2642.

[15] Chen, Z.-Q., Hu, E., Xie, L. and Zhang, X.: Heat kernels for non-symmetric diffusions operators with jumps, *J. Differential Equations* **263** (2017), 6576–6634.

[16] Chen, Z.-Q. and Kumagai, T.: Heat kernel estimates for stable-like processes on d-sets, *Stoch. Proc. Appl.* **108** (2003), 27–62.

[17] Chen, Z.-Q. and Kumagai, T.: Heat kernel estimates for jump processes of mixed types on metric measure spaces, *Probab. Theory Related Fields* **140** (2008), 277–317.

[18] Chen, Z.-Q., Song, R. and Zhang, X.: Stochastic flows for Lévy processes with Hölder drifts, *Rev. Mat. Iberoamericana* **34** (2018), 1755–1788.

[19] Chen, Z.-Q. and Wang, J.-M.: Perturbation by non-local operators, *Ann. Inst. Henri Poincaré Probab. Statist.* **54** (2018), 606-639.

[20] Chen, Z.-Q. and Zhang, X.: Heat kernels and analyticty of non-symmetric jump diffusion semigroups, *Probab. Theory Related Fields* **165** (2016), 267–312.

[21] Chen, Z.-Q. and Zhang, X.: Heat kernels for non-symmetric non-local operators, in: *Recent Developments in Nonlocal Theory*, 24–51, De Gruyter, Berlin, 2018.

[22] Chen, Z.-Q. and Zhang, X.: Heat kernels for time-dependent non-symmetric stable-like operators, *J. Math. Anal. Appl.* **465** (2018), 1–21.

[23] Chen, Z.-Q. and Zhang, X.: Uniqueness of stable-like processes, arXiv:1604.02681

[24] Friedman, A.: *Partial Differential Equations of Parabolic Type*, Prentice-Hall, New York, 1964.

[25] Hoh, W.: *Pseudo differential operators generating Markov processes*, Habilitationsschrift, Bielefeld, 1998.

[26] Grzywny, T. and Szczypkowski, K.: Heat kernels of non-symmetric Lévy-type operators, *J. Differential Equations* **267** (2019), 6004–6064.

[27] Jakubowski, T. and Szczypkowski, K.: Time-dependent gradient perturbations of fractional Laplacian, *J. Evol. Equ.* **10** (2010), 319–339.

[28] Jin, P.: Uniqueness in law for stable-like processes of variable order, arXiv:1802.01151

[29] Kassmann, M.: A priori estimates for integro-differential operators with measurable kernels, *Calc. Var. Partial Differential Equations*, **34** (2009), 1–21.

[30] Kikuchi, K. and Negoro, A.: On Markov processes generated by pseudodifferential operator of variable order, *Osaka J. Math.* **34** (1997), 319–335.

[31] Kim, P. and Lee, J.: Heat kernels of non-symmetric jump processes with exponentially decaying jumping kernel, *Stochastic Process. Appl.*, **129** (2019), 2130–2173.

[32] Kim, P., Song, R. and Vondraček, Z.: Heat kernels of non-symmetric jump processes: beyond the stable case, *Potential Analysis* **49** (2018), 37–90.

[33] Knopova, V. and Kulik, A.: Parametrix construction for certain Lévy-type processes, *Random Oper. Stoch. Equ.* **23** (2015), 111–136.

[34] Knopova, V. and Kulik, A.: Intrinsic compound kernel estimates for the transition probability density of Lévy-type processes and their applications, *Probab. Math. Statist.* **37** (2017), 53–100.

[35] Kolokoltsov, V.N.: Symmetric stable laws and stable-like jump-diffusions, *Proc. London Math. Soc.* **80** (2000), 725–768.

[36] Kühn, F.: Transition probabilities of Lévy-type processes: parametrix construction, *Math. Nachr.* **292** (2019), 358–376.

[37] Kulczycki, T. and Ryznar, M.: Gradient estimates of harmonic functions and transition densities for Lévy processes, *Trans. Amer. Math. Soc.* **368** (2016), 281–318.

[38] Kulczycki, T. and Ryznar, M.: Transition density estimates for diagonal systems of SDEs driven by cylindrical α-stable processes, *ALEA Lat. Am. J. Probab. Math. Stat.* **15** (2018), 1335–1375.

[39] Kulik, A.: Approximation in law of locally α-stable Lévy-type processes by non-linear regressions, *Electron. J. Probab.* **24** (2019), paper no. 83, 45 pp.

[40] Kulik, A.: Parametrix method for one-dimensional locally α-stable Lévy-type processes, arXiv:1808.06779

[41] Luo, D. and Wang, J.: Uniform Hölder estimates on semigroups generated by non-local operators of variable order, *Trans. Amer. Math. Soc.* **371** (2019), 431–459.

[42] Maekawa, Y. and Miura, H.: Upper bounds for fundamental solutions to non-local diffusion equations with divergence free drift, *J. Funct. Anal.* **264** (2013), 2245–2268.

[43] Maekawa, Y. and Miura, H.: On fundamental solutions for non-local parabolic equations with divergence free drift, *Adv. Math.* **247** (2013), 123–191.

[44] Negoro, A.: Stable-like processes: construction of the transition density and the behaviour of sample paths near $t=0$, *Osaka J. Math.* **31** (1994), 189–214.

[45] Sandrić, N.: Long-time behavior of stable-like processes, *Stochastic Process. Appl.* **123** (2013), 1276–1300.

[46] Schilling, R.L.: Feller processes generated by pseudo-differential operators: on the Hausdorff dimension of their sample paths, *J. Theoret. Probab.* **11** (1998), 303–330.

[47] Silvestre, L.: Hölder estimates for solutions of integro-differential equations like the fractional Laplace, *Indiana Univ. Math. J.* **55** (2006), 1155–1174.

[48] Silvestre, L.: On the differentiability of the solution to an equation with drift and fractional diffusion, *Indiana Univ. Math. J.* **61** (2012), 557–584.

[49] Szczypkowski, K.: Fundamental solution for super-critical non-symmetric Lévy-type operators, arXiv:1807.04257

[50] Wang, J.-M.: Laplacian perturbed by non-local operators, *Math. Z.* **279** (2015), 521–556.

[51] Xie, L. and Zhang, X.: Heat kernel estimates for critical fractional diffusion operators, *Studia Mathematica* **224** (2014), 221–263.

[52] Zhang, X.: L^p-maximal regularity of non-local parabolic equations and applications, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30** (2013), 573–614.