

# HEAT KERNELS FOR TIME-DEPENDENT NON-SYMMETRIC MIXED LÉVY-TYPE OPERATORS

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**ABSTRACT.** In this paper we establish the existence and uniqueness of heat kernels to a large class of time-inhomogenous non-symmetric nonlocal operators with Dini's continuous kernels. Moreover, quantitative estimates including two-sided estimates, gradient estimate and fractional derivative estimate of the heat kernels are obtained.

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## 1. INTRODUCTION

The purpose of this paper is to study fundamental solutions (also called heat kernels) and their estimates for a large class of time-inhomogenous non-symmetric non-local operators on Euclidean spaces with Dini-continuous coefficients.

Throughout this paper,  $\phi(r)$  is a strictly increasing continuous function on  $\mathbb{R}_+$  with the property that  $\phi(0) = 0$ ,  $\phi(1) = 1$ , and

$$c_0^\phi := \int_0^\infty \frac{r^2 \wedge 1}{r\phi(r)} dr < \infty. \quad (1.1)$$

Here and below, we use  $:=$  as a way of definition. Condition (1.1) is necessary and sufficient for  $\nu(dz) := \frac{1}{|z|^d \phi(|z|)} dz$  to be the Lévy measure of a Lévy process on  $\mathbb{R}^d$ . We can divide  $\phi$  into three separate cases:

$$\begin{cases} \text{Case}_1^\phi : \int_{0+} \frac{1}{\phi(r)} dr < \infty, \\ \text{Case}_2^\phi : \int_{0+} \frac{1}{\phi(r)} dr = \infty \text{ and } \int_1^\infty \frac{1}{\phi(r)} dr = \infty, \\ \text{Case}_3^\phi : \int_{0+} \frac{1}{\phi(r)} dr = \infty \text{ and } \int_1^\infty \frac{1}{\phi(r)} dr < \infty, \end{cases} \quad (1.2)$$

and define  $z^{(\phi)} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  by

$$z^{(\phi)} := z \mathbf{1}_{\{|z| \leq 1\}} \cdot \mathbf{1}_{\text{Case}_2^\phi} + z \cdot \mathbf{1}_{\text{Case}_3^\phi}. \quad (1.3)$$

When  $\phi(r) = r^\alpha$  with  $\alpha \geq 0$ , (1.1) holds if and only if  $0 < \alpha < 2$ ; in this case,  $\text{Case}_1^\phi$ ,  $\text{Case}_2^\phi$  and  $\text{Case}_3^\phi$  correspond to  $\alpha < 1$ ,  $\alpha = 1$  and  $\alpha > 1$ . When  $\phi(r) = r^\alpha \mathbf{1}_{(0,1)} + r^\beta \mathbf{1}_{[1,\infty)}$  with positive  $\alpha$  and  $\beta$ , condition (1.1) holds if and only if  $0 < \alpha < 2$ ; in this case,  $\text{Case}_1^\phi$  holds if and only if  $\alpha < 1$ ,  $\text{Case}_2^\phi$  holds if and only if  $\alpha \geq 1$  and  $0 < \beta \leq 1$ , and  $\text{Case}_3^\phi$  holds if and only if  $\alpha \geq 1$  and  $\beta > 1$ .

Let  $d \geq 1$  and  $\kappa(t, x, z) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a function that is bounded between two positive constants that is Dini-type continuous in  $x$  uniformly in  $(t, z)$ . In this paper,

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we study, under some mild conditions on  $\kappa(t, x, z)$  and  $\phi(r)$ , the existence, uniqueness and two-sided estimates of heat kernels for the following nonlocal operator on  $\mathbb{R}^d$ :

$$\mathcal{L}_t^\kappa f(x) := \int_{\mathbb{R}^d} \left( f(x+z) - f(x) - z^{(\phi)} \cdot \nabla f(x) \right) \frac{\kappa(t, x, z)}{|z|^d \phi(|z|)} dz, \quad (1.4)$$

if  $\kappa(t, x, z)$  is not symmetric in  $z$ , and

$$\mathcal{L}_t^\kappa f(x) := \frac{1}{2} \int_{\mathbb{R}^d} (f(x+z) + f(x-z) - 2f(x)) \frac{\kappa(t, x, z)}{|z|^d \phi(|z|)} dz, \quad (1.5)$$

if  $\kappa(t, x, z)$  is symmetric in  $z$  for every  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ . For notational simplification, unless otherwise specified, we define

$$\Delta_f^{(\phi)}(x, z) := \begin{cases} \frac{f(x+z) - f(x) - z^{(\phi)} \cdot \nabla f(x)}{|z|^d \phi(|z|)} & \text{if } \kappa(t, x, z) \text{ is not symmetric in } z, \\ \frac{f(x+z) + f(x-z) - 2f(x)}{2|z|^d \phi(|z|)} & \text{if } \kappa(t, x, z) \text{ is symmetric in } z. \end{cases} \quad (1.6)$$

Then we can write the non-local operator in (1.4)-(1.5) in a unified way as

$$\mathcal{L}_t^\kappa f(x) = \int_{\mathbb{R}^d} \Delta_f^{(\phi)}(x, z) \kappa(t, x, z) dz. \quad (1.7)$$

**Remark 1.1.** (i) When  $\kappa(t, x, z)$  is symmetric in  $z$ , if  $f$  is differentiable and the integral in (1.4) is absolutely convergent, then so is the integral in (1.5) and these two integrals give the same value. The definition in (1.5) has the advantage that it does not a priori require  $f$  to be differentiable.

(ii) The reason we use  $z^{(\phi)}$  in (1.4) instead of the more common  $z \mathbf{1}_{\{|z| \leq 1\}}$  in the first order correction term in (1.4) is that this is the form for general  $\alpha$ -stable Lévy processes where  $\phi(r) = r^\alpha$  for  $0 < \alpha < 2$  and  $\kappa(t, x, z)$  is independent of  $t$  and  $x$ .

We consider the following conditions on  $\phi$ . There exist constants  $0 < c_1^\phi \leq c_2^\phi$ , and  $0 < \beta_1 \leq \beta_2 < \infty$  such that for all  $0 < r < R < \infty$ ,

$$c_1^\phi \left( \frac{R}{r} \right)^{\beta_1} \mathbf{1}_{\{0 < r < R \leq 1\}} \leq \frac{\phi(R)}{\phi(r)} \leq c_2^\phi \left( \frac{R}{r} \right)^{\beta_2}. \quad (1.8)$$

Note that the lower bound in (1.8) implies that for any  $r \in (0, 1)$ ,

$$\int_r^1 \frac{1}{s\phi(s)} ds = \frac{1}{\phi(r)} \int_r^1 \frac{\phi(r)}{s\phi(s)} ds \leq \frac{1}{c_1^\phi \phi(r)} \int_r^1 \frac{r^{\beta_1}}{s^{1+\beta_1}} ds = \frac{1 - r^{\beta_1}}{c_1^\phi \beta_1 \phi(r)} \leq \frac{1}{c_1^\phi \beta_1 \phi(r)}. \quad (1.9)$$

We point out that we do not assume weak lower scaling condition on  $\phi$  at infinity; that is, the lower bound in (1.8) is only assumed for  $0 < r < R \leq 1$ . Let

$$\gamma_\phi^{(0)}(r) := r^2 \wedge 1, \quad \gamma_\phi^{(1)}(r) := (r \wedge 1) \mathbf{1}_{\text{Case}_1^\phi} + (r^2 \wedge 1) \mathbf{1}_{\text{Case}_2^\phi} + (r^2 \wedge r) \mathbf{1}_{\text{Case}_3^\phi}. \quad (1.10)$$

We will also consider the condition

$$\mathcal{A}_\phi^{(i)} := \sup_{\lambda \in (0, 1]} \int_0^\infty \frac{\phi(\lambda) \gamma_\phi^{(i)}(r)}{r\phi(\lambda r)} dr < \infty, \quad i = 0, 1. \quad (\mathbf{A}_\phi^{(i)})$$

The above condition is quite natural if we want to have some approximate scaling properties about the heat kernel (see Proposition 2.1 and Lemma 2.7 below). Note that  $(\mathbf{A}_\phi^{(1)})$ , which will be used in the case that  $\kappa(t, x, z)$  is not symmetric in  $z$ , implies  $(\mathbf{A}_\phi^{(0)})$ , and  $(\mathbf{A}_\phi^{(0)})$  implies (1.1). It is easy to verify the following:

- (i) for  $\phi(r) = r^\alpha$ ,  $(\mathbf{A}_\phi^{(1)})$  holds for every  $\alpha \in (0, 2)$ ;
- (ii) for  $\phi(r) = r^\alpha \mathbf{1}_{(0, 1)} + r^\beta \mathbf{1}_{[1, \infty)}$ ,  $(\mathbf{A}_\phi^{(0)})$  holds for every  $\alpha \in (0, 2)$  and  $\beta > 0$ , while  $(\mathbf{A}_\phi^{(1)})$  holds for every  $\alpha \in (0, 2)$  and  $\beta > 0$  except when  $\alpha = 1$  and  $\beta > 1$ .

Some additional examples are given in Section 5 of this paper that satisfy (1.8) and  $(\mathbf{A}_\phi^{(i)})$ .

We recall the following definitions about Dini and slowly varying functions.

**Definition 1.2.** Let  $\ell : (0, 1] \rightarrow (0, \infty)$  be a continuous function. We call it a *slowly varying function* at zero if

$$\lim_{t \rightarrow 0} \ell(\lambda t)/\ell(t) = 1 \quad \text{for every } \lambda > 0.$$

We call it a *Dini function* if  $\ell$  is increasing and

$$\int_0^1 \frac{\ell(t)}{t} dt < \infty.$$

We denote by  $\mathcal{S}_0$  (resp.  $\mathcal{D}_0$ ) the set of all slowly varying functions at zero that is bounded away from zero on  $[\varepsilon, 1]$  for any  $\varepsilon \in (0, 1)$  (resp. Dini functions). For  $\alpha \geq 0$ , we denote by  $\mathcal{R}_\alpha$  the set of all functions  $\ell(t) = t^\alpha \ell_0(t)$  for  $t \in (0, 1]$ , where  $\ell_0 \in \mathcal{S}_0$ . Clearly,  $\mathcal{R}_\alpha \subset \mathcal{D}_0$  for  $\alpha > 0$ . In the following, we use the convention that the definition of functions  $\ell$  in  $\mathcal{S}_0$ ,  $\mathcal{R}_\alpha$  and  $\mathcal{D}_0$  are extended from  $(0, 1]$  to  $(0, \infty)$  by setting  $\ell(t) = \ell(1)$  for  $t \geq 1$ .

**Example 1.3.** Clearly,  $1/\ell \in \mathcal{S}_0$  for  $\ell \in \mathcal{S}_0$ , and  $1 \in \mathcal{S}_0$  but not in  $\mathcal{D}_0$ . Let

$$\ell(t) := (\log(1 + 1/(t \wedge 1)))^\alpha, \quad \alpha \in \mathbb{R}.$$

It is easy to see that  $\ell \in \mathcal{S}_0$ , and  $\ell \in \mathcal{S}_0 \cap \mathcal{D}_0$  when  $\alpha < -1$ .

Throughout this paper, we assume the function  $\kappa(t, x, z)$  in (1.4) satisfies that for some  $\kappa_0 \geq 1$ ,

$$\kappa_0^{-1} \leq \kappa(t, x, z) \leq \kappa_0, \quad |\kappa(t, x, z) - \kappa(t, y, z)| \leq \ell^2(|x - y|), \quad (1.11)$$

where  $\ell \in \mathcal{S}_0 \cap \mathcal{D}_0$  or  $\ell \in \mathcal{R}_\alpha$  with  $\alpha \in (0, 1]$ , and in the above Case<sub>2</sub><sup>ϕ</sup>,

$$\int_{|z| \leq r} z \kappa(t, x, z) dz = 0 \quad \text{for every } r > 0. \quad (1.12)$$

Note that if  $\kappa(t, x, z)$  is symmetric in  $z$ , then condition (1.12) is automatically satisfied. In some situations, we will also need the following condition

$$M_\ell^\phi(t) := \int_0^t \frac{1}{r} \left( \frac{\ell(r)}{\ell(t)} + \frac{\phi(r)}{\phi(t)} \right) d\phi(r) < \infty \quad \text{for } t \in (0, 1]. \quad (1.13)$$

When  $\ell(r) = r^\eta$  and  $\phi(r) = r^\alpha$  on  $[0, 1]$  with  $\eta \in (0, 1]$  and  $\alpha \in (0, 2)$ , condition (1.13) holds if and only if  $\alpha > 1/2$  and  $\alpha + \eta > 1$ . In this case

$$M_\ell^\phi(t) = \left( \frac{\alpha}{\alpha + \eta - 1} + \frac{\alpha}{2\alpha - 1} \right) t^{\alpha-1} \quad \text{for } t \in (0, 1]. \quad (1.14)$$

In comparison, Case<sub>2</sub><sup>ϕ</sup> and Case<sub>3</sub><sup>ϕ</sup> correspond to  $\alpha \in [1, 2)$ . Some additional concrete conditions for (1.13) to hold can be found in Remark 1.5 and in Example 5.6 below. Condition (1.13) is needed for the gradient estimate (1.24) of the heat kernel  $p_{t,s}^\kappa(x, y)$  in Theorem 1.4 as well as for the  $\mathcal{L}_t^\kappa$ -differentiability of the heat kernel constructed in Case<sub>2</sub><sup>ϕ</sup> and Case<sub>3</sub><sup>ϕ</sup> when  $\kappa(t, x, z)$  is not symmetric in  $z$ .

The study of heat kernels and their estimates is an active research area in analysis and in probability theory. It has a long history for second order differential operators. We refer the reader to the Introduction of [9] for a brief history on the study of heat kernels for nonlocal operators. When  $\phi(r) = r^\alpha$  with  $\alpha \in (0, 2)$ ,  $\kappa(t, x, z) = \kappa(x, z)$  is time-independent, symmetric in  $z$  and Hölder continuous in  $x$ , the heat kernel of  $\mathcal{L}_t^\kappa$  is constructed and its sharp two-sided estimates, gradient estimate and fractional derivative estimate are obtained in [9]. Recently, this result has been strengthened in [10] by dropping the symmetry condition on  $\kappa(x, z)$  in  $z$  and allowing  $\kappa$  to be time dependent. The ideas and approach of [9]

are quite robust and they have been adopted to study heat kernels for non-local operators with more general Lévy kernels; see [13, 14, 11, 12, 16, 4] and the references therein. In these works,  $\kappa(t, x, z)$  are all assumed to be independent of  $t$  and Hölder continuous in  $x$ , and, symmetry in  $z$  is assumed in [13, 14, 4]. In [5], we studied heat kernel and its regularity and estimates for time-inhomogeneous diffusion with jumps, whose infinitesimal generators have both diffusive and non-local parts.

The main feature and contributions of this paper are

- (i)  $\kappa(t, x, z)$  is only assumed to be Dini continuous in  $x$  and can be time-inhomogeneous;
- (ii)  $\kappa(t, x, z)$  does not need to be symmetric in  $z$ ;
- (iii) the Lévy kernel  $\frac{1}{|z|^d \phi(|z|)}$  is quite general with  $\phi(r)$  satisfying (1.8) and can have light tails;
- (iv) the lower bound and the upper bound in our two-sided heat kernel estimate are comparable;
- (v) We fully utilize the rough scaling property of the non-local operator  $\mathcal{L}_t^\kappa$ , which makes our approach in this paper more direct;

In this paper, we use the following notations and conventions.

- For  $a, b \in \mathbb{R}$ ,  $a \wedge b := \min\{a, b\}$  and  $a \vee b := \max\{a, b\}$ . Notation  $f \asymp g$  means that there are positive constants  $c_1$  and  $c_2$  so that  $c_1 f \leq g \leq c_2 f$  on their common domain of definitions.
- The space of bounded functions on  $\mathbb{R}^d$  with bounded first and second derivatives is denoted by  $C_b^2(\mathbb{R}^d)$ .
- The inverse function of  $\phi(r)$  is denoted by  $\phi^{-1}(r)$ .
- For  $\ell \in \mathcal{D}_0$ , we introduce

$$\ell_\phi(t) := \ell(\phi^{-1}(t)) \quad \text{and} \quad \Gamma_\ell(t) := \int_0^t \frac{\ell(s)}{s} ds. \quad (1.15)$$

- For any  $T \in (0, \infty]$  and  $\varepsilon \in [0, T]$ , write

$$\mathbb{D}_\varepsilon^T := \{(t, x; s, y) : x, y \in \mathbb{R}^d \text{ and } s, t \geq 0 \text{ with } \varepsilon < s - t < T\}.$$

- $\Theta$  and  $\Theta_1$  stand for sets of parameters:

$$\Theta_1 := (d, \kappa_0, c_0^\phi, c_1^\phi, c_2^\phi, \beta_1, \beta_2), \quad \Theta := (\Theta_1, \mathcal{A}_\phi^{(0)}, \mathcal{A}_\phi^{(1)}).$$

- For  $t > 0$  and  $x \in \mathbb{R}^d$ , we define

$$\rho_\phi(t, x) := \frac{1}{t\phi^{-1}(t)^d + |x|^d \phi(|x|)} \quad \text{and} \quad \rho_\phi(x) := \rho_\phi(1, x) = \frac{1}{1 + |x|^d \phi(|x|)}. \quad (1.16)$$

Clearly

$$\rho_\phi(t, x) \asymp \frac{1}{t\phi^{-1}(t)^d} \wedge \frac{1}{|x|^d \phi(|x|)}.$$

Note that for  $t \in (0, 1]$ ,

$$t \int_{\mathbb{R}^d} \rho_\phi(t, x) dx \asymp \int_{\{|x| \leq \phi^{-1}(t)\}} \frac{1}{\phi^{-1}(t)^d} dx + \int_{\{|x| > \phi^{-1}(t)\}} \frac{t}{|x|^d \phi(|x|)} dx \asymp 1 + \int_{\phi^{-1}(t)}^\infty \frac{t}{r\phi(r)} dr. \quad (1.17)$$

By (1.1) and (1.9), for  $t \in (0, 1]$ ,

$$\int_{\phi^{-1}(t)}^\infty \frac{t}{r\phi(r)} dr = \int_1^\infty \frac{t}{r\phi(r)} dr + \int_{\phi^{-1}(t)}^1 \frac{t}{r\phi(r)} dr \lesssim 1.$$

Thus we have by (1.17),

$$\int_{\mathbb{R}^d} \rho_\phi(t, x) dx \asymp 1/t \quad \text{for } t \in (0, 1]. \quad (1.18)$$

Since  $\phi$  is increasing,

$$t\phi^{-1}(t)^d + |x|^d \phi(|x|) \leq 2 \left( \phi^{-1}(t) + |x| \right)^d \left( \phi(\phi^{-1}(t) + |x|) \right),$$

and since  $\phi(2r) \leq c\phi(r)$ ,

$$\left( \phi^{-1}(t) + |x| \right)^d \left( \phi(\phi^{-1}(t) + |x|) \right) \leq c \left( t\phi^{-1}(t)^d + |x|^d \phi(|x|) \right).$$

Thus

$$\rho_\phi(t, x) \asymp \frac{1}{\left( \phi^{-1}(t) + |x| \right)^d \phi(\phi^{-1}(t) + |x|)}. \quad (1.19)$$

It is known from [7] that the transition density function  $p(t, x, y)$  for a pure jump symmetric Lévy process on  $\mathbb{R}^d$  with Lévy measure  $\frac{1}{|z|^d \phi(|z|)} dz$  has the two-sided estimates:

$$p(t, x, y) \asymp t\rho_\phi(t, |x - y|) \quad \text{for all } t > 0 \text{ and } x, y \in \mathbb{R}^d.$$

We will show in this paper that the above estimate also holds for purely discontinuous non-symmetric Lévy processes whose Lévy measure is comparable to that of isotropic Lévy process on  $\mathbb{R}^d$ . In fact, more is true; see Theorem 2.2 for a precise statement.

The following is the main result of this paper.

**Theorem 1.4.** *Suppose that one of the following two assumptions holds:*

**(H1)** *If  $\kappa(t, x, z) = \kappa(t, x, -z)$ , we assume (1.8),  $(\mathbf{A}_\phi^{(0)})$  and (1.11).*

**(H2)** *If  $\kappa(t, x, z) \neq \kappa(t, x, -z)$ , we assume (1.8),  $(\mathbf{A}_\phi^{(1)})$ , (1.11) and (1.12). In addition, we assume (1.13) for  $\text{Case}_2^\phi$  and  $\text{Case}_3^\phi$ .*

*Then there is a unique continuous function  $p_{t,s}^\kappa(x, y) = p^\kappa(t, x; s, y)$  on  $\mathbb{D}_0^\infty$  (called the fundamental solution or heat kernel of  $\mathcal{L}_t^\kappa$ ) so that for all  $f \in C_b^2(\mathbb{R}^d)$ ,*

$$P_{t,s}^\kappa f(x) := \int_{\mathbb{R}^d} p_{t,s}^\kappa(x, y) f(y) dy$$

*has the following properties:  $\mathcal{L}_r^\kappa P_{t,s}^\kappa f(x)$  exists for each  $s > t \geq 0$  and  $x \in \mathbb{R}^d$ ,*

$$P_{t,s}^\kappa f(x) = f(x) + \int_t^s \mathcal{L}_r^\kappa P_{r,s}^\kappa f(x) dr \quad \text{for every } (t, x) \in [0, s] \times \mathbb{R}^d, \quad (1.20)$$

*and*

(i) *for each  $t_0 \in [0, s)$  and  $x \in \mathbb{R}^d$ , it holds that*

$$\lim_{t \downarrow t_0} \left| \mathcal{L}_t^\kappa P_{t,s}^\kappa f(x) - \mathcal{L}_{t_0}^\kappa P_{t_0,s}^\kappa f(x) \right| = 0;$$

(ii) *when  $\kappa(t, x, z)$  is not symmetric in  $z$ ,  $x \mapsto \nabla P_{t,s}^\kappa f(x)$  exists and is continuous on  $\mathbb{R}^d$  in  $\text{Case}_2^\phi$  and  $\text{Case}_3^\phi$  for each  $0 \leq t < s$ ;*

(iii) *for any bounded and uniformly continuous function  $f$  on  $\mathbb{R}^d$ ,*

$$\lim_{|t-s| \rightarrow 0} \|P_{t,s}^\kappa f - f\|_\infty = 0. \quad (1.21)$$

*Moreover, the heat kernel  $p_{t,s}^\kappa(x, y)$  enjoys the following properties:*

**(a)** *(Two-sided estimate) For any  $T > 0$ , there is a  $c_1 = c_1(T, \ell, \Theta) > 1$ , such that on  $\mathbb{D}_0^T$ ,*

$$c_1^{-1}(s-t)\rho_\phi(s-t, x-y) \leq p_{t,s}^\kappa(x, y) \leq c_1(s-t)\rho_\phi(s-t, x-y). \quad (1.22)$$

(b) (Fractional derivative estimate) For any  $T > 0$ , there is a constant  $c_2 = c_2(T, \ell, \Theta) > 0$  such that on  $\mathbb{D}_0^T$ ,

$$\int_{\mathbb{R}^d} |\Delta_{p_{t,s}^\kappa(\cdot, y)}^{(\phi)}(x, z)| dz \leq c_2 \left( \frac{\Gamma_{\ell_\phi}(s-t)}{\ell_\phi(s-t)} \right) \rho_\phi(s-t, x-y). \quad (1.23)$$

(c) (Gradient estimate) Suppose that (1.13) holds. Then  $x \mapsto p_{t,s}^\kappa(x, y)$  is continuously differentiable for each  $0 \leq t < s$ , and for every  $T > 0$  there is a constant  $c_3 = c_3(T, \ell, \Theta) > 0$  so that on  $\mathbb{D}_0^T$ ,

$$|\nabla p_{t,s}^\kappa(\cdot, y)(x)| \leq c_3 \left( \frac{s-t}{\phi^{-1}(s-t)} + M_\ell^\phi \circ \phi^{-1}(s-t) \right) \rho_\phi(s-t, x-y). \quad (1.24)$$

(d) (Conservativeness) For every  $0 < t < s$  and  $x \in \mathbb{R}^d$ ,

$$\int_{\mathbb{R}^d} p_{t,s}^\kappa(x, y) dy = 1.$$

(e) (Chapman-Kolmogorov equation) For all  $0 < t < r < s < \infty$  and  $x, y \in \mathbb{R}^d$ ,

$$\int_{\mathbb{R}^d} p_{t,r}^\kappa(x, z) p_{r,s}^\kappa(z, y) dz = p_{t,s}^\kappa(x, y). \quad (1.25)$$

(f) (Generator) For any  $f \in C_b^2(\mathbb{R}^d)$ , we have

$$P_{t,s}^\kappa f(x) = f(x) + \int_t^s P_{t,r}^\kappa \mathcal{L}_r^\kappa f(x) dr. \quad (1.26)$$

**Remark 1.5.** (i) If  $\ell \in \mathcal{R}_\alpha$  for some  $\alpha > 0$ , then  $\lim_{t \rightarrow 0} \Gamma_{\ell_\phi}(t)/\ell_\phi(t) = 1/\alpha$  by (3.3) below. In this case, (1.23) implies that

$$|\mathcal{L}_t^\kappa p_{t,s}^\kappa(\cdot, y)(x)| \leq \tilde{c}_2 \rho_\phi(s-t, x-y) \quad \text{on } \mathbb{D}_0^T. \quad (1.27)$$

(ii) We emphasize that the jumping kernel  $j(t, x, z) := \frac{\kappa(t, x, z)}{|z|^\alpha \phi(|z|)}$  for  $\mathcal{L}_t^\kappa$  of (1.4) can have light tail in  $z$  at infinity. For instance, in the example when  $\phi(r) = r^\alpha \mathbf{1}_{(0,1)} + r^\beta \mathbf{1}_{[1,\infty)}$ , one can check (see Example 5.3 below) that (1.8) holds with  $\beta_1 = \alpha$  and  $\beta_2 = \alpha \vee \beta$  and  $(\mathbf{A}_\phi^{(0)})$  is satisfied for any  $\alpha \in (0, 2)$  and  $\beta > 0$ ; while  $(\mathbf{A}_\phi^{(1)})$  holds for all  $\alpha \in (0, 2)$  and  $\beta > 0$  except when  $\alpha = 1$  and  $\beta > 1$ . In other words,  $\beta$  can be any positive number and so  $\beta$  can be larger than or equal 2. Two-sided heat kernel estimates for symmetric pure jump processes with light polynomial decay jumping kernels have recently been studied in [1, 8]. In particular, see some symmetric analogous estimates of (1.22) in [8, (1.4)] as well as [1, Theorem 1.2 and Theorem 1.4(i)].

(iii) As mentioned earlier, when  $\ell(s) = s^\beta$  and  $\phi(s) = s^\alpha$  on  $[0, 1]$  for  $\beta \in (0, 1]$  and  $\alpha \in (0, 2)$ , condition (1.13) holds if and only if  $\alpha > 1/2$  and  $\alpha + \beta > 1$ . In this case,  $M_\ell^\phi(t)$  is given by (1.14). So the gradient estimate (1.24) takes the following form

$$|\nabla p_{t,s}^\kappa(\cdot, y)(x)| \leq \tilde{c}_3 (s-t)^{1-(1/\alpha)} \rho_\phi(s-t, x-y) \quad \text{on } \mathbb{D}_0^T, \quad (1.28)$$

which recovers and extends the gradient estimate in [9, Theorem 1.1(5)] and [10, Theorem 1.1(v)]. Gradient estimate for  $\phi(r) = r^\alpha$  with  $\alpha \leq 1/2$  would need more restrictive assumption on the kernel  $\kappa(t, x, z)$ ; see [15]. Condition (1.13) is also satisfied when  $\ell \in \mathcal{D}_0 \cap \mathcal{S}_0$  and  $\phi(r) = r$ . In this case,

$$M_\ell^\phi(t) = \frac{1}{\ell(t)} \int_0^t \frac{\ell(r)}{r} dr + 1 \asymp \frac{\Gamma_\ell(t)}{\ell(t)} \quad (1.29)$$

in view of Proposition 3.1(ii) and so the gradient estimate (1.24) has the form

$$|\nabla p_{t,s}^\kappa(\cdot, y)(x)| \leq \tilde{c}_3 \frac{\Gamma_\ell(s-t)}{\ell(s-t)} \rho_\phi(s-t, x-y) \quad \text{on } \mathbb{D}_0^T. \quad (1.30)$$

- (iv) Recall  $\beta_1$  is the exponent in (1.8) for  $\phi$ . More generally, when  $\ell \in \mathcal{R}_\eta$  for some  $\eta \geq 0$ , we will show in Example 5.6 below that condition (1.13) holds if  $\beta_1 > 1/2$  and  $\beta_1 + \eta > 1$ ; in this case,  $M_t^\phi(t) \asymp \phi(t)/t$  on  $(0, 1]$ , and consequently the gradient estimate (1.24) takes the following form:

$$|\nabla p_{t,s}^\kappa(\cdot, y)(x)| \leq \tilde{c}_3 \frac{s-t}{\phi^{-1}(s-t)} \rho_\phi(s-t, x-y) \quad \text{on } \mathbb{D}_0^T. \quad (1.31)$$

This extends the gradient estimate [13, Theorem 1.2(4)], where  $\frac{1}{|z|^d \phi(|z|)} dz$  is assumed to be the Lévy measure of a subordinate Brownian motion, the lower scaling exponent  $\beta_1$  of  $\phi$  in (1.8) is within  $(2/3, 2)$ , and  $\kappa(t, x, z)$  is time-independent, symmetric in  $z$  and uniformly  $\eta$ -Hölder continuous in  $x$  with  $\beta_1 + \eta > 1$ ,

- (v) Our approach exploits the rough scaling property of the operator  $\mathcal{L}_t^\kappa$ ; see (2.3) and Proposition 2.1. This allows us to reduce the study of heat kernel  $p_{t,s}^\kappa(x, y)$  for general time  $0 \leq t < s \leq T$  to  $p_{0,1}^\kappa(x, y)$ . In particular, this rough scaling property combined with an idea from T. Watanabe [17] allows us to derive two-sided estimates as well as derivative estimates for heat kernel estimate of time-dependent Lévy process in Theorem 2.2 via (2.8), which is of independent interest and plays a key role in our investigation of heat kernels of the space and time dependent non-local operator  $\mathcal{L}_t^\kappa$ .
- (vi) Under conditions **(H1)** and (1.13), one can see from the proof of Theorem 1.4 (in particular, Lemma 2.7) below that in fact (1.23) also holds for  $\Delta_{p_{t,s}^\kappa}^{(\phi)}$  being defined using the first expression in (1.6). Consequently,  $x \mapsto p_{t,s}^\kappa(x, y)$  is pointwisely  $\mathcal{L}_t^\kappa$ -differentiable for every fixed  $s > t \geq 0$  and  $y \in \mathbb{R}^d$  with  $\mathcal{L}_t^\kappa$  being defined by (1.4), and Theorem 1.4(i) holds in this sense as well.

The rest of the paper is organized as follows. In Section 2, we study heat kernel estimates for  $\mathcal{L}^\kappa$  when  $\kappa$  does not depend on the state variable  $x$ , or equivalently, transition density functions of time-inhomogeneous Lévy processes. In particular, the derivative estimates as well as the continuous dependence of the heat kernel in  $\kappa$  are derived. In Section 3, the properties of slowly varying functions and the basic convolution inequality are presented. In Section 4, we prove our main result Theorem 1.4 using the classical Levi method in time-inhomogeneous and non-local operator setting. In Section 5, several examples are provided to illustrate the main result of this paper and its scope.

## 2. HEAT KERNEL ESTIMATES OF $\mathcal{L}_t^\kappa$ WITH $\kappa(t, x, z) = \kappa(t, z)$

Throughout this section,  $\phi$  is a strictly increasing continuous function on  $\mathbb{R}_+$  with  $\phi(0) = 0$  and  $\phi(1) = 1$  satisfying conditions (1.1) and (1.8), and  $\kappa(t, x, z) = \kappa(t, z)$  is independent of  $x$  and we assume that for some  $\kappa_0 \geq 1$  and all  $r > 0$ ,

$$\kappa_0^{-1} \leq \kappa(t, z) \leq \kappa_0, \quad \mathbf{1}_{\text{Case}_2^\phi} \int_{|z| \leq r} z \kappa(t, z) dz = 0. \quad (2.1)$$

Note that if  $\kappa(t, z)$  is symmetric in  $z$ , then  $\int_{|z| \leq r} z \kappa(t, z) dz = 0$  is automatically satisfied.

**2.1. Scaling property.** Let  $N(dt, dz)$  be a time-inhomogeneous Poisson random measure with intensity measure  $\frac{\kappa(t, z)}{|z|^d \phi(|z|)} dz dt$ . Define

$$X_{t,s}^\kappa := \int_t^s \int_{\mathbb{R}^d} z \tilde{N}(dr, dz) + \int_t^s \int_{\mathbb{R}^d} (z - z^{(\phi)}) \frac{\kappa(r, z)}{|z|^d \phi(|z|)} dz dr, \quad (2.2)$$

where  $\tilde{N}(dt, dz) := N(dt, dz) - \frac{\kappa(t, z)}{|z|^d \phi(|z|)} dz dt$ . Note that the process  $X_{t,s}^\kappa$  is a time-inhomogeneous Lévy process on  $\mathbb{R}^d$  in the sense that it has independent increments.

Sometimes we write  $X_{t,s}^{\kappa,\phi}$  for  $X_{t,s}^\kappa$  if we want to emphasize its dependence on  $\phi$  as well. By Itô's formula, we have

$$\mathbb{E}f(X_{t,s}^\kappa) = \mathbb{E} \int_t^s \mathcal{L}_r^\kappa f(X_{t,r}^\kappa) dr, \quad f \in C_b^2(\mathbb{R}^d).$$

In particular, if we take  $f(x) = e^{i\xi \cdot x}$ , then one finds that the characteristic function of  $X_{t,s}^\kappa$  is given by

$$\mathbb{E}e^{i\xi \cdot X_{t,s}^\kappa} = \exp \left( \int_t^s \int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(\phi)}) \frac{\kappa(r, z)}{|z|^d \phi(|z|)} dz dr \right).$$

By the definition of  $z^{(\phi)}$  and a change of variable, as well as (2.1), we conclude from the last display that for every  $\lambda > 0$ ,

$$\{(\phi^{-1}(\lambda))^{-1} X_{\lambda t, \lambda s}^{\kappa, \phi}, s > t\} \text{ has the same distribution as } \{X_{t,s}^{\kappa_\lambda, \phi_\lambda}, s > t\}, \quad (2.3)$$

where

$$\kappa_\lambda(r, z) := \kappa(\lambda r, \phi^{-1}(\lambda)z) \quad \text{and} \quad \phi_\lambda(r) := \phi(\phi^{-1}(\lambda)r)/\lambda,$$

and

$$\mathbb{E}e^{i\xi \cdot X_{t,s}^\kappa} = \exp \left( (s-t) \int_0^1 \int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(\phi)}) \frac{\kappa(t + (s-t)r, z)}{|z|^d \phi(|z|)} dz dr \right). \quad (2.4)$$

By (2.1) and (1.8), for  $|\xi| \geq 1$ , letting  $\bar{\xi} := \xi/|\xi|$ , we have

$$\begin{aligned} |\mathbb{E}e^{i\xi \cdot X_{t,s}^\kappa}| &\leq \exp \left( -\frac{s-t}{\kappa_0} \int_{\mathbb{R}^d} \frac{1 - \cos(\xi \cdot z)}{|z|^d \phi(|z|)} dz \right) = \exp \left( -\frac{s-t}{\kappa_0} \int_{\mathbb{R}^d} \frac{1 - \cos(\bar{\xi} \cdot z)}{|z|^d \phi(|z|/|\xi|)} dz \right) \\ &\leq \exp \left( -\frac{c_1^\phi |\xi|^{\beta_1} (s-t)}{\kappa_0} \int_{|z| \leq 1} \frac{1 - \cos(\bar{\xi} \cdot z)}{|z|^d \phi(|z|)} dz \right) \leq \exp(-c|\xi|^{\beta_1}(s-t)), \end{aligned} \quad (2.5)$$

where  $c = c(\Theta_1) > 0$ . Hence,  $X_{t,s}^{\kappa,\phi}$  admits a smooth density function  $p_{t,s}^{\kappa,\phi}(x)$  given by the inverse Fourier transform

$$p_{t,s}^{\kappa,\phi}(x) = (2\pi)^d \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \mathbb{E}e^{i\xi \cdot X_{t,s}^{\kappa,\phi}} d\xi = (2\pi)^d \int_{\mathbb{R}^d} \mathbb{E}e^{i\xi \cdot (X_{t,s}^{\kappa,\phi} - x)} d\xi. \quad (2.6)$$

Moreover, with  $\bar{p}_{t,s}^{\kappa,\phi}(x) := p_{t,s}^{\kappa,\phi}(-x)$ ,

$$\partial_t \bar{p}_{t,s}^{\kappa,\phi}(x) + \mathcal{L}_t^\kappa \bar{p}_{t,s}^{\kappa,\phi}(x) = 0 \text{ for } s > t \text{ with } \lim_{t \uparrow s} \bar{p}_{t,s}^{\kappa,\phi}(x) dx = \delta_0(dx), \quad (2.7)$$

where the limit is taken in the weak sense.

By (2.3), we have the following scaling property, which will play a basic role in the sequel and simplify many calculations.

**Proposition 2.1.** *For  $0 \leq t < s \leq 1$ , define*

$$\bar{\kappa}(r, z) := \kappa(t + (s-t)r, \phi^{-1}(s-t)z), \quad \bar{\phi}(u) := \phi(u\phi^{-1}(s-t))/(s-t).$$

- (i)  $\bar{\kappa}$  satisfies (2.1) with the same  $\kappa_0$ , and  $\bar{\phi}$  satisfies (1.8) with the same parameters as  $\phi$ .
- (ii) For  $x \in \mathbb{R}^d$ , it holds

$$p_{t,s}^{\kappa,\phi}(x) = (\phi^{-1}(s-t))^{-d} p_{0,1}^{\bar{\kappa}, \bar{\phi}}(x/\phi^{-1}(s-t)), \quad (2.8)$$

and for another bounded measurable  $\kappa'$  satisfying (2.1),

$$(\mathcal{L}^{\kappa', \phi} p_{t,s}^{\kappa,\phi})(x) = (s-t)^{-1} (\phi^{-1}(s-t))^{-d} (\mathcal{L}^{\bar{\kappa}', \bar{\phi}} p_{0,1}^{\bar{\kappa}, \bar{\phi}})(x/\phi^{-1}(s-t)),$$

where  $\bar{\kappa}'(z) := \kappa'(\phi^{-1}(s-t)z)$ .



**2.2. Two-sided estimate of  $p_{0,1}^{\kappa,\phi}$ .** In this subsection we show the sharp two-sided estimate of  $p_{0,1}^{\kappa,\phi}$  by a purely probabilistic argument. Recall that the function  $\rho_\phi(x)$  is defined by (1.16).

**Theorem 2.2.** *Under (2.1), there is a constant  $c_0 = c_0(\Theta_1) > 1$  such that for all  $x \in \mathbb{R}^d$ ,*

$$c_0^{-1} \rho_\phi(x) \leq p_{0,1}^{\kappa,\phi}(x) \leq c_0 \rho_\phi(x), \quad (2.9)$$

and for each  $j \in \mathbb{N}$ , there is a constant  $c_j = c_j(\Theta_1) > 0$  so that for all  $x \in \mathbb{R}^d$ ,

$$|\nabla^j p_{0,1}^{\kappa,\phi}(x)| \leq c_j \rho_\phi(x). \quad (2.10)$$

Define

$$\Upsilon_1 := \int_0^1 \int_{|z| \leq 1} z \widetilde{N}(dr, dz) + \int_0^1 \int_{|z| \leq 1} (z - z^{(\phi)}) \frac{\kappa(r, z)}{|z|^d \phi(|z|)} dz dr,$$

$$\Upsilon_2 := \int_0^1 \int_{|z| > 1} z \widetilde{N}(dr, dz) + \int_0^1 \int_{|z| > 1} (z - z^{(\phi)}) \frac{\kappa(r, z)}{|z|^d \phi(|z|)} dz dr.$$

Note that  $\Upsilon_1$  and  $\Upsilon_2$  are independent and have the characteristic functions

$$\mathbb{E} e^{i\xi \cdot \Upsilon_1} = \exp \left( \int_0^1 \int_{|z| \leq 1} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(\phi)}) \frac{\kappa(r, z)}{|z|^d \phi(|z|)} dz dr \right) =: e^{\varphi_1(\xi)}, \quad (2.11)$$

$$\mathbb{E} e^{i\xi \cdot \Upsilon_2} = \exp \left( \int_0^1 \int_{|z| > 1} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(\phi)}) \frac{\kappa(r, z)}{|z|^d \phi(|z|)} dz dr \right) =: e^{\varphi_2(\xi)}. \quad (2.12)$$

By a similar calculation as that for (2.5), we have

$$|\mathbb{E} e^{i\xi \cdot \Upsilon_1}| = |e^{\varphi_1(\xi)}| \leq e^{-c|\xi|^{\beta_1}}.$$

Consequently,  $\Upsilon_1$  has a smooth density  $p_1(x)$  with respect to the Lebesgue measure on  $\mathbb{R}^d$ . Since  $X_{0,1}^\kappa$  is the independent sum of  $\Upsilon_1$  and  $\Upsilon_2$ , we have

$$p_{0,1}^{\kappa,\phi}(x) = \mathbb{E} [p_1(x - \Upsilon_2)]. \quad (2.13)$$

To get the two-sided estimate of  $p_1^k(x)$ , we prepare the following two lemmas.

**Lemma 2.3.** (i) *For any  $R > 0$ , there is a  $\delta = \delta(R, \Theta_1) > 0$  so that*

$$\inf_{x \in B_R} p_1(x) \geq \delta.$$

(ii) *For any integer  $m, j \in \mathbb{N}_0$ , there is a constant  $c = c(m, j, \Theta_1) > 0$  so that*

$$|\nabla^j p_1(x)| \leq c(1 + |x|^2)^{-m}.$$

*Proof.* (i) Let  $\Upsilon_{11}$  and  $\Upsilon_{12}$  be two independent random variables with the characteristic functions

$$\mathbb{E} e^{i\xi \cdot \Upsilon_{11}} = \exp \left( \int_{|z| \leq 1} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(\phi)}) \left( \frac{\kappa(z)}{|z|^d \phi(|z|)} - \frac{\kappa_0 c_1^\phi}{2|z|^{d+\beta_1}} \right) dz \right) =: e^{\varphi_{11}(\xi)}, \quad (2.14)$$

$$\mathbb{E} e^{i\xi \cdot \Upsilon_{12}} = \exp \left( \int_{|z| \leq 1} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(\phi)}) \frac{\kappa_0 c_1^\phi}{2|z|^{d+\beta_1}} dz \right) =: e^{\varphi_{12}(\xi)},$$

where  $\kappa(z) := \int_0^1 \kappa(r, z) dr \geq \kappa_0$  by (2.1). Let  $p_{11}$  and  $p_{12}$  be the continuous density functions of  $\Upsilon_{11}$  and  $\Upsilon_{12}$ , respectively. Clearly,

$$p_1(x) = \int_{\mathbb{R}^d} p_{11}(x - z) p_{12}(z) dz. \quad (2.15)$$

Since  $\Upsilon_{12}$  is a truncated rotationally symmetric  $\beta_1$ -stable random variable, it is well known (see, e.g., [6]) that  $p_{12}$  is strictly positive on  $\mathbb{R}^d$ . On the other hand, we have by (2.14) and (2.5) that

$$\mathbb{E}|\Upsilon_{11}| \leq c(\Theta_1) < \infty.$$

Hence, by (2.15), we have for any  $R_1 > R$  and  $x \in B_R$ ,

$$\begin{aligned} p_{11}(x) &= \int_{\mathbb{R}^d} p_{11}(x-z)p_{12}(z)dz \geq \inf_{z \in B_{R_1}} p_{12}(z) \int_{|z| \leq R_1} p_{11}(x-z)dz \\ &= \inf_{z \in B_{R_1}} p_{12}(z) \left(1 - \mathbb{P}(|\Upsilon_{11} - x| > R_1)\right) \\ &\geq \inf_{z \in B_{R_1}} p_{12}(z) \left(1 - (\mathbb{E}|\Upsilon_{11}| + R)/R_1\right), \end{aligned}$$

which yields (i) by taking  $R_1$  large enough.

(ii) Using the inverse Fourier transform, for every integer  $m \geq 1$ , we have by (2.11)

$$(1 + |x|^2)^m |\nabla^j p_{11}(x)| \leq (2\pi)^d \int_{\mathbb{R}^d} |\xi|^j |(\mathbb{I} - \Delta)^m e^{\varphi_1(\xi)}| d\xi \leq c(\Theta_1) < \infty.$$

The proof is complete.  $\square$

**Lemma 2.4.** *For any  $R > 2$ , there is a constant  $c_1 = c_1(R, \Theta_1) > 0$  so that for all  $x \in \mathbb{R}^d$ ,*

$$\frac{c_1^{-1}}{(1 + |x|)^d \phi(1 + |x|)} \leq \mathbb{P}(\Upsilon_2 \in B_R(x)) \leq \frac{c_1}{(1 + |x|)^d \phi(1 + |x|)}. \quad (2.16)$$

*Proof.* Observe that by (2.12),

$$\mathbb{E}e^{i\xi \cdot \Upsilon_2} = \exp\left(\int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1)\nu(dz)\right) \exp(-i\xi \cdot b),$$

where  $\nu(dz) := \mathbf{1}_{\{|z|>1\}} \left(\int_0^1 \kappa(r, z)dr\right) \frac{1}{|z|^d \phi(|z|)} dz$  and  $b := \int_{\mathbb{R}^d} z^{(\phi)} \nu(dz)$ . Let  $\eta := \{\eta_n, n \in \mathbb{N}\}$  be a family of i.i.d. random variables in  $\mathbb{R}^d$  with distribution  $\nu/\lambda$ , where

$$\lambda := \nu(\mathbb{R}^d) \leq c(\Theta_1) < \infty.$$

Let  $S_0 = 0$  and  $S_n := \eta_1 + \dots + \eta_n$ . Let  $N$  be a Poisson random variable with parameter  $\lambda$ , which is independent of  $\eta$ . It is easy to see that

$$S_N \stackrel{(d)}{=} \Upsilon_2 + b.$$

Now, by definition we have

$$\begin{aligned} \mathbb{P}(\Upsilon_2 \in B_R(x)) &= \mathbb{P}(S_N \in B_R(x+b)) = \sum_{n=1}^{\infty} \mathbb{P}(S_n \in B_R(x+b)) \mathbb{P}(N = n) \\ &= e^{-\lambda} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{nd}} \mathbf{1}_{\sum_{j=1}^n z_j \in B_R(x+b)} \nu(dz_1) \cdots \nu(dz_n). \end{aligned}$$

When  $|x+b| < R+1$ , the upper bound in (2.16) for  $\mathbb{P}(\Upsilon_2 \in B_R(x))$  trivially holds. Thus we assume that  $|x+b| \geq R+1$ . Notice that  $\sum_{j=1}^n z_j \in B_R(x+b)$  implies that there is at least one  $i$  such that  $|z_i| > (|x+b| - R)/n$ . Hence,

$$\mathbb{P}(\Upsilon_2 \in B_R(x)) \leq e^{-\lambda} \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{i=1}^n \int_{\mathbb{R}^{nd}} \mathbf{1}_{\sum_{j=1}^n z_j \in B_R(x+b)} \mathbf{1}_{|z_i| > (|x+b| - R)/n} \nu(dz_1) \cdots \nu(dz_n) \right).$$

Recalling  $\nu(dz_i) = \mathbf{1}_{|z_i|>1}(|z_i|^d \phi(|z_i|))^{-1} \left( \int_0^1 \kappa(r, z_i) dr \right) dz_i$ , we have by (2.1)

$$\begin{aligned} \mathbb{P}(\Upsilon_2 \in B_R(x)) &\leq e^{-\lambda} \sum_{n=1}^{\infty} \frac{\kappa_0 n^d}{(|x+b|-R)^d \phi((|x+b|-R)/n)} \frac{1}{n!} \\ &\quad \times \left( \sum_{i=1}^n \int_{\mathbb{R}^{nd}} \mathbf{1}_{\sum_{j=1}^n z_j \in B_R(x+b)} \nu(dz_1) \cdots dz_i \cdots \nu(dz_n) \right) \\ &\leq e^{-\lambda} \sum_{n=1}^{\infty} \frac{\kappa_0 c_2^\phi n^{d+\beta_2}}{(|x+b|-R)^d \phi(|x+b|-R)} |B_R|^{\frac{\lambda^{n-1}}{n!}} \\ &\leq \frac{c_1(\Theta_1)}{(|x+b|-R)^d \phi(|x+b|-R)} |B_R| \leq \frac{c_2(R, \Theta_1)}{(1+|x|)^d \phi(1+|x|)}, \end{aligned}$$

where in the second inequality we used (1.8) and the translation invariance property of the Lebesgue measure in  $z_i$ -variable. On the other hand, for any  $x \in \mathbb{R}^d$ , since  $R > 2$ ,

$$\begin{aligned} \mathbb{P}(\Upsilon_2 \in B_R(x)) &\geq e^{-\lambda} \int_{\mathbb{R}^d} \mathbf{1}_{\{z_1 \in B_R(x+b)\}} \nu(dz_1) \\ &\geq \frac{\kappa_0^{-1} e^{-\lambda} |B_R(x+b) \cap B_1^c|}{(|x|+|b|+R)^d \phi((|x|+|b|+R))} \geq \frac{c_2(R, \Theta_1)}{(1+|x|)^d \phi(1+|x|)}. \end{aligned}$$

Combining the above calculations, we get the desired estimate.  $\square$

Now we can give

*Proof of Theorem 2.2.* Our proof is adapted from [17]. Let  $R > 2$ . For the lower bound, by (i) of Lemma 2.3, we have

$$\delta := \inf_{z \in B_R} p_1(z) > 0.$$

Hence, by (2.13) and Lemma 2.4,

$$p_{0,1}^{\kappa,\phi}(x) = \mathbb{E}[p_1(x - \Upsilon_2)] \geq \delta \mathbb{P}(|x - \Upsilon_2| \leq R) \geq \frac{\delta c_1^{-1}}{(1+|x|)^d \phi(1+|x|)}. \quad (2.17)$$

For the upper bound, by (2.13) again, we have

$$p_{0,1}^{\kappa,\phi}(x) \leq \mathbb{E}[p_1(x - \Upsilon_2) \mathbf{1}_{|x - \Upsilon_2| \leq |x|/2}] + \sup_{|z| > |x|/2} p_1(z). \quad (2.18)$$

By (ii) of Lemma 2.3, we can choose  $N$ -points  $z_1, \dots, z_N \in B_{|x|/2}$  and  $\varepsilon > 0$  such that

$$B_{|x|/2} \subset \cup_{j=1}^N B_\varepsilon(z_j) \quad \text{and} \quad \sum_{j=1}^N \sup_{z \in B_\varepsilon(z_j)} p_1(z) \leq c_4,$$

where  $c_4$  only depends on  $\varepsilon, \kappa_0, d, \alpha$ . Hence, by Lemma 2.4, we have

$$\begin{aligned} \mathbb{E}[p_1(x - \Upsilon_2) \mathbf{1}_{|x - \Upsilon_2| \leq |x|/2}] &\leq \sum_{j=1}^N \mathbb{E}[p_1(x - \Upsilon_2) \mathbf{1}_{x - \Upsilon_2 \in B_\varepsilon(z_j)}] \\ &\leq \sum_{j=1}^N \sup_{z \in B_\varepsilon(z_j)} p_1(z) \mathbb{P}(|x - \Upsilon_2 - z_j| \leq \varepsilon) \\ &\leq \sum_{j=1}^N \sup_{z \in B_\varepsilon(z_j)} p_1(z) \frac{c_1}{(1+|x - z_j|)^d \phi(1+|x - z_j|)} \\ &\leq \frac{c_1 c_4}{(1+|x|/2)^d \phi(1+|x|/2)}. \end{aligned}$$

This together with (1.8), (2.18) and Lemma 2.3(ii) yields that

$$p_{0,1}^{\kappa,\phi}(x) \leq \frac{c_5}{(1+|x|)^d \phi(1+|x|)} \quad \text{for } x \in \mathbb{R}^d.$$

Combining with (2.17), we get the desired estimate (2.9).  $\square$

**Remark 2.5.** The strong Markov process  $X_{t,s}^\kappa$  of (2.2) has infinitesimal generator

$$\mathcal{L}_t^\kappa f(x) = \int_{\mathbb{R}^d} \left( f(x+z) - f(x) - \nabla f(x) \cdot z^{(\phi)} \right) \frac{\kappa(t,z)}{|z|^d \phi(|z|)} dz.$$

Suppose

$$\widetilde{X}_{t,s}^\kappa := \int_t^s \int_{\mathbb{R}^d} z \widetilde{N}(dr, dz) + \int_t^s \int_{\{|z|>1\}} z \frac{\kappa(r,z)}{|z|^d \phi(|z|)} dz dr, \quad (2.19)$$

which has infinitesimal generator

$$\widetilde{\mathcal{L}}_t^\kappa f(x) := \int_{\mathbb{R}^d} \left( f(x+z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{\{|z|\leq 1\}} \right) \frac{\kappa(t,z)}{|z|^d \phi(|z|)} dz.$$

Clearly,

$$\widetilde{X}_{t,s}^\kappa = X_{t,s}^\kappa + \int_t^s b(r) dr \quad \text{and} \quad \widetilde{\mathcal{L}}_t^\kappa = \mathcal{L}_t^\kappa + b(r) \cdot \nabla, \quad (2.20)$$

where

$$b(r) = \begin{cases} - \int_{\{|z|\leq 1\}} z \frac{\kappa(r,z)}{|z|^d \phi(|z|)} dz & \text{in Case}_1^\phi, \\ 0 & \text{in Case}_2^\phi, \\ \int_{\{|z|>1\}} z \frac{\kappa(r,z)}{|z|^d \phi(|z|)} dz & \text{in Case}_3^\phi. \end{cases} \quad (2.21)$$

Denote by  $\widetilde{p}_{t,s}^\kappa(x)$  the density function of  $\widetilde{X}_{t,s}^\kappa$ . Then by (2.20),

$$\widetilde{p}_{t,s}^\kappa(x) = p_{t,s}^\kappa \left( x - \int_t^s b(r) dr \right).$$

Thus under conditions (2.1), one can get two-sided estimates on  $\widetilde{p}_{t,s}^\kappa(x)$  from that of  $p_{t,s}^\kappa(x)$ .

**2.3. Fractional derivative estimates of  $p_{0,1}^{\kappa,\phi}$ .** In this subsection we show the fractional derivative estimates of  $p_{0,1}^{\kappa,\phi}$ , that will be used to construct the heat kernel with variable coefficients by Levi's method.

**Lemma 2.6.** *Let  $\phi$  be as in (1.8) and  $p(x) := p_{0,1}^{\kappa,\phi}(x)$ . Define*

$$\delta_p^{(1)}(x, z) := p(x+z) - p(x), \quad \delta_p^{(2)}(x, z) := p(x+z) + p(x-z) - 2p(x)$$

and

$$\delta_p^{(3)}(x, z) := p(x+z) - p(x) - z \cdot \nabla p(x).$$

Under assumption (2.1), there is a constant  $c = c(\Theta_1) > 0$  such that for all  $x, x', z \in \mathbb{R}^d$  and  $i = 1, 2, 3$ ,

$$|\delta_p^{(i)}(x, z)| \leq c \left( \mathbf{1}_{\{|z|>1\}} (\rho_\phi(x+z) + \mathbf{1}_{\{i=2\}} \rho_\phi(x-z)) + (|z|^{i\wedge 2} \wedge |z|^{(i-2)\vee 0}) \rho_\phi(x) \right), \quad (2.22)$$

and

$$\begin{aligned} & |\delta_p^{(i)}(x, z) - \delta_p^{(i)}(x', z)| \\ & \leq c (|x - x'| \wedge 1) \left( \mathbf{1}_{\{|z|>1\}} (\rho_\phi(x+z) + \rho_\phi(x'+z) + \mathbf{1}_{\{i=2\}} (\rho_\phi(x-z) + \rho_\phi(x'-z))) \right. \\ & \quad \left. + (|z|^{i\wedge 2} \wedge |z|^{(i-2)\vee 0}) (\rho_\phi(x) + \rho_\phi(x')) \right). \end{aligned} \quad (2.23)$$

*Proof.* We only present the proof of (2.23) for  $i = 3$ . The proofs for the other two cases are similar. Notice that

$$\begin{aligned} |\delta_p^{(2)}(x, z) - \delta_p^{(2)}(x', z)| &\leq |z|^2 |x - x'| \int_{[0,1]^3} |\nabla^3 p(x + \theta_1 z + \theta_1 \theta_2 z + \theta_3(x - x'))| d\theta_1 d\theta_2 d\theta_3 \\ &\stackrel{(2.10)}{\lesssim} |z|^2 |x - x'| \int_{[0,1]^3} \rho_\phi(x + \theta_1 z + \theta_1 \theta_2 z + \theta_3(x - x')) d\theta_1 d\theta_2 d\theta_3. \end{aligned}$$

If  $|x| > 4$ ,  $|x - x'| \leq 1$  and  $|z| \leq 1$ , then due to  $|x + \theta_1 z + \theta_1 \theta_2 z + \theta_3(x - x')| \asymp |x|$ , we have

$$\rho_\phi(x + \theta_1 z + \theta_1 \theta_2 z + \theta_3(x - x')) \lesssim \rho_\phi(x).$$

If  $|x| \leq 4$ ,  $|x - x'| \leq 1$  and  $|z| \leq 1$ , then

$$\rho_\phi(x + \theta_1 z + \theta_1 \theta_2 z + \theta_3(x - x')) \leq 1 \lesssim \rho_\phi(x).$$

Hence, for  $|x - x'| \leq 1$  and  $|z| \leq 1$ ,

$$|\delta_p^{(2)}(x, z) - \delta_p^{(2)}(x', z)| \lesssim |z|^2 |x - x'| \rho_\phi(x).$$

Similarly, if  $|x - x'| > 1$  and  $|z| \leq 1$ , then

$$|\delta_p^{(2)}(x, z) - \delta_p^{(2)}(x', z)| \leq |\delta_p^{(2)}(x, z)| + |\delta_p^{(2)}(x', z)| \lesssim |z|^2 (\rho_\phi(x) + \rho_\phi(x'));$$

if  $|x - x'| \leq 1$  and  $|z| > 1$ , then

$$|\delta_p^{(2)}(x, z) - \delta_p^{(2)}(x', z)| \lesssim |x - x'| (\rho_\phi(x + z) + (|z| + 1)\rho_\phi(x));$$

if  $|x - x'| > 1$  and  $|z| > 1$ , then

$$|\delta_p^{(2)}(x, z) - \delta_p^{(2)}(x', z)| \lesssim \rho_\phi(x + z) + \rho_\phi(x' + z) + (|z| + 1)(\rho_\phi(x') + \rho_\phi(x)).$$

Combining the above cases, we obtain (2.23) for  $i = 3$ .  $\square$

Using Lemma 2.6, it is easy to derive the following from definition, which is a counterpart of [9, Theorem 2.4].

**Lemma 2.7.** *Let  $\phi$  be as in (1.8) and  $z^{(\phi)}$  be defined by (1.3). Let  $\Delta_p^{(\phi)}$  be defined by any one of (1.6) (regardless whether  $\kappa(t, z)$  is symmetric in  $z$  or not), and  $p(x) := p_{0,1}^{\kappa, \phi}(x)$ . Under condition (2.1), there is a constant  $c = c(\Theta_1) > 0$  such that for all  $x \in \mathbb{R}^d$ ,*

$$\int_{\mathbb{R}^d} |\Delta_p^{(\phi)}(x, z)| dz \leq c \rho_\phi(x), \quad (2.24)$$

and for all  $x, x' \in \mathbb{R}^d$ ,

$$\int_{\mathbb{R}^d} |\Delta_p^{(\phi)}(x, z) - \Delta_p^{(\phi)}(x', z)| dz \leq c (|x - x'| \wedge 1) (\rho_\phi(x) + \rho_\phi(x')). \quad (2.25)$$

*Proof.* We first consider the case that  $\Delta_p^{(\phi)}$  is defined by the first expression of (1.6). By Lemma 2.6, it is easy to see that

$$|\Delta_p^{(\phi)}(x, z)| \leq c \frac{\mathbf{1}_{\{|z|>1\}} \rho_\phi(x + z) + \gamma_\phi^{(1)}(|z|) \rho_\phi(x)}{|z|^d \phi(z)},$$

where  $\gamma_\phi^{(1)}(r)$  is defined by (1.10), and

$$|\Delta_p^{(\phi)}(x; z) - \Delta_p^{(\phi)}(x'; z)| \leq c (|x - x'| \wedge 1) \frac{\mathbf{1}_{\{|z|>1\}} (\rho_\phi(x + z) + \rho_\phi(x' + z)) + \gamma_\phi^{(1)}(|z|) (\rho_\phi(x) + \rho_\phi(x'))}{|z|^d \phi(z)}.$$

Thus, to prove (2.24) and (2.25), it suffices to show

$$I_1 := \int_{\mathbb{R}^d} \frac{\gamma_\phi^{(1)}(|z|)}{|z|^d \phi(|z|)} dz < \infty \quad \text{and} \quad I_2(x) := \int_{|z|>1} \frac{\rho_\phi(x + z)}{|z|^d \phi(|z|)} dz \lesssim \rho_\phi(x).$$

Clearly, by definition

$$I_1 = c \int_0^\infty \frac{\gamma_\phi^{(1)}(r)}{r\phi(r)} dr < \infty.$$

For  $|x| > 2$ , we have

$$\begin{aligned} I_2(x) &\leq \rho_\phi(x/2) \int_{1 < |z| \leq |x|/2} \frac{dz}{|z|^d \phi(|z|)} + \int_{|z| > |x|/2} \frac{\rho_\phi(x+z)}{|z|^d \phi(|z|)} dz \\ &\lesssim \rho_\phi(x) \int_{|z| > 1} \frac{dz}{|z|^d \phi(|z|)} + \frac{1}{|x|^d \phi(|x|)} \int_{\mathbb{R}^d} \rho_\phi(z) dz \lesssim \rho_\phi(x), \end{aligned}$$

which together with  $\sup_{|x| \leq 2} I_2(x) < \infty$  yields  $I_2 \lesssim \rho_\phi$  on  $\mathbb{R}^d$ .

When  $\Delta_p^{(\phi)}$  is defined by the second expression of (1.6) (here we do not need to assume that  $\kappa(t, z)$  is symmetric in  $z$ ), we can establish (2.24) and (2.25) in a similar way as above. This completes the proof of the lemma.  $\square$

**2.4. Continuous dependence of  $p_{0,1}^{\kappa, \phi}(x)$  with respect to  $\kappa$ .** In this subsection we show the continuous dependence of  $p_{0,1}^{\kappa, \phi}(x)$  in the point-wise sense with respect to  $\kappa$ .

**Lemma 2.8.** *Let  $\kappa_1$  and  $\kappa_2$  be two kernels satisfying (2.1) with the same constant  $\kappa_0$ . Let  $\phi$  be as in (1.8) and*

$$p_1(x) := p_{0,1}^{\kappa_1, \phi}(x), \quad p_2(x) := p_{0,1}^{\kappa_2, \phi}(x).$$

*There exists a constant  $c = c(\Theta_1) > 0$  such that for all  $x \in \mathbb{R}^d$ ,*

$$|\nabla^j p_1(x) - \nabla^j p_2(x)| \leq c \|\kappa_1 - \kappa_2\|_\infty \rho_\phi(x) \quad \text{for } j = 0, 1, \quad (2.26)$$

$$\int_{\mathbb{R}^d} |\Delta_{p_1}^{(\phi)}(x, z) - \Delta_{p_2}^{(\phi)}(x, z)| dz \leq c \|\kappa_1 - \kappa_2\|_\infty \rho_\phi(x), \quad (2.27)$$

where  $\Delta_{p_i}^{(\phi)}(x, z)$  is defined by any one of (1.6) regardless whether  $\kappa(t, z)$  is symmetric in  $z$  or not.

*Proof.* Noticing that by (2.4) and (2.6),

$$q_\lambda(x) := p_{0,1}^{\lambda\kappa_1 + (1-\lambda)\kappa_2}(x) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \exp\left(\int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(\phi)}) \frac{\lambda\kappa_1(z) + (1-\lambda)\kappa_2(z)}{|z|^d \phi(|z|)} dz\right) d\xi,$$

where  $\kappa_1(z) := \int_0^1 \kappa_1(r, z) dr$  and  $\kappa_2(z) := \int_0^1 \kappa_2(r, z) dr$ . We claim that

$$\partial_\lambda q_\lambda(x) = (\mathcal{L}^{\kappa_1} - \mathcal{L}^{\kappa_2}) q_\lambda(x). \quad (2.28)$$

Indeed, since

$$\widehat{\mathcal{L}^{\kappa_i} f}(\xi) = \left( \int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(\phi)}) \frac{\kappa_i(z)}{|z|^d \phi(|z|)} dz \right) \widehat{f}(\xi)$$

and

$$\widehat{q}_\lambda(\xi) = \exp\left(\int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1 - i\xi \cdot z^{(\phi)}) \frac{\lambda\kappa_1(z) + (1-\lambda)\kappa_2(z)}{|z|^d \phi(|z|)} dz\right),$$

we have

$$\partial_\lambda \widehat{q}_\lambda(\xi) = \widehat{\mathcal{L}^{\kappa_1} q_\lambda}(\xi) - \widehat{\mathcal{L}^{\kappa_2} q_\lambda}(\xi).$$

By the uniqueness of Fourier transform, we get (2.28).

By the definition of  $q_\lambda(x)$  and (2.28),

$$\begin{aligned} |p_1(x) - p_2(x)| &= \left| \int_0^1 \partial_\lambda q_\lambda(x) d\lambda \right| = \left| \int_0^1 \int_{\mathbb{R}^d} \Delta_{q_\lambda}^{(\phi)}(x, z) (\kappa_1(z) - \kappa_2(z)) dz d\lambda \right| \\ &\leq \|\kappa_1 - \kappa_2\|_\infty \int_0^1 \int_{\mathbb{R}^d} |\Delta_{q_\lambda}^{(\phi)}(x, z)| dz. \end{aligned}$$

Thus we obtain by (2.24) that

$$|p_1(x) - p_2(x)| \leq c \|\kappa_1 - \kappa_2\|_\infty \rho_\phi(x), \quad (2.29)$$

which establishes (2.26) for  $j = 0$ .

We next use the convolution technique to show (2.26) for  $j = 1$  and (2.27). Let

$$p_0(x) := p_{0,1}^{\kappa_0/2,\phi}(x), \quad \bar{p}_i(x) := p_{0,1}^{\kappa_i - \kappa_0/2,\phi}(x) \text{ for } i = 1, 2.$$

Then we have

$$p_i(x) = \int_{\mathbb{R}^d} p_0(x-y) \bar{p}_i(y) dy \text{ for } i = 1, 2.$$

We have by (2.10) and (2.29) applied to  $\bar{p}_1$  and  $\bar{p}_2$  that

$$\begin{aligned} |\nabla p_1(x) - \nabla p_2(x)| &\leq \int_{\mathbb{R}^d} |\nabla p_0(x-y)| \cdot |\bar{p}_1(y) - \bar{p}_2(y)| dy \\ &\lesssim \|\kappa_1 - \kappa_2\|_\infty \int_{\mathbb{R}^d} \rho_\phi(x-y) \cdot \rho_\phi(y) dy \lesssim \|\kappa_1 - \kappa_2\|_\infty \rho_\phi(x). \end{aligned}$$

Similarly, by (2.24) and (2.29),

$$\begin{aligned} \int_{\mathbb{R}^d} |\Delta_{p_1}^{(\phi)}(x, z) - \Delta_{p_2}^{(\phi)}(x, z)| dz &\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\Delta_{p_0}^{(\phi)}(x-y, z)| dz \right) |\bar{p}_1(y) - \bar{p}_2(y)| dy \\ &\lesssim \|\kappa_1 - \kappa_2\|_\infty \int_{\mathbb{R}^d} \rho_\phi(x-y) \cdot \rho_\phi(y) dy \lesssim \|\kappa_1 - \kappa_2\|_\infty \rho_\phi(x). \end{aligned}$$

This completes the proof of the Lemma.  $\square$

### 3. BASIC CONVOLUTION INEQUALITIES

We first list some important properties about slowly varying functions.

**Proposition 3.1.** (i) *Let  $\ell \in \mathcal{S}_0$  with the convention that  $\ell(t) = \ell(1)$  for  $t \geq 1$ . For any  $\delta > 0$ , there is a constant  $C = C(\ell, \delta) \geq 1$  such that for all  $s, t > 0$ ,*

$$\frac{\ell(s)}{\ell(t)} \leq C \max \left\{ \left( \frac{s}{t} \right)^\delta, \left( \frac{s}{t} \right)^{-\delta} \right\}. \quad (3.1)$$

(ii) *If  $\ell \in \mathcal{S}_0 \cap \mathcal{D}_0$ , then  $\Gamma_\ell(\cdot) = \int_0^\cdot \ell(s)/s ds \in \mathcal{S}_0$  and*

$$\lim_{t \rightarrow 0} \Gamma_\ell(t)/\ell(t) = \infty. \quad (3.2)$$

*If  $\ell \in \mathcal{R}_\alpha$  for some  $\alpha > 0$ , then*

$$\lim_{t \rightarrow 0} \Gamma_\ell(t)/\ell(t) = 1/\alpha. \quad (3.3)$$

(iii) *Let  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous function with  $g(0) = 0$ . Suppose that*

$$0 < \liminf_{t \rightarrow 0} g(\lambda t)/g(t) \leq \overline{\lim}_{t \rightarrow 0} g(\lambda t)/g(t) < \infty, \quad \lambda > 0.$$

*For any  $\ell \in \mathcal{S}_0$ , we have  $\ell \circ g \in \mathcal{S}_0$ .*

(iv) *For  $\ell \in \mathcal{S}_0 \cap \mathcal{D}_0$  and an increasing positive function  $\phi$  on  $[0, \infty)$  that satisfies (1.8), we have  $\ell_\phi \in \mathcal{S}_0 \cap \mathcal{D}_0$ .*

*Proof.* (i) Estimate (3.1) follows by Potter's theorem (see [3, (ii) of Theorem 1.5.6]).

(ii) It follows by [3, Proposition 1.5.9b and Proposition 1.5.10].

(iii) For any  $0 < \lambda_0 < \lambda_1 < \infty$ , by UCT theorem for slowly varying functions (see [3, Theorem 1.2.1]), it holds that

$$\lim_{t \rightarrow 0} \sup_{\lambda \in [\lambda_0, \lambda_1]} |\ell(\lambda t)/\ell(t) - 1| = 0.$$

Fix  $\lambda > 0$ . By the assumption, there are  $t_0 > 0$  and interval  $[\lambda_0, \lambda_1]$  such that

$$g(\lambda t)/g(t) \in [\lambda_0, \lambda_1], \quad t \in (0, t_0].$$

Hence,

$$\lim_{t \rightarrow 0} \ell(g(\lambda t))/\ell(g(t)) = 1.$$

(iv) By (1.8) we have

$$\left(\frac{R}{c_2^\phi r}\right)^{1/\beta_2} \leq \frac{\phi^{-1}(R)}{\phi^{-1}(r)} \leq \left(\frac{R}{c_1^\phi r}\right)^{1/\beta_1} \quad \text{for } 0 < r \leq R \leq 1. \quad (3.4)$$

By (iii), we immediately have  $\ell_\phi = \ell \circ \phi^{-1} \in \mathcal{S}_0$ . Moreover, the above estimate implies

$$\phi^{-1}(s) \leq (c_2^\phi s)^{1/\beta_2},$$

and so, by the increase of  $\ell$  and a change of variable,

$$\Gamma_{\ell_\phi}(t) = \int_0^t \ell \circ \phi^{-1}(s)/s ds \leq \int_0^t \ell((c_2^\phi s)^{1/\beta_2})/s ds = \beta_2 \Gamma_\ell((c_2^\phi t)^{1/\beta_2}).$$

The proof is complete.  $\square$

The following lemma plays a crucial role in our construction of the heat kernel of  $\mathcal{L}_t^\kappa$  by Levi's method. Recall the definition of  $\ell_\phi$  from (1.15) and  $\rho_\phi(t, x)$  from (1.16).

**Lemma 3.2.** *Let  $\phi$  be as in (1.8) and  $\rho_\phi(t, x)$  as in (1.16). For a function  $\ell$  on  $\mathbb{R}_+$ , define*

$$h_\phi^\ell(t, x) := \ell(\phi^{-1}(t) + |x|)\rho_\phi(t, x). \quad (3.5)$$

(i) *If  $\ell \in \cup_{\alpha \in [0, 1)} \mathcal{R}_\alpha$ , then there is some  $C = C(\ell, \Theta_1) > 0$  so that*

$$\int_{\mathbb{R}^d} h_\phi^\ell(t, x) dx \leq C \frac{\ell_\phi(t)}{t} \quad \text{for all } t \in (0, 1]. \quad (3.6)$$

(ii) *For any  $\ell_1, \ell_2 \in \cup_{\alpha \in [0, 1)} \mathcal{R}_\alpha$ , there is a constant  $C = C(\ell_1, \ell_2, \Theta_1) \geq 1$  such that for all  $0 < s < t < \infty$  and  $x, y \in \mathbb{R}^d$ ,*

$$\int_{\mathbb{R}^d} h_\phi^{\ell_1}(t-s, x-y) h_\phi^{\ell_2}(s, y) dy \leq C \left( \frac{\ell_\phi(t-s)}{t-s} + \frac{\ell_\phi(s)}{s} \right) h_\phi^{\ell_1 \vee \ell_2}(t, x). \quad (3.7)$$

*Proof.* (i) By the definition, (1.19) and the use of polar coordinate, we have

$$\begin{aligned} \int_{\mathbb{R}^d} h_\phi^\ell(t, x) dx &\lesssim \int_0^\infty \ell(\phi^{-1}(t) + r) (\phi^{-1}(t) + r)^{-d} (\phi(\phi^{-1}(t) + r))^{-1} r^{d-1} dr \\ &\leq \int_0^\infty \frac{\ell(\phi^{-1}(t) + r)}{(\phi^{-1}(t) + r)\phi(\phi^{-1}(t) + r)} dr = \int_{\phi^{-1}(t)}^\infty \frac{\ell(r)}{r\phi(r)} dr \\ &= \ell(1) \int_1^\infty \frac{1}{r\phi(r)} dr + \int_{\phi^{-1}(t)}^1 \frac{\ell(r)}{r\phi(r)} dr. \end{aligned} \quad (3.8)$$

Here we have used the convention that  $\ell(r) = \ell(1)$  for  $r \geq 1$ . By the lower bound of (1.8) and (3.1) with  $\delta = \beta_1/2$ , for  $t \in (0, 1]$ ,

$$\begin{aligned} \int_{\phi^{-1}(t)}^1 \frac{\ell(r)}{r\phi(r)} dr &= \frac{\ell(\phi^{-1}(t))}{t} \int_{\phi^{-1}(t)}^1 \frac{\ell(r)}{\ell(\phi^{-1}(t))} \frac{\phi(\phi^{-1}(t))}{r\phi(r)} dr \\ &\lesssim \frac{\ell(\phi^{-1}(t))}{t} \int_{\phi^{-1}(t)}^1 \frac{r^\delta}{\phi^{-1}(t)^\delta} \frac{\phi^{-1}(t)^{\beta_1}}{r^{1+\beta_1}} dr \lesssim \frac{\ell(\phi^{-1}(t))}{t}. \end{aligned}$$

Taking  $\delta = \beta_1/2$  again in (3.1), we have by (1.8) that  $t \lesssim \ell(\phi^{-1}(t))$  for  $t \in (0, 1]$ . The above display together with (3.8) and (1.1) yields the desired estimate (3.6).



(ii) Without loss of generality, we assume  $\ell_1, \ell_2 \in \mathcal{S}_0$ . By the definition of slowly varying function, it is easy to see that  $\ell_1 \vee \ell_2 \in \mathcal{S}_0$ . By (3.1) with  $\delta = 1$ , there is a constant  $c_0 > 0$  such that for all  $u < w$ ,

$$\frac{u^d(\ell_1 \vee \ell_2)(w)}{w^d(\ell_1 \vee \ell_2)(u)} \leq c_0 \left( \left( \frac{u}{w} \right)^{d+1} \vee \left( \frac{u}{w} \right)^{d-1} \right) \leq c_0.$$

Thus, if we let  $\gamma(u) := u^d \phi(u)$ , then by the increase of  $\phi$ , we have for all  $\lambda \geq 1$  and  $0 < u < \lambda w$ ,

$$\frac{\gamma(u)}{(\ell_1 \vee \ell_2)(u)} \leq c_0 \frac{\gamma(\lambda w)}{(\ell_1 \vee \ell_2)(\lambda w)} \leq c_\lambda \frac{\gamma(w)}{(\ell_1 \vee \ell_2)(w)}. \quad (3.9)$$

On the other hand, as

$$u \vee w \leq u + w \leq 2(u \vee w) \quad \text{for } u, w > 0,$$

we have by (3.1) with  $\delta = 1$  again that for all  $u, w > 0$ ,

$$\frac{(\ell_1 \vee \ell_2)(u \vee w)}{(\ell_1 \vee \ell_2)(u + w)} \leq c_0 \max \left\{ \frac{u + w}{u \vee w}, \frac{u \vee w}{u + w} \right\} \leq 2c_0.$$

This together with  $\gamma(u + w) \leq \gamma(2(u \vee w)) \leq 2^{d+\beta_2} c_2^\phi \gamma(u \vee w)$  yields

$$\frac{\gamma(u + w)}{(\ell_1 \vee \ell_2)(u + w)} \lesssim \left( \frac{\gamma}{\ell_1 \vee \ell_2} \right)(u \vee w) \leq \frac{\gamma(u)}{\ell_1(u)} + \frac{\gamma(w)}{\ell_2(w)}. \quad (3.10)$$

Now let  $g(t, x) := \phi^{-1}(t) + |x|$ . Since for  $0 < s < t < \infty$  and  $x, y \in \mathbb{R}^d$ ,

$$g(t, x) \lesssim g(t - s, x - y) + g(s, y),$$

we have by (3.9) and (3.10),

$$\begin{aligned} \frac{\gamma(g(t, x))}{(\ell_1 \vee \ell_2)(g(t, x))} &\lesssim \left( \frac{\gamma}{\ell_1 \vee \ell_2} \right)(g(t - s, x - y) + g(s, y)) \\ &\lesssim \frac{\gamma(g(t - s, x - y))}{\ell_1(g(t - s, x - y))} + \frac{\gamma(g(s, y))}{\ell_2(g(s, y))}. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{\ell_1(g(t - s, x - y))}{\gamma(g(t - s, x - y))} \times \frac{\ell_2(g(s, y))}{\gamma(g(s, y))} \\ &\lesssim \left( \frac{\ell_1(g(t - s, x - y))}{\gamma(g(t - s, x - y))} + \frac{\ell_2(g(s, y))}{\gamma(g(s, y))} \right) \times \frac{(\ell_1 \vee \ell_2)(g(t, x))}{\gamma(g(t, x))}, \end{aligned}$$

which together with (1.19) yields

$$h_\phi^{\ell_1}(t - s, x - y) h_\phi^{\ell_2}(s, y) \lesssim \left( h_\phi^{\ell_1}(t - s, x - y) + h_\phi^{\ell_2}(s, y) \right) h_\phi^{\ell_1 \vee \ell_2}(t, x).$$

Integrating both sides in  $y$  and by (3.6), we obtain (3.7).  $\square$

#### 4. PROOF OF THEOREM 1.4

Now we consider the space and time dependent nonlocal operator  $\mathcal{L}_t^\kappa$  defined by (1.4), and give a proof for Theorem 1.4. For each fixed  $y \in \mathbb{R}^d$ , let  $\mathcal{L}_t^{k_y}$  be the freezing operator

$$\mathcal{L}_t^{k_y} f(x) = \int_{\mathbb{R}^d} \Delta_f^{(\phi)}(x, z) \kappa(t, y, z) dz, \quad (4.1)$$

where  $\Delta_f^{(\phi)}$  is the difference operator defined by (1.6). Let  $p_{t,s}^{(y)}(x) := p_{t,s}^{k_y}(-x)$  be the heat kernel of operator  $\mathcal{L}_t^{k_y}$  as given by (2.6). Equivalently,  $p_{t,s}^{(y)}(x)$  is the probability transition

density of the time-inhomogenous Lévy process associated with  $\mathcal{L}_t^{\kappa_y}$  starting from position  $x$  at time  $t$  to be at the origin 0 at time  $s$ . We know from (2.7) that it satisfies

$$\partial_t p_{t,s}^{(y)}(x) + \mathcal{L}_t^{\kappa_y} p_{t,s}^{(y)}(x) = 0 \text{ for } s > t \text{ with } \lim_{t \uparrow s} p_{t,s}^{(y)}(x) = \delta_{\{0\}}(x), \quad (4.2)$$

where  $\delta_{\{0\}}(x)$  denotes the usual Dirac measure concentrated at the origin 0.

Following Levi's idea, we seek heat kernel  $p_{t,s}^\kappa(x, y)$  of  $\mathcal{L}_t^\kappa$  of the following form:

$$p_{t,s}^\kappa(x, y) = p_{t,s}^{(y)}(x - y) + \int_t^s \int_{\mathbb{R}^d} p_{t,r}^{(z)}(x - z) q_{r,s}(z, y) dz dr, \quad (4.3)$$

where  $q_{r,s}(z, y)$  is some suitable function to be determined. If the above  $p_{t,s}^\kappa(x, y)$  is a heat kernel for  $\mathcal{L}_t^\kappa$ , that is, for each  $t < s$  and  $x, y \in \mathbb{R}^d$ ,

$$\partial_t p_{t,s}^\kappa(x, y) + \mathcal{L}_t^\kappa p_{t,s}^\kappa(\cdot, y)(x) = 0,$$

formally differentiate both sides of (4.3) with respect to  $t$  would yield

$$\mathcal{L}_t^\kappa p_{t,s}^\kappa(\cdot, y)(x) = (\mathcal{L}_t^{\kappa_y} p_{t,s}^{(y)})(x - y) + q_{t,s}(x, y) + \int_t^s \int_{\mathbb{R}^d} (\mathcal{L}_t^{\kappa_z} p_{t,r}^{(z)})(x - z) q_{r,s}(z, y) dz dr.$$

Applying  $\mathcal{L}_t^\kappa$  on both sides of (4.3) in  $x$ -variable formally gives

$$\mathcal{L}_t^\kappa p_{t,s}^\kappa(\cdot, y)(x) = (\mathcal{L}_t^{\kappa_x} p_{t,s}^{(y)})(x - y) + \int_t^s \int_{\mathbb{R}^d} (\mathcal{L}_t^{\kappa_x} p_{t,r}^{(z)})(x - z) q_{r,s}(z, y) dz dr.$$

Subtracting the above two displays and defining

$$q_{t,s}^{(0)}(x, y) := (\mathcal{L}_t^{\kappa_x} - \mathcal{L}_t^{\kappa_y}) p_{t,s}^{(y)}(\cdot)(x - y), \quad (4.4)$$

we conclude that  $q_{t,s}(x, y)$  must satisfy

$$q_{t,s}(x, y) = q_{t,s}^{(0)}(x, y) + \int_t^s \int_{\mathbb{R}^d} q_{t,r}^{(0)}(x, z) q_{r,s}(z, y) dz dr \quad (4.5)$$

for any  $t < s$  and  $x, y \in \mathbb{R}^d$ . For  $n \in \mathbb{N}$ , define  $q_{t,s}^{(n)}(x, y)$  recursively by

$$q_{t,s}^{(n)}(x, y) := \int_t^s \int_{\mathbb{R}^d} q_{t,r}^{(0)}(x, z) q_{r,s}^{(n-1)}(z, y) dz dr. \quad (4.6)$$

Iterating the identity (4.5) repeatedly, we get for  $N \geq 1$ ,

$$q_{t,s}(x, y) = \sum_{n=0}^N q_{t,s}^{(n)}(x, y) + \int_t^s \int_{\mathbb{R}^d} q_{t,r}^{(N)}(x, z) q_{r,s}(z, y) dz dr. \quad (4.7)$$

If the remainder tends to zero as  $N \rightarrow \infty$ , we would get

$$q_{t,s}(x, y) := \sum_{n=0}^{\infty} q_{t,s}^{(n)}(x, y). \quad (4.8)$$

Our approach is that, instead of showing the remainder in (4.7) tends to zero, we show that the infinite sum in the above converges absolutely and locally uniformly and the function  $q_{t,s}(x, y)$  defined by (4.8) satisfies the integral equation (4.7), using the estimates obtained in the last two sections. We then establish rigorously that the function  $p_{t,s}^\kappa(x, y)$  defined by (4.3) in terms of  $q_{t,s}(x, y)$  of (4.8) is indeed a time-inhomogeneous heat kernel for  $\mathcal{L}_t^\kappa$  with desired regularity and estimates. The positivity of  $q_{t,s}(x, y)$  and the uniqueness of the heat kernel are obtained through a maximum principle for non-local operator  $\mathcal{L}_t^\kappa$ .

*Throughout the remaining of this section, we assume  $\ell \in \mathcal{S}_0 \cap \mathcal{D}_0$  and one of the following holds:*

**(H1)** If  $\kappa(t, x, z) = \kappa(t, x, -z)$ , we assume (1.8),  $(\mathbf{A}_\phi^{(0)})$  and (1.11).

**(H2')** If  $\kappa(t, x, z) \neq \kappa(t, x, -z)$ , we assume (1.8),  $(\mathbf{A}_\phi^{(1)})$ , (1.11) and (1.12).

**4.1. Solving integral equation (4.5).** Our first step is to show that the function  $q_{t,s}(x, y)$  given by (4.8) solves the integral equation (4.5). We will use scaling to reduce the consideration of heat kernel  $p_{t,s}^x$  to the case of  $t = 0$  and  $s = 1$ . For this, we define for each fixed  $t < s$  with  $s - t \leq 1$ ,

$$\bar{\kappa}_y(r, z) := \kappa(t + r(s - t), y, \phi^{-1}(s - t)z), \quad \bar{\phi}(u) := \phi(u\phi^{-1}(s - t))/(s - t). \quad (4.9)$$

Note that  $\bar{\phi}$  satisfies (1.8) with the same constants  $c_1^\phi, c_2^\phi$  and  $0 < \beta_1 \leq \beta_2 < \infty$ , and, in view of  $(\mathbf{A}_\phi^{(i)})$ ,

$$\int_0^\infty \frac{\gamma_\phi^{(i)}(r)}{r\bar{\phi}(r)} dr \leq \mathcal{A}_\phi^{(i)} < \infty \quad \text{for } i = 0, 1. \quad (4.10)$$

Clearly,  $\bar{\kappa}_y(r, z)$  satisfies (1.12), and we have by (1.11) that

$$\kappa_0^{-1} \leq \bar{\kappa}_y(r, z) \leq \kappa_0, \quad |\bar{\kappa}_x(r, z) - \bar{\kappa}_y(r, z)| \leq \ell^2(|x - y|). \quad (4.11)$$

By Proposition 2.1, we have for any  $x, y$  and  $t < s$ ,

$$q_{t,s}^{(0)}(x, y) = \frac{\left( (\mathcal{L}_0^{\bar{\kappa}_x, \bar{\phi}} - \mathcal{L}_0^{\bar{\kappa}_y, \bar{\phi}}) p_{0,1}^{\bar{\kappa}_y, \bar{\phi}}(\cdot) \right) \left( (x - y)/\phi^{-1}(s - t) \right)}{(s - t) (\phi^{-1}(s - t))^d}. \quad (4.12)$$

Noticing that by definition (4.1), (2.24) and (4.11),

$$\left| (\mathcal{L}_0^{\bar{\kappa}_x, \bar{\phi}} - \mathcal{L}_0^{\bar{\kappa}_y, \bar{\phi}}) p_{0,1}^{\bar{\kappa}_y, \bar{\phi}} \right| (z) \lesssim \ell^2(|x - y|) \rho_{\bar{\phi}}(z),$$

and also by definition (1.16) and (4.9),

$$\rho_{\bar{\phi}}((x - y)/\phi^{-1}(s - t)) = (s - t) (\phi^{-1}(s - t))^d \rho_\phi(s - t, x - y), \quad (4.13)$$

we conclude that there is a positive constant  $C_0 = C_0(\Theta_1, \mathcal{A}_\phi^{(0)})$  so that for any  $(t, x; s, y) \in \mathbb{D}_0^1$ ,

$$|q_{t,s}^{(0)}(x, y)| \leq C_0 \ell^2(|x - y|) \rho_\phi(s - t, x - y) \leq C_0 h_\phi^{\ell^2}(s - t, x - y), \quad (4.14)$$

where  $h_\phi^{\ell^2}$  is defined by (3.5).

The following theorem extends [9, Theorem 3.1] to the time-dependent and mixed stable-like non-local operator setting of this paper, and relaxes the Hölder continuous assumption on  $x \mapsto \kappa(t, x, z)$  to Dini continuity. Recall the definition of  $\ell_\phi$  from (1.15).

**Theorem 4.1.** *Let  $q_{t,s}^{(n)}(x, y)$  be defined as by (4.4) and (4.6). Under either **(H1)** or **(H2')**, there is an  $\varepsilon_0 > 0$  such that the series  $q_{t,s}(x, y) := \sum_{n=0}^\infty q_{t,s}^{(n)}(x, y)$  is absolutely and locally uniformly convergent on  $\mathbb{D}_0^{\varepsilon_0}$  and solves the integral equation (4.5). Moreover, for each  $t > 0$ ,  $(s, x, y) \mapsto q_{t,s}(x, y)$  is jointly continuous in  $\mathbb{K}_t^{\varepsilon_0} := (t, t + \varepsilon_0) \times \mathbb{R}^d \times \mathbb{R}^d$ , and has the following estimates: there is a constant  $c_1 = c_1(\varepsilon_0, \Theta) > 0$  so that on  $\mathbb{D}_0^{\varepsilon_0}$ ,*

$$|q_{t,s}(x, y)| \leq c_1 h_\phi^{\ell^2}(s - t, x - y) \leq c_1 \|\ell\|_\infty h_\phi^\ell(s - t, x - y), \quad (4.15)$$

where  $h_\phi^\ell$  is defined by (3.5), and

$$|q_{t,s}(x, y) - q_{t,s}(x', y)| \leq c_1 \frac{\ell(|x - x'|)}{\ell_\phi(s - t)} \left( h_\phi^\ell(s - t, x - y) + h_\phi^\ell(s - t, x' - y) \right). \quad (4.16)$$

*Proof.* (i) Let  $C_2 := 2C_1C_0$ , where  $C_1$  is the constant in (3.7) associated to  $\ell_1 = \ell_2 = \ell^2$ , and  $C_0$  is from (4.14). We use induction method to show that for all  $(t, x; s, y) \in \mathbb{D}_0^1$ ,

$$|q_{t,s}^{(n)}(x, y)| \leq C_2^{n+1} \left( \Gamma_{\ell_\phi}^{\ell^2}(s - t) \right)^n h_\phi^{\ell^2}(s - t, x - y),$$

where  $\Gamma_{\ell_\phi^2}(t) := \int_0^t \ell_\phi^2(s) ds/s$ . First of all, for  $n = 0$ , it is true by (4.14). Suppose now that it has been proven for some  $n \in \mathbb{N}$ . Then by (4.14), the induction hypothesis and (3.7), we have

$$\begin{aligned} |q_{t,s}^{(n+1)}(x,y)| &\leq C_2^{n+1} \left( \Gamma_{\ell_\phi^2}(s-t) \right)^n C_0 \int_t^s \int_{\mathbb{R}^d} h_\phi^{\ell^2}(r-t, x-z) h_\phi^{\ell^2}(s-r, z-y) dz dr \\ &\leq C_2^{n+1} \left( \Gamma_{\ell_\phi^2}(s-t) \right)^n C_0 \left( C_1 \int_t^s \left( \frac{\ell_\phi^2(r-t)}{r-t} + \frac{\ell_\phi^2(s-r)}{s-r} \right) dr \right) h_\phi^{\ell^2}(s-t, x-y) \\ &= C_2^{n+1} \left( \Gamma_{\ell_\phi^2}(s-t) \right)^n C_0 \left( 2C_1 \Gamma_{\ell_\phi^2}(s-t) \right) h_\phi^{\ell^2}(s-t, x-y) \\ &= C_2^{n+2} \left( \Gamma_{\ell_\phi^2}(s-t) \right)^{n+1} h_\phi^{\ell^2}(s-t, x-y). \end{aligned}$$

Now by (iv) of Proposition 3.1, we can choose  $\varepsilon_0 \in (0, 1)$  small enough so that

$$\Gamma_{\ell_\phi^2}(\varepsilon_0) \leq 1/(2C_2).$$

Thus  $q_{t,s}(x,y) = \sum_{n=0}^{\infty} q_{t,s}^{(n)}(x,y)$  converges absolutely and locally uniformly on  $\mathbb{D}_0^{\varepsilon_0}$  and

$$|q_{t,s}(x,y)| \leq \sum_{n=0}^{\infty} |q_{t,s}^{(n)}(x,y)| \leq 2C_2 h_\phi^{\ell^2}(s-t, x-y).$$

(ii) By (2.2), (2.5) and (2.6), one sees that  $\{(t, s, x) \mapsto p_{t,s}^{(y)}(x) : y \in \mathbb{R}^d\}$  is equi-continuous in any compact subsets of  $\{(t, s, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d : t < s\}$ . On the other hand, by (2.26) and Proposition 2.1, it is easy to see that  $y \mapsto p_{t,s}^{(y)}(x)$  is continuous for each  $t < s$  and  $x \in \mathbb{R}^d$ . Hence,  $(t, x; s, y) \mapsto p_{t,s}^{(y)}(x-y)$  is continuous on  $\mathbb{D}_0^1$ . Moreover, by the definition of  $q^{(0)}$ , one sees that for each  $t > 0$ ,  $(s, x, y) \mapsto q_{t,s}^{(0)}(x, y)$  is continuous on  $(t, t+1) \times \mathbb{R}^d \times \mathbb{R}^d$ . Furthermore, by definition (4.6) and induction method, for each  $n \in \mathbb{N}$ ,  $(s, x, y) \mapsto q_{t,s}^{(n)}(x, y)$  is continuous on  $(t, t+1) \times \mathbb{R}^d \times \mathbb{R}^d$ . So  $(s, x, y) \mapsto q_{t,s}(x, y)$  is continuous on  $\mathbb{K}_t^{\varepsilon_0}$ .

(iii) In this step we show that for all  $(t, x; s, y), (t, x'; s, y) \in \mathbb{D}_0^1$ ,

$$|q_{t,s}^{(0)}(x, y) - q_{t,s}^{(0)}(x', y)| \lesssim \frac{\ell(|x-x'|)}{\ell_\phi(s-t)} \left( h_\phi^{\ell^2}(s-t, x-y) + h_\phi^{\ell^2}(s-t, x'-y) \right). \quad (4.17)$$

First of all, if  $|x-x'| \geq \phi^{-1}(s-t)$ , then by (4.14) and the increase of  $\ell$ , we clearly have the above estimate. Next we assume

$$|x-x'| \leq \phi^{-1}(s-t).$$

Write  $p(x) := p_{0,1}^{\bar{\kappa}_y, \bar{\phi}}(x)$ . Recall the definition of  $\Delta_p^{(\bar{\phi})}$  from (1.6). We have by definition (4.9), Lemma 2.7 and (4.11),

$$\begin{aligned} &\left| \left( \mathcal{L}_0^{\bar{\kappa}_x, \bar{\phi}} - \mathcal{L}_0^{\bar{\kappa}_y, \bar{\phi}} \right) p(z_1) - \left( \mathcal{L}_0^{\bar{\kappa}_{x'}, \bar{\phi}} - \mathcal{L}_0^{\bar{\kappa}_y, \bar{\phi}} \right) p(z_2) \right| \\ &\leq \int_{\mathbb{R}^d} \left| \Delta_p^{(\bar{\phi})}(z_1, z) - \Delta_p^{(\bar{\phi})}(z_2, z) \right| |\bar{\kappa}_x(0, z) - \bar{\kappa}_y(0, z)| dz + \int_{\mathbb{R}^d} \left| \Delta_p^{(\bar{\phi})}(z_2, z) \right| |\bar{\kappa}_x(0, z) - \bar{\kappa}_{x'}(0, z)| dz \\ &\lesssim \ell^2(|x-y|)(|z_1 - z_2| \wedge 1) \left( \rho_{\bar{\phi}}(z_1) + \rho_{\bar{\phi}}(z_2) \right) + \ell^2(|x-x'|) \rho_{\bar{\phi}}(z_2), \end{aligned}$$

where the implicit constant depends on  $\Theta$ , especially on  $\mathcal{A}_\phi^{(0)}$  (resp.  $\mathcal{A}_\phi^{(1)}$ ) in the symmetric (resp. non-symmetric) case of  $z \mapsto \kappa(t, x, z)$ . By (4.12) and (4.13) and taking  $z_1 = \frac{x-y}{\phi^{-1}(s-t)}$

and  $z_2 = \frac{x'-y}{\phi^{-1}(s-t)}$ , we get for  $|x-x'| \leq \phi^{-1}(s-t)$ ,

$$\begin{aligned} |q_{t,s}^{(0)}(x,y) - q_{t,s}^{(0)}(x',y)| &\lesssim \ell^2(|x-y|) \frac{|x-x'|}{\phi^{-1}(s-t)} \rho_\phi(s-t, x-y) \\ &+ \left( \ell^2(|x-y|) \frac{|x-x'|}{\phi^{-1}(s-t)} + \ell^2(|x-x'|) \right) \rho_\phi(s-t, x'-y). \end{aligned} \quad (4.18)$$

Since  $\ell \in \mathcal{S}_0 \cap \mathcal{D}_0$  is bounded, by Proposition 3.1 we have

$$\frac{r}{R} \lesssim \frac{\ell(r)}{\ell(R)} \lesssim 1 \text{ for } r \leq R \text{ and } \ell(s+t) \lesssim \ell(s) + \ell(t).$$

Thus, by the definition (3.5) of  $h_\phi^{\ell^2}$ , from (4.18), we immediately have (4.17).

(iv) By (4.17), (4.15) and Lemma 3.2, we have

$$\begin{aligned} &\int_t^s \int_{\mathbb{R}^d} |q_{t,r}^{(0)}(x,z) - q_{t,r}^{(0)}(x',z)| |q_{r,s}(z,y)| dz dr \\ &\lesssim \ell(|x-x'|) \int_t^s \frac{1}{\ell_\phi(r-t)} \left[ \frac{\ell_\phi^2(r-t)}{r-t} + \frac{\ell_\phi^2(s-r)}{s-r} \right] dr \\ &\quad \times \left( h_\phi^{\ell^2}(s-t, x-y) + h_\phi^{\ell^2}(s-t, x'-y) \right). \end{aligned} \quad (4.19)$$

Clearly, we have

$$\int_t^s \frac{\ell_\phi(r-t)}{r-t} dr = \int_0^{s-t} \frac{\ell_\phi(r)}{r} dr = \Gamma_{\ell_\phi}(s-t) < \infty.$$

Write

$$\int_t^s \frac{\ell_\phi^2(s-r)}{\ell_\phi(r-t)(s-r)} dr = \left( \int_{(s+t)/2}^s + \int_t^{(s+t)/2} \right) \frac{\ell_\phi^2(s-r)}{\ell_\phi(r-t)(s-r)} dr =: I_1 + I_2.$$

For  $I_1$ , since  $\ell_\phi \in \mathcal{S}_0 \cap \mathcal{D}_0$ , we have

$$I_1 \lesssim \frac{1}{\ell_\phi((s-t)/2)} \int_0^{(s-t)/2} \frac{\ell_\phi^2(r)}{r} dr \leq \int_0^{(s-t)/2} \frac{\ell_\phi(r)}{r} dr = \Gamma_{\ell_\phi}((s-t)/2).$$

For  $I_2$ , since  $s \mapsto s/\ell_\phi(s) \in \mathcal{R}_1$  by  $1/\ell_\phi \in \mathcal{S}_0$ , we have by (3.3) that

$$I_2 \lesssim \frac{\ell_\phi^2(s-t)}{s-t} \int_0^{(s-t)/2} \frac{dr}{\ell_\phi(r)} \lesssim \ell_\phi(s-t).$$

Combining these with (4.19), (4.17) and (4.5), we obtain (4.16).  $\square$

**Remark 4.2.** In order to obtain estimate (4.16), we need to borrow some regularity from the spatial variable to compensate the time singularity (see (4.19)). This is the only reason that we have to assume (1.11) for the square of some Dini's function.

**Corollary 4.3.** *Suppose either assumption (H1) or (H2') holds. Let  $p_{t,s}^k(x,y)$  be defined by (4.3) and  $\varepsilon_0$  be as in Theorem 4.1. Then  $p_{t,s}^k(x,y)$  is continuous on  $\mathbb{D}_0^{\varepsilon_0}$  and there are constants  $c_0, c_1 > 0$  and  $\delta > 0$  such that*

$$p_{t,s}^k(x,y) \leq c_0 (s-t) \rho_\phi(s-t, x-y) \quad \text{on } \mathbb{D}_0^{\varepsilon_0}, \quad (4.20)$$

and if  $|x-y| \leq \phi^{-1}(s-t) \leq \delta$ , then

$$p_{t,s}^k(x,y) \geq c_1 \phi^{-1}(s-t)^{-d}. \quad (4.21)$$

*Proof.* First of all, by Proposition 2.1 and Theorem 2.2, there is a constant  $c_2 > 1$  such that on  $\mathbb{D}_0^1$ ,

$$c_2^{-1}(s-t)\rho_\phi(s-t, x-y) \leq p_{t,s}^{(y)}(x, y) \leq c_2(s-t)\rho_\phi(s-t, x-y). \quad (4.22)$$

Define

$$\tilde{\ell}(t) := \int_0^{\phi(t)} \ell \circ \phi^{-1}(s)/s ds = \Gamma_{\ell_\phi} \circ \phi(t). \quad (4.23)$$

By Proposition 3.1, we know that  $\ell_\phi \in \mathcal{S}_0 \cap \mathcal{D}_0$  and  $\tilde{\ell} \in \mathcal{S}_0$ . Moreover, by (3.2),

$$\ell_\phi(t) \leq \Gamma_{\ell_\phi}(t) = \tilde{\ell} \circ \phi^{-1}(t) = \tilde{\ell}_\phi(t), \quad t \in [0, 1]. \quad (4.24)$$

Thus, by (4.15), (4.22) and (3.7), we have

$$\begin{aligned} \int_t^s \int_{\mathbb{R}^d} |p_{t,r}^{(z)}(x, z) q_{r,s}(z, y)| dz dr &\leq \int_t^s \int_{\mathbb{R}^d} (r-t)\rho_\phi(r-t, x-z) h_\phi^\ell(s-r, z-y) dz dr \\ &\leq \int_t^s \frac{r-t}{\tilde{\ell}_\phi(r-t)} \left( \int_{\mathbb{R}^d} h_\phi^{\tilde{\ell}}(r-t, x-z) h_\phi^\ell(s-r, z-y) dz \right) dr \\ &\leq h_\phi^{\tilde{\ell}}(s-t, x-y) \int_t^s \frac{r-t}{\tilde{\ell}_\phi(r-t)} \left( \frac{\tilde{\ell}_\phi(r-t)}{r-t} + \frac{\ell_\phi(s-r)}{s-r} \right) dr \\ &\leq h_\phi^{\tilde{\ell}}(s-t, x-y) \left( (s-t) + \int_t^s \frac{r-t}{\tilde{\ell}_\phi(r-t)} \frac{\ell_\phi(s-r)}{s-r} dr \right). \end{aligned}$$

Note that by (4.24) and (4.23),

$$\begin{aligned} \int_t^s \frac{r-t}{\tilde{\ell}_\phi(r-t)} \frac{\ell_\phi(s-r)}{s-r} dr &= \left( \int_t^{(s+t)/2} + \int_{(s+t)/2}^s \right) \left( \frac{r-t}{\tilde{\ell}_\phi(r-t)} \frac{\ell_\phi(s-r)}{s-r} \right) dr \\ &\leq \frac{\ell_\phi(s-t)}{s-t} \int_0^{s-t} \frac{r}{\tilde{\ell}_\phi(r)} dr + \frac{s-t}{\tilde{\ell}_\phi(s-t)} \int_0^{s-t} \frac{\ell_\phi(r)}{r} dr \\ &\leq \frac{\tilde{\ell}_\phi(s-t)}{s-t} \int_0^{s-t} \frac{r}{\tilde{\ell}_\phi(r)} dr + (s-t) \lesssim s-t, \end{aligned}$$

where the last step is due to  $s \mapsto s^2/\tilde{\ell}_\phi(s) \in \mathcal{R}_2$  and (3.3). Hence,

$$\int_t^s \int_{\mathbb{R}^d} |p_{t,r}^{(z)}(x, z) q_{r,s}(z, y)| dz dr \leq c_3(s-t) h_\phi^{\tilde{\ell}}(s-t, x-y), \quad (4.25)$$

which together with (4.3) and (4.31) yields (4.20).

On the other hand, if  $|x-y| \leq \phi^{-1}(s-t)$ , then by (4.22) and (4.25),

$$p_{t,s}^k(x, y) \geq (s-t)\rho_\phi(s-t, x-y) \left( c_2^{-1} - c_3 \tilde{\ell}_\phi(s-t) \right).$$

Choosing  $\delta$  be small enough, we get (4.21).

Finally, since  $(t, x; s, y) \mapsto p_{t,s}^{(y)}(x-y)$  is continuous on  $\mathbb{D}_0^1$ , by (4.3), Theorem 4.1 and the dominated convergence theorem, one sees that  $(t, x; s, y) \mapsto p_{t,s}^k(x, y)$  is continuous on  $\mathbb{D}_0^{\varepsilon_0}$   $\square$

**4.2. Gradient and fractional derivative estimates of  $p_{t,s}^k$ .** This section is similar to [9, Sections 3.2 and 3.3]. We only point out the main points. The following lemma follows easily from (2.5), (2.6), Theorem 2.2 and equation (4.2).

**Lemma 4.4.** *Suppose either (H1) or (H2') holds. For each  $j \in \mathbb{N}$ ,  $s > 0$  and  $y \in \mathbb{R}^d$ , the mapping  $(t, x) \mapsto \nabla^j p_{t,s}^{(y)}(x-y)$  is continuous on  $[0, s) \times \mathbb{R}^d$ . Moreover, there is a constant  $C > 0$  such that for all  $(t, x; s, y) \in \mathbb{D}_0^1$ ,*

$$|\nabla^j p_{t,s}^{(y)}(x-y)| \leq \frac{C(s-t)}{(\phi^{-1}(s-t))^j} \rho_\phi(s-t, x-y) \quad (4.26)$$

and

$$\limsup_{t \uparrow s} \left| \int_{\mathbb{R}^d} p_{t,s}^{(y)}(x-y) dy - 1 \right| = 0. \quad (4.27)$$

*Proof.* The estimate (4.26) follows from (2.8), (2.10) and (4.13). We next show (4.27). By (2.26), (2.8), (4.11), (4.13) and (3.6), we have

$$\begin{aligned} \left| \int_{\mathbb{R}^d} p_{t,s}^{(y)}(x-y) dy - 1 \right| &= \left| \int_{\mathbb{R}^d} (p_{t,s}^{(y)}(x-y) - p_{t,s}^{(x)}(x-y)) dy \right| \\ &\lesssim (s-t) \int_{\mathbb{R}^d} \ell^2(|x-y|) \rho_\phi(s-t, x-y) dy \\ &\leq (s-t) \int_{\mathbb{R}^d} h_\phi^{\ell^2}(s-t, x-y) dy \lesssim \ell_\phi^2(s-t), \end{aligned}$$

where the implicit constant  $C$  is independent of  $x$  and  $s-t$ . Thus we get (4.27).  $\square$

To show the gradient and fractional derivative estimates, by (4.3) we write

$$\begin{aligned} p_{t,s}^\kappa(x, y) &= p_{t,s}^{(y)}(x-y) + \int_{\frac{s+t}{2}}^s \int_{\mathbb{R}^d} p_{t,r}^{(z)}(x-z) q_{r,s}(z, y) dz dr \\ &\quad + \int_t^{\frac{s+t}{2}} \int_{\mathbb{R}^d} p_{t,r}^{(z)}(x-z) q_{r,s}(z, y) dz dr \\ &=: \sum_{i=1}^3 J_i(t, x; s, y). \end{aligned} \quad (4.28)$$

Recall from (1.15) that  $\ell_\phi(t) := \ell(\phi^{-1}(t))$  and  $\Gamma_\ell(t) := \int_0^t \frac{\ell(s)}{s} ds$ .

**Lemma 4.5.** *Assume that in addition to the assumption **(H1)** or **(H2')**, condition (1.13) holds as well. Then for each  $0 \leq t < s$  with  $s-t \leq 1$  and  $y \in \mathbb{R}^d$ ,  $p_{t,s}^\kappa(x, y)$  is continuously differentiable in  $x \in \mathbb{R}^d$ . Moreover, there is a constant  $C > 0$  such that for all  $(t, x; s, y) \in \mathbb{D}_0^1$ ,*

$$|\nabla_x p_{t,s}^\kappa(x, y)| \leq C \left( \frac{s-t}{\phi^{-1}(s-t)} + M_\ell^\phi \circ \phi^{-1}(s-t) \right) \rho_\phi(s-t, x-y), \quad (4.29)$$

where  $M_\ell^\phi(t)$  is the function defined by (1.13).

*Proof.* For  $J_1(t, x; s, y)$  in (4.28), we have by (4.26)

$$|\nabla J_1(t, \cdot; s, y)(x)| \lesssim \frac{s-t}{\phi^{-1}(s-t)} \rho_\phi(s-t, x-y).$$

For  $J_2(t, x; s, y)$  in (4.28), we have by (4.31), (4.15) and (3.7),

$$\begin{aligned} |\nabla J_2(t, \cdot; s, y)(x)| &\lesssim \int_{\frac{s+t}{2}}^s \int_{\mathbb{R}^d} \frac{r-t}{\phi^{-1}(r-t)} \rho_\phi(r-t, x-z) h_\phi^{\ell^2}(s-r, z-y) dz dr \\ &\lesssim \rho_\phi(s-t, x-y) \int_{\frac{s+t}{2}}^s \frac{r-t}{\phi^{-1}(r-t)} \left( \frac{1}{r-t} + \frac{\ell_\phi(s-r)}{s-r} \right) dr \\ &\lesssim \frac{s-t}{\phi^{-1}(s-t)} (1 + \Gamma_{\ell_\phi}(s-t)) \rho_\phi(s-t, x-y). \end{aligned}$$

For  $J_3(t, x; s, y)$ , we approximate it by

$$J_3^{(\varepsilon)}(t, x; s, y) := \int_{t+\varepsilon}^{\frac{s+t}{2}} \int_{\mathbb{R}^d} p_{t,r}^{(z)}(x-z) q_{r,s}(z, y) dz dr, \quad \varepsilon \in (0, \frac{s-t}{2}). \quad (4.30)$$

Fix  $\varepsilon \in (0, \frac{s-t}{2})$ . By (4.26) and (4.15), we can exchange  $\nabla$  with the integral and arrive at

$$\begin{aligned} \nabla J_3^{(\varepsilon)}(t, \cdot, s, y)(x) &= \int_{t+\varepsilon}^{\frac{s+t}{2}} \int_{\mathbb{R}^d} \nabla p_{t,r}^{(z)}(x-z) q_{r,s}(z, y) dz dr \\ &= \int_{t+\varepsilon}^{\frac{s+t}{2}} \int_{\mathbb{R}^d} \nabla p_{t,r}^{(z)}(x-z) (q_{r,s}(z, y) - q_{r,s}(x, y)) dz dr \\ &\quad + \int_{t+\varepsilon}^{\frac{s+t}{2}} \left( \int_{\mathbb{R}^d} (\nabla p_{t,r}^{(z)} - \nabla p_{t,r}^{(x)})(x-z) dz \right) q_{r,s}(x, y) dr \\ &=: K_1^{(\varepsilon)}(t, x; s, y) + K_2^{(\varepsilon)}(t, x; s, y), \end{aligned}$$

where in the second equality we have used

$$\int_{\mathbb{R}^d} \nabla p_{t,r}^{(x)}(x-z) dz = 0,$$

For  $K_1^{(\varepsilon)}(t, x; s, y)$ , by (4.16) and (3.6), (3.7), we have

$$\begin{aligned} &|K_1^{(\varepsilon)}(t, x; s, y)| \\ &\lesssim \int_{t+\varepsilon}^{\frac{s+t}{2}} \int_{\mathbb{R}^d} \frac{r-t}{\phi^{-1}(r-t)} \rho_\phi(r-t, x-z) \frac{\ell(|x-z|)}{\ell_\phi(s-r)} (h_\phi^\ell(s-r, z-y) + h_\phi^\ell(s-r, x-y)) dz dr \\ &\lesssim \frac{1}{\ell_\phi(s-t)} \int_t^{\frac{s+t}{2}} \frac{r-t}{\phi^{-1}(r-t)} \int_{\mathbb{R}^d} h_\phi^\ell(r-t, x-z) (h_\phi^\ell(s-r, z-y) + h_\phi^\ell(s-r, x-y)) dz dr \\ &\lesssim \left( \frac{h_\phi^\ell(s-t, x-y)}{\ell_\phi(s-t)} \right) \int_t^{\frac{s+t}{2}} \left( \frac{r-t}{\phi^{-1}(r-t)} \right) \left( \frac{\ell_\phi(r-t)}{r-t} + \frac{\ell_\phi(s-r)}{s-r} \right) dr \\ &\lesssim \left( \frac{1}{\ell_\phi(s-t)} \int_0^{s-t} \frac{\ell_\phi(r)}{\phi^{-1}(r)} dr + \frac{1}{s-t} \int_0^{s-t} \frac{r}{\phi^{-1}(r)} dr \right) \rho_\phi(s-t, x-y) \\ &= \left( \int_0^{\phi^{-1}(s-t)} \left( \frac{\ell(r)}{\ell_\phi(s-t)r} + \frac{\phi(r)}{(s-t)r} \right) d\phi(r) \right) \rho_\phi(s-t, x-y) \\ &= M_\ell^\phi \circ \phi^{-1}(s-t) \rho_\phi(s-t, x-y). \end{aligned}$$

For  $K_2^{(\varepsilon)}(t, x; s, y)$ , noting that by (2.26),

$$|\nabla p_{0,1}^{\tilde{\kappa}_y, \tilde{\phi}} - \nabla p_{0,1}^{\tilde{\kappa}_x, \tilde{\phi}}|(z) \lesssim \|\tilde{\kappa}_x - \tilde{\kappa}_y\|_\infty \rho_{\tilde{\phi}}(z) \lesssim \ell(|x-y|) \rho_{\tilde{\phi}}(z),$$

by Proposition 2.1 and (4.13), we have

$$\begin{aligned} |K_2^{(\varepsilon)}(t, x; s, y)| &\lesssim \int_t^{\frac{s+t}{2}} \frac{r-t}{\phi^{-1}(r-t)} \left( \int_{\mathbb{R}^d} h_\phi^\ell(r-t, x-z) dz \right) h_\phi^\ell(s-r, x-y) dr \\ &\lesssim \left( \int_t^{\frac{s+t}{2}} \frac{\ell_\phi(r-t)}{\phi^{-1}(r-t)} dr \right) h_\phi^\ell(s-t, x-y) \lesssim \left( \int_0^{s-t} \frac{\ell_\phi(r)}{\phi^{-1}(r)} dr \right) \rho_\phi(s-t, x-y) \\ &= \left( \int_0^{\phi^{-1}(s-t)} \frac{\ell(r)}{r} d\phi(r) \right) \rho_\phi(s-t, x-y) \lesssim M_\ell^\phi \circ \phi^{-1}(s-t) \rho_\phi(s-t, x-y), \end{aligned}$$

where the above implicit constant is independent of  $\varepsilon$ . Moreover, from the above proof, it is also easy to see that

$$\lim_{\varepsilon \downarrow 0} K_i^{(\varepsilon)}(t, x; s, y) = K_i^{(0)}(t, x; s, y), \quad i = 1, 2,$$

locally uniformly. Moreover, by the dominated convergence theorem,

$$x \mapsto K_i^{(0)}(t, x; s, y) \text{ is continuous.}$$



As  $J_3^{(\varepsilon)}(t, x; s, y)$  converges to  $J_3(t, x; s, y)$  pointwise and  $\nabla_x J_3^{(\varepsilon)}(t, x; s, y) = K_1^{(\varepsilon)}(t, x; s, y) + K_i^{(\varepsilon)}(t, x; s, y)$ , we conclude that  $J_3(t, x; s, y)$  is differentiable in  $x$  and  $\nabla_x J_3(t, x; s, y) = K_1^{(0)}(t, x; s, y) + K_i^{(0)}(t, x; s, y)$ , which is continuous in  $x$ . Summing the above up, we have shown that  $p_{t,s}^k(x, y)$  is continuously differentiable in  $x \in \mathbb{R}^d$  whose gradient has the desired estimate (4.29).  $\square$

**Lemma 4.6.** *Suppose either (H1) or (H2') holds. For each  $0 \leq t < s$  and  $y \in \mathbb{R}^d$ , the mapping  $x \mapsto \mathcal{L}_t^{\kappa_x} p_{t,s}^{(y)}(x)$  is continuous and for fixed  $t_0 < s$  and  $x \in \mathbb{R}^d$ ,*

$$\lim_{t \downarrow t_0} \left| \mathcal{L}_t^{\kappa_x} p_{t,s}^{(y)}(x) - \mathcal{L}_{t_0}^{\kappa_x} p_{t_0,s}^{(y)}(x) \right| = 0.$$

Moreover, there is a constant  $C > 0$  such that for all  $(t, x; s, y) \in \mathbb{D}_0^1$ ,

$$\int_{\mathbb{R}^d} \left| \Delta_{p_{t,s}^{(y)}}^{(\phi)}(x - y, z) \right| dz \leq C \rho_\phi(s - t, x - y), \quad (4.31)$$

where  $\Delta_p^{(\phi)}$  is defined in (1.6).

*Proof.* We first show (4.31). By Proposition 2.1, (2.24) and (4.13), we have

$$\begin{aligned} \int_{\mathbb{R}^d} \left| \Delta_{p_{t,s}^{(y)}}^{(\phi)}(x - y, z) \right| dz &= (s - t)^{-1} \left( \phi^{-1}(s - t) \right)^{-d} \int_{\mathbb{R}^d} \left| \Delta_{p_{0,1}^{\bar{\kappa}, \phi}}^{(\phi)}((x - y)/\phi^{-1}(s - t), z) \right| dz \\ &\lesssim (s - t)^{-1} \left( \phi^{-1}(s - t) \right)^{-d} \rho_{\bar{\phi}}((x - y)/\phi^{-1}(s - t)) = \rho_\phi(s - t, x - y). \end{aligned}$$

This proves (4.31). Note that by Remark 1.1(i),

$$\mathcal{L}_t^{\kappa_x} p_{t,s}^{(y)}(x) = \int_{\mathbb{R}^d} \Delta_{p_{t,s}^{(y)}}^{(\phi)}(x, z) dz.$$

The desired continuity follows by the dominated convergence theorem and Lemma 4.4.  $\square$

By Lemma 4.6, the following lemma can be proved in a similar way as that for Lemma 4.5.

**Lemma 4.7.** *Suppose the condition of Theorem 1.4 holds. For each  $0 \leq t < s$  with  $s - t \leq 1$  and  $y \in \mathbb{R}^d$ ,  $x \mapsto p_{t,s}^k(x, y)$  is pointwisely  $\mathcal{L}_t^k$ -differentiable in the sense that the integral in (1.4) and (1.5) is absolutely convergent for every  $x \in \mathbb{R}^d$ . Moreover,  $x \mapsto \mathcal{L}_t^k p_{t,s}^k(\cdot, y)(x)$  is continuous and for fixed  $t_0 < s$  and  $x \in \mathbb{R}^d$ ,*

$$\lim_{t \downarrow t_0} \left| \mathcal{L}_t^k p_{t,s}^k(x, y) - \mathcal{L}_{t_0}^k p_{t_0,s}^k(x, y) \right| = 0.$$

Furthermore, there is a constant  $C > 0$  such that for all  $(t, x; s, y) \in \mathbb{D}_0^1$ ,

$$\int_{\mathbb{R}^d} \left| \Delta_{p_{t,s}^k}^{(\phi)}(x - y, z) \right| dz \leq C \left( \frac{\Gamma_{\ell_\phi}(s - t)}{\ell_\phi(s - t)} \right) \rho_\phi(s - t, x - y). \quad (4.32)$$

*Proof.* Recall from (4.28),  $p_{t,s}^k(x, y) = \sum_{k=1}^3 J_k(t, x; s, y)$ . By (4.31),

$$\int_{\mathbb{R}^d} \left| \Delta_{J_1(t, \cdot; s, y)}^{(\phi)}(x, z) \right| dz \lesssim \rho_\phi(s - t, x - y).$$

For  $J_2(t, x; s, y)$ , we have by (4.31), (4.15) and (3.7),

$$\begin{aligned} \int_{\mathbb{R}^d} \left| \Delta_{J_2(t, \cdot; s, y)}^{(\phi)}(x, z) \right| dz &\leq \int_{\frac{s+t}{2}}^s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \Delta_{p_{t,s}^{(\bar{\kappa})}}^{(\phi)}(x - \bar{z}, z) \right| dz |q_{r,s}(\bar{z}, y)| d\bar{z} dr \\ &\lesssim \int_{\frac{s+t}{2}}^s \int_{\mathbb{R}^d} \rho_\phi(r - t, x - \bar{z}) h_\phi^\ell(s - r, \bar{z} - y) d\bar{z} dr \\ &\lesssim \rho_\phi(s - t, x - y) \int_{\frac{s+t}{2}}^s \left( \frac{1}{r - t} + \frac{\ell_\phi(s - r)}{s - r} \right) dr \end{aligned}$$

$$\lesssim \rho_\phi(s-t, x-y) \left(1 + \Gamma_{\ell_\phi} \left(\frac{s-t}{2}\right)\right) \lesssim \rho_\phi(s-t, x-y).$$

For  $J_3(t, x; s, y)$  in (4.28), we can use the same approximation as used in Lemma 4.5. Here we omit approximation procedure and only make the following calculations. Note that

$$\begin{aligned} \Delta_{J_3(t, \cdot; s, y)}^{(\phi)}(x, z) &= \int_t^{\frac{s+t}{2}} \int_{\mathbb{R}^d} \Delta_{P_{t,r}^{(\phi)}}^{(\phi)}(x - \bar{z}, z) q_{r,s}(\bar{z}, y) d\bar{z} dr \\ &= \int_t^{\frac{s+t}{2}} \int_{\mathbb{R}^d} \Delta_{P_{t,r}^{(\phi)}}^{(\phi)}(x - \bar{z}, z) (q_{r,s}(\bar{z}, y) - q_{r,s}(x, y)) d\bar{z} dr \\ &\quad + \int_t^{\frac{s+t}{2}} \left( \int_{\mathbb{R}^d} \left( \Delta_{P_{t,r}^{(\phi)}}^{(\phi)}(x - \bar{z}, z) - \Delta_{P_{t,r}^{(\phi)}}^{(\phi)}(x - \bar{z}, z) \right) d\bar{z} \right) q_{r,s}(x, y) dr, \end{aligned} \quad (4.33)$$

where we have used

$$\int_{\mathbb{R}^d} \Delta_{P_{t,r}^{(\phi)}}^{(\phi)}(x - \bar{z}, z) d\bar{z} = 0.$$

We therefore have

$$\begin{aligned} \int_{\mathbb{R}^d} \left| \Delta_{J_3(t, \cdot; s, y)}^{(\phi)}(x, z) \right| dz &\leq \int_t^{\frac{s+t}{2}} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left| \Delta_{P_{t,r}^{(\phi)}}^{(\phi)}(x - \bar{z}, z) \right| d\bar{z} \right) |q_{r,s}(\bar{z}, y) - q_{r,s}(x, y)| d\bar{z} dr \\ &\quad + \int_t^{\frac{s+t}{2}} \int_{\mathbb{R}^d} d\bar{z} \left( \int_{\mathbb{R}^d} \left| \Delta_{P_{t,r}^{(\phi)}}^{(\phi)}(x - \bar{z}, z) \right| d\bar{z} \right) |q_{r,s}(x, y)| dr \\ &=: K_1(t, x; s, y) + K_2(t, x; s, y). \end{aligned} \quad (4.34)$$

For  $K_1(t, x; s, y)$ , we have by (4.31) and (4.16),

$$\begin{aligned} |K_1(t, x; s, y)| &\leq \int_t^{\frac{s+t}{2}} \int_{\mathbb{R}^d} \rho_\phi(r-t, x-z) \frac{\ell(|x-z|)}{\ell_\phi(s-r)} \left( h_\phi^\ell(s-r, z-y) + h_\phi^\ell(s-r, x-y) \right) dz dr \\ &\leq \frac{1}{\ell_\phi(s-t)} \int_t^{\frac{s+t}{2}} \int_{\mathbb{R}^d} h_\phi^\ell(r-t, x-z) \left( h_\phi^\ell(s-r, z-y) + h_\phi^\ell(s-r, x-y) \right) dz dr \\ &\stackrel{(3.7)}{\leq} \left( \frac{h_\phi^\ell(s-t, x-y)}{\ell_\phi(s-t)} \right) \int_t^{\frac{s+t}{2}} \left( \frac{\ell_\phi(r-t)}{r-t} + \frac{\ell_\phi(s-r)}{s-r} \right) dr \\ &\leq \frac{2\Gamma_{\ell_\phi}(s-t)}{\ell_\phi(s-t)} h_\phi^\ell(s-t, x-y) \leq \frac{\Gamma_{\ell_\phi}(s-t)}{\ell_\phi(s-t)} \rho_\phi(s-t, x-y), \end{aligned}$$

where the last inequality is due to the fact that  $\ell(t) = \ell(1)$  for  $t \geq 1$  so  $\ell \lesssim 1$  on  $[0, \infty)$  and consequently  $h_\phi^\ell(s-t, x-y) \lesssim \rho_\phi(s-t, x-y)$ . For  $K_2(t, x; s, y)$ , noting that by Proposition 2.1 and (2.27),

$$\int_{\mathbb{R}^d} \left| \Delta_{P_{t,r}^{(\phi)}}^{(\phi)}(x - \bar{z}, z) \right| dz \lesssim \rho_\phi(r-t, x - \bar{z}) \ell(|x - \bar{z}|) \leq h_\phi^\ell(r-t, x - \bar{z}),$$

we have by (4.15), Lemma 3.2(i) and Proposition 3.1(ii) that

$$\begin{aligned} |K_2(t, x; s, y)| &\lesssim \int_t^{\frac{s+t}{2}} \left( \int_{\mathbb{R}^d} h_\phi^\ell(r-t, x - \bar{z}) d\bar{z} \right) h_\phi^\ell(s-r, x-y) dr \\ &\leq \left( \int_t^{\frac{s+t}{2}} \frac{\ell_\phi(r-t)}{r-t} dr \right) h_\phi^\ell(s-t, x-y) \\ &\leq \Gamma_{\ell}(s-t)/2 \rho_\phi(s-t, x-y) \\ &\lesssim \rho_\phi(s-t, x-y). \end{aligned}$$

Combining the above calculations, we obtain (4.32). As for the desired continuity of  $\mathcal{L}_t^\kappa p_{t,s}^\kappa(x, y)$  in  $t$ , it follows from Theorem 4.1, (4.5), Lemma 4.6, (4.15), (3.7) and the dominated convergence theorem. The proof is now complete.  $\square$

**4.3. A maximum principle.** In this subsection we establish a maximum principle for operator  $\mathcal{L}^\kappa$ , which will be used to obtain the uniqueness and positivity of heat kernels.

**Theorem 4.8.** *For  $T > 0$ , let  $u(t, x) \in C_b([0, T] \times \mathbb{R}^d)$  satisfy the following equation: for all  $x \in \mathbb{R}^d$  and Lebesgue almost all  $t \in [0, T)$ ,*

$$\partial_t u(t, x) + \mathcal{L}_t^\kappa u(t, x) \leq 0, \quad \liminf_{t \uparrow T} u(t, x) \geq 0.$$

Assume that for each  $t \in [0, T)$  and  $x \in \mathbb{R}^d$ ,

$$\lim_{s \downarrow t} |\mathcal{L}_s^\kappa u(s, x) - \mathcal{L}_s^\kappa u(t, x)| = 0, \quad (4.35)$$

and that in Case<sub>2</sub><sup>ϕ</sup> and Case<sub>3</sub><sup>ϕ</sup>, when  $\kappa(t, x, z)$  is not symmetric in  $z$ , for each  $t \in [0, T)$ ,

$$x \mapsto \nabla u(t, x) \text{ is continuous on } \mathbb{R}^d. \quad (4.36)$$

Then we have

$$u(t, x) \geq 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d. \quad (4.37)$$

*Proof.* We only give the proof for the case when  $\kappa(t, x, z)$  is not symmetric in  $z$ . The proof when  $\kappa(t, x, z)$  is symmetric in  $z$  is similar except that we use (1.5) for the expression of  $\mathcal{L}_t^\kappa$ . First of all, we assume that for all  $t \in [0, T)$ ,

$$\lim_{|x| \rightarrow \infty} u(t, x) = \infty$$

and there is some constant  $\delta < 0$  so that for each  $x \in \mathbb{R}^d$  and Lebesgue almost all  $t \in [0, T)$ ,

$$\partial_t u(t, x) + \mathcal{L}_t^\kappa u(t, x) \leq \delta < 0 \quad (4.38)$$

Suppose that (4.37) is not true. Since  $\lim_{|x| \rightarrow \infty} u(t, x) = \infty$  and  $\liminf_{t \uparrow T} u(t, x) \geq 0$ , there must be a point  $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$  such that

$$u(t_0, x_0) = \inf_{(t, x) \in [0, T) \times \mathbb{R}^d} u(t, x) < 0.$$

In Case<sub>2</sub><sup>ϕ</sup> and Case<sub>3</sub><sup>ϕ</sup>, since  $x_0$  is a minimum point of  $x \mapsto u(t_0, x)$ , by (4.36) we have

$$\nabla u(t_0, x_0) = 0.$$

Therefore, for each  $s > 0$ ,

$$\mathcal{L}_s^\kappa u(t_0, x_0) = \int_{\mathbb{R}^d} \left( u(t_0, x_0 + z) - u(t_0, x_0) - z^{(\phi)} \cdot \nabla u(t_0, x_0) \right) \frac{\kappa(s, x_0, z)}{|z|^d \phi(|z|)} dz \geq 0,$$

and integrating both sides of (4.38) from  $t_0$  to  $t$ , we have

$$\begin{aligned} u(t, x_0) - u(t_0, x_0) &\leq (t - t_0)\delta - \int_{t_0}^t \mathcal{L}_s^\kappa u(s, x_0) ds \\ &\leq (t - t_0)\delta - \int_{t_0}^t (\mathcal{L}_s^\kappa u(s, x_0) - \mathcal{L}_s^\kappa u(t_0, x_0)) ds. \end{aligned} \quad (4.39)$$

Dividing both sides by  $t - t_0$  and letting  $t \downarrow t_0$ , we obtain by (4.35)

$$0 \leq \delta + \lim_{t \downarrow t_0} \frac{1}{t - t_0} \int_{t_0}^t |\mathcal{L}_s^\kappa u(s, x_0) - \mathcal{L}_s^\kappa u(t_0, x_0)| ds = \delta < 0,$$

which is impossible. In other words, the infimum is achieved at the terminal time  $T$ , and (4.37) holds.

Next, we drop the restriction (4.38). For this, let

$$f(x) := (1 + |x|^2)^\alpha, \quad \alpha \in (0, \beta_1/2).$$

For  $\varepsilon, \delta > 0$ , define

$$u_{\varepsilon, \delta}(t, x) := u(t, x) + \delta(T - t) + \varepsilon e^{-t} f(x).$$

By easy calculations, one sees that for some  $C > 0$ ,

$$|\mathcal{L}_t^\kappa f(x)| \leq C(1 + |x|^\alpha),$$

and

$$\partial_t u_{\delta,\varepsilon}(t,x) + \mathcal{L}_t^\kappa u_{\delta,\varepsilon}(t,x) \leq -\delta + \varepsilon e^{-t}(\mathcal{L}_t^\kappa f(x) - f(x)) \leq -\delta/2 < 0,$$

provided  $\varepsilon$  being small enough so that  $\varepsilon e^{-t}(\mathcal{L}_t^\kappa f(x) - f(x)) < \delta/2$ . Clearly,

$$\lim_{x \rightarrow \infty} |u_{\varepsilon,\delta}(t,x)| = \infty.$$

Hence, by what we have proved,

$$u_{\varepsilon,\delta}(t,x) \geq 0.$$

By letting  $\varepsilon \rightarrow 0$  and then  $\delta \rightarrow 0$ , we obtain (4.37).  $\square$

**4.4. Proof of Theorem 1.4.** Assume the conditions of Theorem 1.4 hold. Let  $\varepsilon_0 > 0$  be defined as in Theorem 4.1 and  $p_{t,s}^\kappa(x,y)$  be defined by (4.3). For  $f \in C_b^2(\mathbb{R}^d)$ , define

$$u(t,x) := P_{t,s}^\kappa f(x) := \int_{\mathbb{R}^d} p_{t,s}^\kappa(x,y) f(y) dy.$$

(i) It follows from Lemma 4.7 that  $x \rightarrow u(t,x)$  is pointwisely  $\mathcal{L}_t^\kappa$ -differentiable in the sense that the integrals in (1.4) and (1.5) are absolutely convergent for every  $x \in \mathbb{R}^d$ , and that for fixed  $t_0 < s$  and  $x \in \mathbb{R}^d$ ,

$$\lim_{t \downarrow t_0} |\mathcal{L}_t^\kappa u(t,x) - \mathcal{L}_{t_0}^\kappa u(t_0,x)| = 0. \quad (4.40)$$

(ii) It follows from Lemma 4.5 that when  $\kappa(t,x,z)$  is not symmetric, in Case<sub>2</sub><sup>ϕ</sup> and Case<sub>3</sub><sup>ϕ</sup> under condition (1.13),

$$x \mapsto \nabla u(t,x) \text{ is continuous on } \mathbb{R}^d. \quad (4.41)$$

(iii) For any bounded and uniformly continuous function  $f$ , by (4.27) and (4.25), it is not hard to see that

$$\lim_{t \uparrow s} \|P_{t,s}^\kappa f - f\|_\infty = 0.$$

Moreover, by Lemma 4.7, Lemma 4.5 and the discussion at the beginning of this section, one has that (see [9] for more details)

$$u(t,x) = f(x) + \int_t^s \mathcal{L}_r^\kappa u(r,x) dr, \quad \forall (t,x) \in [0,s) \times \mathbb{R}^d. \quad (4.42)$$

The maximum principle from Theorem 4.8 gives the uniqueness of  $p_{t,s}^\kappa(x,y)$  as well as the properties that

$$p_{t,s}^\kappa(x,y) \geq 0 \text{ and } \int_{\mathbb{R}^d} p_{t,s}^\kappa(x,y) dy = 1 \text{ on } \mathbb{D}_0^{\varepsilon_0}, \quad (4.43)$$

and for all  $0 \leq t < r < s < \infty$  with  $s - t \leq \varepsilon_0$  and  $x, y \in \mathbb{R}^d$ ,

$$\int_{\mathbb{R}^d} p_{t,r}^\kappa(x,z) p_{r,s}^\kappa(z,y) dz = p_{t,s}^\kappa(x,y). \quad (4.44)$$

Now we are in a position to give

*Proof of Theorem 1.4.* Let  $\varepsilon_0 > 0$  be the constant from Theorem 4.1. We have established in the above the existence and uniqueness of heat kernel  $p_{t,s}^\kappa(x,y)$  on  $\mathbb{D}_0^{\varepsilon_0}$  that satisfies (i)-(iii) of Theorem 1.4 on  $\mathbb{D}_0^{\varepsilon_0}$ . We now extend the definition of  $p_{t,s}^\kappa(x,y)$  and its properties in (i)-(iii) from  $\mathbb{D}_0^{\varepsilon_0}$  to  $\mathbb{D}_0^\infty$  by (4.44) as follows: If  $\varepsilon_0 < s - t \leq 2\varepsilon_0$ , we define

$$p_{t,s}^\kappa(x,y) = \int_{\mathbb{R}^d} p_{t, \frac{t+s}{2}}^\kappa(x,z) p_{\frac{t+s}{2}, s}^\kappa(z,y) dz. \quad (4.45)$$

Proceeding this procedure, we can extend  $p^\kappa$  to  $\mathbb{D}_0^\infty$  and the Chapman-Kolmogorov equation (4.44) holds for all  $0 \leq t < r < s < \infty$  and  $x, y \in \mathbb{R}^d$ . In particular, equation (1.20) and (i), (ii), (iii) hold for  $p_{t,s}^\kappa(x, y)$ .

Next we show that the heat kernel  $p_{t,s}^\kappa(x, y)$  enjoys properties **(a)-(f)**.

- (a)** The upper bound estimate follows by (4.45), (4.20) and (3.7). Moreover, by (4.43) and (4.45) we also have

$$p_{t,s}^\kappa(x, y) \geq 0.$$

The lower bound will be proved in the next subsection.

- (b)** It follows by (4.45), (4.31) and (3.7).  
**(c)** It follows by (4.45), (4.29) and (3.7).  
**(d)** It follows by (4.45) and (4.43).  
**(e)** It follows by (4.45) and (4.44).  
**(f)** Fix  $s > 0$ . Define

$$\tilde{u}(t, x) := f(x) + \int_t^s P_{t,r}^\kappa \mathcal{L}_r^\kappa f(x) dr.$$

By Fubini's theorem, it is easy to see that  $\tilde{u}$  also satisfies equation (4.42) and (4.40), (4.41) (see [9] and [5]). Thus by the maximum principle, we have  $\tilde{u}(t, x) = u(t, x) = P_{t,s}^\kappa f(x)$ .

This completes the proof of the theorem except for the lower bound in **(a)** on the heat kernel  $p_{t,s}^\kappa(x, y)$ , which will be given separately in next subsection.  $\square$

**4.5. Proof of lower bound in (1.22).** We know from the last subsection that  $\{p_{t,s}^\kappa(x, y) : (t, x; s, y) \in \mathbb{D}_0^\infty\}$  is a family of transition probability density functions. It uniquely determines a Feller process

$$X := \left\{ \Omega, \mathcal{F}, (X_s)_{s \geq 0}; \mathbb{P}_{t,x}, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \right\}$$

on  $\mathbb{R}^d$  with the property that

$$\mathbb{P}_{t,x}(X_s = x, 0 \leq s \leq t) = 1,$$

and for  $r \in [t, s]$  and  $A \in \mathcal{B}(\mathbb{R}^d)$ ,

$$\mathbb{E}_{t,x}[X_s \in A | \mathcal{F}_r] = \int_A p_{r,s}^\kappa(X_r, y) dy, \quad (4.46)$$

where  $\mathcal{F}_s := \sigma\{X_t, t \leq s\}$ ,  $s \geq 0$ , is the filtration generated by the Feller process  $X$ . Moreover, for any  $f \in C_b^2(\mathbb{R}^d)$ , it follows from (1.26) and the Markov property of  $X$  that under  $\mathbb{P}_{t,x}$ , with respect to the filtration  $\{\mathcal{F}_s; s \geq 0\}$

$$M_s^f := f(X_s) - f(X_t) - \int_t^s \mathcal{L}_r f(X_r) dr \text{ is a martingale.} \quad (4.47)$$

In other words,  $\mathbb{P}_{t,x}$  solves the martingale problem for  $(\mathcal{L}_t, C_b^2(\mathbb{R}^d))$ .

For any Borel set  $E$ , let

$$\sigma_E := \inf\{s \geq 0 : X_s \in E\}, \quad \tau_E := \inf\{s \geq 0 : X_s \notin E\},$$

be the first hitting and exit time, respectively, of  $E$ . Below for simplicity, we write

$$\mathcal{J}_\phi(t, x, y) := \frac{\kappa(t, x, y - x)}{|y - x|^d \phi(|y - x|)}.$$

We have the following Lévy system of the Feller process  $X$  (see [5]).

**Lemma 4.9.** *Let  $f$  be a non-negative measurable function on  $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$  that vanishes along the diagonal. Then for every stopping time  $T \geq t$ ,*

$$\mathbb{E}_{t,x} \left[ \sum_{t < r \leq T} f(r, X_{r-}, X_r) \right] = \mathbb{E}_{t,x} \left[ \int_t^T \int_{\mathbb{R}^d} f(r, X_r, y) \mathcal{J}_\phi(r, X_r, y) dy dr \right]. \quad (4.48)$$

We need the following two lemmas.

**Lemma 4.10.** *There is a constant  $\gamma_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, 1)$ ,*

$$\sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} \mathbb{P}_{t,x}(\tau_{B(x,\varepsilon)} \leq t + \gamma_0 \phi(\varepsilon)) \leq 1/2. \quad (4.49)$$

*Proof.* For simplicity, write  $\tau := \tau_{B(x,\varepsilon)}$ . By the strong Markov property of  $X$ , we have

$$\begin{aligned} \mathbb{P}_{t,x}(\tau \leq t+r) &\leq \mathbb{P}_{t,x}(\tau \leq t+r; X_{t+r} \in B(x, \varepsilon/2)) + \mathbb{P}_{t,x}(X_{t+r} \notin B(x, \varepsilon/2)) \\ &= \mathbb{P}_{t,x}(\mathbb{P}_{\tau, X_\tau}(X_{t+r} \in B(x, \varepsilon/2)); \tau \leq t+r) + \mathbb{P}_{t,x}(X_{t+r} \notin B(x, \varepsilon/2)) \\ &\leq \mathbb{P}_{t,x}(\mathbb{P}_{\tau, X_\tau}(|X_{t+r} - X_\tau| \geq \varepsilon/2); \tau \leq t+r) + \mathbb{P}_{t,x}(X_{t+r} \notin B(x, \varepsilon/2)) \\ &\leq 2 \sup_{t \leq s \leq t+r} \sup_{x \in \mathbb{R}^d} \mathbb{P}_{s,x}(|X_{t+r} - x| \geq \varepsilon/2), \end{aligned} \quad (4.50)$$

where the second inequality is due to  $|X_\tau - x| \geq \varepsilon$  and  $|X_{t+r} - x| \leq \varepsilon/2$ . On the other hand, by (4.46) and the heat kernel upper bound estimate in (1.22), there is a constant  $C > 0$  such that for all  $r \in (0, 1)$ ,  $t \leq s \leq t+r$  and  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} \mathbb{P}_{s,x}(|X_{t+r} - x| \geq \varepsilon/2) &= \int_{|x-y| \geq \varepsilon/2} p_{s,t+r}^\kappa(x, y) dy \leq C(t+r-s) \int_{|z| \geq \varepsilon/2} \rho_\phi(t+r-s, z) dz \\ &\leq Cr \int_{\varepsilon/2}^\infty \frac{du}{u\phi(u)} = Cr \int_1^\infty \frac{du}{u\phi(\varepsilon u/2)} \leq \frac{Cr \mathcal{A}_\phi^{(0)}}{\phi(\varepsilon/2)} \leq \frac{C_0 r}{\phi(\varepsilon)}, \end{aligned}$$

where  $\mathcal{A}_\phi^{(0)}$  is defined in  $(\mathbf{A}_\phi^{(i)})$  with  $i = 0$ . Substituting this into (4.50) yields

$$\mathbb{P}_{t,x}(\tau_{B(x,\varepsilon)} \leq t+r) \leq \frac{C_0 r}{\phi(\varepsilon)}. \quad (4.51)$$

Letting  $r = \frac{\phi(\varepsilon)}{2C_0}$  in (4.51), we obtain (4.49) with  $\gamma_0 = \frac{1}{2C_0}$ .  $\square$

**Lemma 4.11.** *Let  $\gamma_0$  be the constant from Lemma 4.10. For all  $\gamma \in (0, \gamma_0]$ , there exists a constant  $c_1 > 0$  such that for all  $t > 0$ ,  $\varepsilon \in (0, 1)$  and  $x, y \in \mathbb{R}^d$  with  $|x - y| \geq 2\varepsilon$ ,*

$$\mathbb{P}_{t,x}(\sigma_{B(y,\varepsilon)} < t + \gamma\phi(\varepsilon)) \geq c_1 \frac{\varepsilon^d \phi(\varepsilon)}{|x - y|^d \phi(|x - y|)}. \quad (4.52)$$

*Proof.* For  $\varepsilon \in (0, 1)$  and  $\gamma \in (0, \gamma_0]$ , by (4.49) we have

$$\mathbb{E}_{t,x} [(t + \gamma\phi(\varepsilon)) \wedge \tau_{B(x,\varepsilon)} - t] \geq \gamma\phi(\varepsilon) \mathbb{P}_{t,x}(\tau_{B(x,\varepsilon)} \geq t + \gamma\phi(\varepsilon)) \geq \frac{\gamma\phi(\varepsilon)}{2}. \quad (4.53)$$

Noticing that under  $\mathbb{P}_{t,x}$ ,

$$X_r \notin B(y, \varepsilon) \quad \text{when } t < r < (t + \gamma\phi(\varepsilon)) \wedge \tau_{B(x,\varepsilon)},$$

we have

$$\mathbf{1}_{X_{(t+\gamma\phi(\varepsilon)) \wedge \tau_{B(x,\varepsilon)}} \in B(y,\varepsilon)} = \sum_{t < r \leq (t+\gamma\phi(\varepsilon)) \wedge \tau_{B(x,\varepsilon)}} \mathbf{1}_{X_r \in B(y,\varepsilon)}.$$

By the Lévy system formula (4.48) and the definition of  $\mathcal{J}_\phi$ , we have

$$\begin{aligned} \mathbb{P}_{t,x}(\sigma_{B(y,\varepsilon)} < t + \gamma\phi(\varepsilon)) &\geq \mathbb{P}_{t,x}(X_{(t+\gamma\phi(\varepsilon)) \wedge \tau_{B(x,\varepsilon)}} \in B(y, \varepsilon)) \\ &= \mathbb{E}_{t,x} \int_t^{(t+\gamma\phi(\varepsilon)) \wedge \tau_{B(x,\varepsilon)}} \int_{B(y,\varepsilon)} \frac{\kappa(t, X_r, z - X_r)}{|z - X_r|^d \phi(|z - X_r|)} dz dr. \end{aligned} \quad (4.54)$$

Since  $|x - y| \geq 2\varepsilon$ , we have for all  $z \in B(y, \varepsilon)$  and  $X_r \in B(x, \varepsilon)$ ,

$$|z - X_r| \leq |y - z| + |x - y| + |X_r - x| < 2|x - y|.$$

Thus by (4.54) and (4.53), we have

$$\mathbb{P}_{t,x}(\sigma_{B(y,\varepsilon)} < t + \gamma\phi(\varepsilon)) \geq \frac{\gamma\phi(\varepsilon)}{2} \int_{B(y,\varepsilon)} \frac{\kappa_0^{-1}}{(2|x-y|)^d \phi(2|x-y|)} dz \geq c_2 \frac{\varepsilon^d \phi(\varepsilon)}{|x-y|^d \phi(|x-y|)}.$$

This proves the lemma.  $\square$

Now we can give

*Proof of lower bound in Theorem 1.4(a).* Let  $\delta > 0$  be the constant in Corollary 4.3. We claim that by (4.21), for any  $0 \leq t < s \leq T$ ,  $x, y \in \mathbb{R}^d$  and  $n \in \mathbb{N}$ , there is a constant  $C_n > 0$  such that

$$p_{t,s}^k(x, y) \geq C_n \phi^{-1}(s-t)^{-d} \quad \text{whenever } |x-y| \leq \phi^{-1}((s-t)/2^n) \leq \delta. \quad (4.55)$$

Indeed, if  $|x-y| \leq \phi^{-1}((s-t)/2) \leq \delta$ , then by the Chapman-Kolmogorov equation,

$$\begin{aligned} p_{t,s}^k(x, y) &= \int_{\mathbb{R}^d} p_{t, \frac{t+s}{2}}^k(x, z) p_{\frac{t+s}{2}, s}^k(z, y) dz \\ &\geq \int_{B(\frac{x+y}{2}, \phi^{-1}((s-t)/2))} p_{t, \frac{t+s}{2}}^k(x, z) p_{\frac{t+s}{2}, s}^k(z, y) dz \\ &\stackrel{(4.21)}{\geq} c_1^2 \phi^{-1}((s-t)/2)^{-2d} \text{Vol}\left(B\left(\frac{x+y}{2}, \phi^{-1}((s-t)/2)\right)\right) \\ &\stackrel{(1.8)}{\geq} \phi^{-1}(s-t)^{-d}. \end{aligned}$$

Iterating the above estimates establishes the claim (4.55).

Now fix  $T > 0$  and choose  $n$  large enough so that

$$T/2^n \leq \phi(\delta), \quad \text{or equivalent, } \phi^{-1}(T/2^n) \leq \delta.$$

Consider  $x, y \in \mathbb{R}^d$  and  $0 \leq t < s \leq T$ . If  $|x-y| \leq \phi^{-1}((s-t)/2^n)$ , we immediately get from (4.55) the lower bound for  $p_{t,s}^k(x, y)$  in (1.22). It remains to consider the case that  $|x-y| > \phi^{-1}((s-t)/2^n)$ . Define

$$\varepsilon = \frac{1}{3} \phi^{-1}((s-t)/2^{n+1}) \quad \text{so } (s-t)/2^n = 2\phi(3\varepsilon). \quad (4.56)$$

Let  $\gamma_0 \in (0, 1)$  be the constant in Lemma 4.10. By the strong Markov property of  $X$  and Lemma 4.11, we have for any  $|x-y| \geq 3\varepsilon$ ,

$$\begin{aligned} \mathbb{P}_{t,x}(X_{t+2\gamma_0\phi(\varepsilon)} \in B(y, 2\varepsilon)) &\geq \mathbb{P}_{t,x}\left(\sigma := \sigma_{B(y,\varepsilon)} < t + \gamma_0\phi(\varepsilon); \sup_{s \in [\sigma, \sigma + \gamma_0\phi(\varepsilon)]} |X_s - X_\sigma| < \varepsilon\right) \\ &= \mathbb{E}_{t,x}\left[\mathbb{P}_{\sigma, X_\sigma}\left(\sup_{s \in [\sigma, \sigma + \gamma_0\phi(\varepsilon)]} |X_s - X_\sigma| < \varepsilon\right); \sigma_{B(y,\varepsilon)} < t + \gamma_0\phi(\varepsilon)\right] \\ &\geq \inf_{r,z} \mathbb{P}_{r,z}(\tau_{B(z,\varepsilon)} > r + \gamma_0\phi(\varepsilon)) \mathbb{P}_{t,x}(\sigma_{B(y,\varepsilon)} < t + \gamma_0\phi(\varepsilon)) \\ &\stackrel{(4.49)}{\geq} \frac{1}{2} \mathbb{P}_{t,x}(\sigma_{B(y,\varepsilon)} < t + \gamma_0\phi(\varepsilon)) \\ &\stackrel{(4.52)}{\geq} \frac{\varepsilon^d \phi(\varepsilon)}{|x-y|^d \phi(|x-y|)}. \end{aligned}$$

Hence we have for any  $x, y \in \mathbb{R}^d$  with  $|x-y| \geq 3\varepsilon$ ,

$$p_{t,s}^k(x, y) \geq \int_{B(y, 2\varepsilon)} p_{t, t+2\gamma_0\phi(\varepsilon)}^k(x, z) p_{t+2\gamma_0\phi(\varepsilon), s}^k(z, y) dz$$

$$\begin{aligned}
&\geq \inf_{z \in B(y, 2\varepsilon)} p_{t+2\gamma_0\phi(\varepsilon), s}^\kappa(z, y) \mathbb{P}_{t, x}(X_{t+2\gamma_0\phi(\varepsilon)} \in B(y, 2\varepsilon)) \\
(4.55) \quad &\geq C_1 \phi^{-1}(s-t)^{-d} \cdot \frac{\varepsilon^d \phi(\varepsilon)}{|x-y|^d \phi(|x-y|)} \\
(1.8) \quad &\geq C_2 (s-t) \rho_\phi(s-t, x-y),
\end{aligned}$$

where in the third inequality when apply (4.55) we used the fact that from (4.56)

$$\phi(\delta) > \frac{s - (t + 2\gamma_0\phi(\varepsilon))}{2^n} \geq 2\phi(3\varepsilon) - \gamma_0\phi(\varepsilon) \geq \phi(2\varepsilon) > \phi(|z-y|) \quad \text{for } z \in B(y, 2\varepsilon).$$

This establishes the lower bound for  $p_{t,s}^\kappa(x, y)$  in (1.22) on  $\mathbb{D}_0^T$ .  $\square$

## 5. EXAMPLES

In this section, we discuss the assumptions (1.8),  $(\mathbf{A}_\phi^{(i)})$  and (1.13), and give some examples that satisfy these conditions and therefore our main results apply.

**Example 5.1.** Let  $\phi(r) = r^\alpha$  with  $\alpha \in (0, 2)$ . It is easy to see that (1.8),  $(\mathbf{A}_\phi^{(i)})$  and (1.13) hold. In particular, Theorem 1.4 in this case extends the main results in [9] and [10] to time-dependent and Dini's continuous kernels  $\kappa(t, x, z)$ .

**Example 5.2.** Let  $0 < \beta_1 \leq \beta_2 < 2$  and  $\phi$  an increasing function on  $[0, \infty)$  so that there are positive constants  $c_2^\phi \geq c_1^\phi$  such that

$$c_1^\phi \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\phi(R)}{\phi(r)} \leq c_2^\phi \left(\frac{R}{r}\right)^{\beta_2} \quad \text{for any } 0 < r < R < \infty.$$

This is the case, for example, when

$$\phi(r) = \int_{\beta_1}^{\beta_2} r^\alpha \nu(d\alpha) \quad \text{or} \quad \phi(r) = 1 / \int_{\beta_1}^{\beta_2} r^{-\alpha} \nu(d\alpha),$$

where  $\nu$  is a probability measure on  $[\beta_1, \beta_2]$ . Clearly,  $\phi$  satisfies (1.8) and  $(\mathbf{A}_\phi^{(0)})$ , as well as  $(\mathbf{A}_\phi^{(1)})$  of Case $_1^\phi$  and Case $_2^\phi$ . Property  $(\mathbf{A}_\phi^{(1)})$  holds when  $\beta_1 > 1$ , under which Case $_3^\phi$  occurs. When  $\kappa(t, x, z)$  is symmetric in  $z$ , Theorem 1.4 in particular extends the main results in [9, 10, 13]. See (iii) and (iv) of Remark 1.5 for the gradient estimate.

**Example 5.3.** Let  $\phi(r) = r^\alpha \mathbf{1}_{\{r \leq 1\}} + r^\beta \mathbf{1}_{\{r > 1\}}$  with  $\alpha > 0$  and  $\beta > 0$ . Note that (1.8) holds with  $\beta_1 := \alpha$  and  $\beta_2 := \alpha \vee \beta$ . One can check that  $(\mathbf{A}_\phi^{(0)})$  is satisfied for any  $\alpha \in (0, 2)$  and  $\beta > 0$ , while  $(\mathbf{A}_\phi^{(1)})$  holds for all  $\alpha \in (0, 2)$  and  $\beta > 0$  except when  $\alpha = 1$  and  $\beta > 1$  (corresponding to Case $_3^\phi$ ). As we noted in (1.14), for  $\ell(r) = r^\eta$  on  $(0, 1]$ , condition (1.13) holds if and only if  $\alpha > 1/2$  and  $\alpha + \eta > 1$ . Note that the function  $\phi$  can have any polynomial growth as  $r \rightarrow \infty$ . When  $\ell \in \mathcal{D}_0$ , condition (1.13) holds if  $\alpha \geq 1$  with

$$M_\ell^\phi(t) = \frac{\alpha}{\ell(t)} \int_0^t \frac{\ell(r)}{r} r^{\alpha-1} dr + \frac{\alpha}{2\alpha-1} t^{\alpha-1} \quad \text{on } (0, 1]$$

When  $\alpha > 1$ , clearly  $M_\ell^\phi(t) \asymp t^{\alpha-1}$  on  $(0, 1]$  and thus the gradient estimate (1.24) takes the form

$$|\nabla p_{t,s}^\kappa(\cdot, y)(x)| \leq \widetilde{c}_3 (s-t)^{1-(1/\alpha)} \rho_\phi(s-t, x-y) \quad \text{on } \mathbb{D}_0^T. \quad (5.1)$$

When  $\alpha = 1$ , by the same calculation as that for (1.29), we have  $M_\ell^\phi(t) \asymp \frac{\Gamma_\ell(t)}{\ell(t)}$  on  $(0, 1]$  and so the gradient estimate (1.24) has the form

$$|\nabla p_{t,s}^\kappa(\cdot, y)(x)| \leq \widetilde{c}_3 \frac{\Gamma_\ell(s-t)}{\ell(s-t)} \rho_\phi(s-t, x-y) \quad \text{on } \mathbb{D}_0^T. \quad (5.2)$$

We have the following more general result.



**Proposition 5.4.** *Suppose that  $\phi$  is an increasing function on  $\mathbb{R}_+$  so that  $\phi \in \mathcal{R}_\alpha$  on  $(0, 1]$  and  $r \mapsto \phi(1/r) \in \mathcal{R}_\beta$  for  $\alpha \in (0, 1)$  and  $\beta < 0$  or for  $\alpha \in (1, 2)$  and  $\beta \in (-\infty, -1) \cup (-1, 0)$ . Then (1.8) and  $(\mathbf{A}_\phi^{(1)})$  hold*

*Proof.* By definition, there are some slowly varying functions  $\ell, \ell' \in \mathcal{S}_0$  so that

$$\phi(r) = r^\alpha \ell(r) \text{ when } r \in (0, 1) \quad \text{and} \quad \phi(r) = r^{-\beta} \ell'(1/r) \text{ when } r \in (1, \infty).$$

Take  $0 < \varepsilon < \alpha \wedge (-\beta)$ . By (3.1), we see that (1.8) holds with  $\beta_1 := \alpha - \varepsilon$  and  $\beta_2 := (\alpha + \varepsilon) \vee (\varepsilon - \beta)$ .

Suppose  $\alpha \in (0, 1)$  and  $\beta < 0$ . Then Case $_1^\phi$  holds. By (3.1) again, we have for  $\delta \in (0, 1 - \alpha)$ ,

$$\sup_{\lambda \in (0, 1]} \int_0^1 \frac{\gamma_\phi(r) \phi(\lambda)}{r \phi(r\lambda)} dr = \sup_{\lambda \in (0, 1]} \int_0^1 \frac{\phi(\lambda)}{\phi(r\lambda)} dr \lesssim \int_0^1 r^{-\delta - \alpha} dr < \infty,$$

and for  $\delta \in (0, -\beta)$

$$\begin{aligned} \sup_{\lambda \in (0, 1]} \int_1^\infty \frac{\gamma_\phi(r) \phi(\lambda)}{r \phi(r\lambda)} dr &= \sup_{\lambda \in (0, 1]} \int_1^\infty \frac{\phi(\lambda)}{r \phi(r\lambda)} dr \\ &= \sup_{\lambda \in (0, 1]} \phi(\lambda) \left( \int_\lambda^1 \frac{1}{s \phi(s)} ds + \int_1^\infty \frac{1}{s \phi(s)} ds \right) \\ &\lesssim \sup_{\lambda \in (0, 1]} \left( 1 + \phi(\lambda) \int_1^\infty \frac{dr}{s^{1-\beta-\delta}} ds \right) < \infty, \end{aligned} \quad (5.3)$$

where in the last inequality we used (1.9). Thus  $(\mathbf{A}_\phi^{(1)})$  holds when  $\alpha \in (0, 1)$  and  $\beta < 0$ .

When  $\alpha \in (1, 2)$ , we have by (3.1) that  $\int_{0+} \frac{1}{\phi(r)} dr = \infty$  and for  $\delta \in (0, 2 - \alpha)$ ,

$$\sup_{\lambda \in (0, 1]} \int_0^1 \frac{\gamma_\phi(r) \phi(\lambda)}{r \phi(r\lambda)} dr = \sup_{\lambda \in (0, 1]} \int_0^1 \frac{r^2 \phi(\lambda)}{r \phi(r\lambda)} dr \lesssim \int_0^1 r^{1-\delta-\alpha} dr < \infty.$$

When  $\beta \in (-1, 0)$ , Case $_2^\phi$  holds. In this case, we have by (5.3),

$$\sup_{\lambda \in (0, 1]} \int_1^\infty \frac{\gamma_\phi(r) \phi(\lambda)}{r \phi(r\lambda)} dr = \sup_{\lambda \in (0, 1]} \int_1^\infty \frac{\phi(\lambda)}{r \phi(r\lambda)} dr < \infty.$$

When  $\beta \in (-\infty, -1)$ , Case $_3^\phi$  holds. In this case, for some  $0 < \delta < \min\{\alpha - 1, -\beta - 1\}$ , we have by (3.1) that

$$\begin{aligned} \sup_{\lambda \in (0, 1]} \int_1^\infty \frac{\gamma_\phi(r) \phi(\lambda)}{r \phi(r\lambda)} dr &= \sup_{\lambda \in (0, 1]} \int_1^\infty \frac{\phi(\lambda)}{\phi(r\lambda)} dr = \sup_{\lambda \in (0, 1]} \frac{1}{\lambda} \int_\lambda^\infty \frac{\phi(\lambda)}{\phi(s)} ds \\ &= \sup_{\lambda \in (0, 1]} \left( \frac{1}{\lambda} \int_\lambda^1 \frac{\phi(\lambda)}{\phi(s)} ds + \frac{\phi(\lambda)}{\lambda} \int_1^\infty \frac{1}{\phi(s)} ds \right) \\ &\lesssim \sup_{\lambda \in (0, 1]} \left( \frac{1}{\lambda} \int_\lambda^1 (\lambda/s)^{\alpha-\delta} ds + \lambda^{\alpha-\delta-1} \int_1^\infty s^{\delta+\beta} ds \right) \\ &\lesssim 1 + 1 < \infty. \end{aligned}$$

Hence  $(\mathbf{A}_\phi^{(1)})$  holds when  $\alpha \in (1, 2)$  and  $\beta \in (-\infty, -1) \cup (-1, 0)$ .  $\square$

**Example 5.5.** Let  $\ell(s) = (\log(1/s))^{-2}$  and  $\phi(s) = s \log(1/s)$  for  $0 < s \ll 1$ . It is easy to see that

$$\ell \in \mathcal{S}_0 \cap \mathcal{D}_0 \quad \text{and} \quad \int_{0+} 1/\phi(s) ds = \infty.$$

However,

$$\int_{0+} \frac{\ell(s)}{s} d\phi(s) = \int_{0+} \frac{(\log(1/s) + 1)}{s(\log(1/s))^2} ds = \infty.$$

Hence in this case, (1.13) does not hold.

**Example 5.6.** Suppose that  $\ell \in \mathcal{R}_\eta$  for some  $\eta \geq 0$  and  $\phi$  is an increasing function on  $[0, 1]$  such that there are  $\beta_1 > 0$  and  $c_1 > 0$  so that

$$\phi(R)/\phi(r) \geq c_1(R/r)^{\beta_1} \quad \text{for any } 0 < r < R \leq 1. \quad (5.4)$$

Then condition (1.13) holds if  $\beta_1 > 1/2$  and  $\beta_1 + \eta > 1$ , and in this case

$$M_\ell^\phi(t) := \int_0^t \frac{1}{r} \left( \frac{\ell(r)}{\ell(t)} + \frac{\phi(r)}{\phi(t)} \right) d\phi(r) \asymp \frac{\phi(t)}{t} \quad \text{for } t \in (0, 1]. \quad (5.5)$$

Indeed, there is  $\ell_0 \in \mathcal{S}_0$  so that  $\ell(r) = r^\eta \ell_0(r)$ . By Proposition 3.1(i), for any  $\delta \in (0, (\beta_1 + \eta - 1) \wedge 1)$ , there is  $c_0 > 0$  so that

$$\frac{\ell(s)}{\ell(t)} \leq c_0 \left( \frac{s}{t} \right)^{\eta - \delta} \quad \text{for } 0 < s < t \leq 1.$$

Thus by integration by parts and the fact that  $\beta_1 + \eta - \delta > 1$ , we have for  $t \in (0, 1]$ ,

$$\int_0^t \frac{\ell(r)}{r\ell(t)} d\phi(r) \lesssim t^{\delta - \eta} \int_0^t r^{\eta - \delta - 1} d\phi(r) = t^{\delta - \eta} \left( t^{\eta - \delta - 1} \phi(t) + (1 - \eta - \delta) \int_0^t \phi(r) r^{\eta - \delta - 2} dr \right). \quad (5.6)$$

By (5.4),

$$\int_0^t \phi(r) r^{\eta - \delta - 2} dr = \phi(t) \int_0^t \frac{\phi(r)}{\phi(t)} r^{\eta - \delta - 2} dr \leq \frac{\phi(t)}{c_1} \int_0^t (r/t)^{\beta_1} r^{\eta - \delta - 2} dr = \frac{\phi(t) t^{\eta - \delta - 1}}{c_1(\beta_1 + \eta - \delta - 1)}.$$

This together with (5.6) shows that

$$\int_0^t \frac{\ell(r)}{r\ell(t)} d\phi(r) \asymp \frac{\phi(t)}{t} \quad \text{for } t \in (0, 1]. \quad (5.7)$$

On the other hand, for  $t \in (0, 1]$ , by integration by parts and the assumption that  $\beta_1 > 1/2$ ,

$$\int_0^t \frac{\phi(r)}{r\phi(t)} d\phi(r) = \frac{1}{2\phi(t)} \int_0^t r^{-1} d(\phi(r)^2) = \frac{1}{2\phi(t)} \left( t^{-1} \phi(t)^2 + \int_0^t r^{-2} \phi(r)^2 dr \right). \quad (5.8)$$

By (5.4),

$$\int_0^t r^{-2} \phi(r)^2 dr = \phi(t)^2 \int_0^t r^{-2} \left( \frac{\phi(r)}{\phi(t)} \right)^2 dr \leq \frac{\phi(t)^2}{c_1} \int_0^t r^{-2} (r/t)^{2\beta_1} dr = \frac{\phi(t)^2}{c_1(2\beta_1 - 1)t}.$$

It follows from (5.8) that

$$\int_0^t \frac{\phi(r)}{r\phi(t)} d\phi(r) \asymp \frac{\phi(t)}{t} \quad \text{for } t \in (0, 1].$$

This together with (5.7) proves the claim (5.5).

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