PERIODS AND REAL NUMBERS

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ABSTRACT

The purpose of this book is to provide an introduction to period theory and then to place it within the matrix of recursive function theory.

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CONTENTS

According to M. Kontsevich and D. Zagier:

* * * * *

"A <u>period</u> is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains in R^n·given by polynomial inequalities with rational coefficients".

Ref: *Mathematics Unlimited – 2001 and Beyond*, Springer, pp. 771–808.

§1. *SEMIALGEBRAIC SETS*

1: NOTATION Put

$$R_{alg} = R \cap \bar{Q}.$$

Therefore R_{alg} is the field of real algebraic numbers, so

$$Q \subset R_{alg} \subset \bar{Q}$$
$$\cap \qquad \cap$$
$$R \quad \subset \quad C.$$

[Note: Viewed as a vector space over Q, R_{alg} is infinite dimensional (the algebraic numbers $\sqrt{p}$ (p a prime) are independent over Q).]

2: N.B. R_{alg} is a real closed field and R_{alg} is the real closure of Q.

[Note: R_{alg} carries the relative topology per R, hence is totally disconnected, i.e., its connected components are points.]

3: DEFINITION A subset $X \subset R^n$ is <u>semialgebraic</u> if there are natural numbers r and $s_1, \ldots, s_r$ and polynomials

$$f_{ij}, \; g_{ij} \in R_{alg} [T_1, \ldots, T_n],$$

where $1 \leq i \leq r$ and $1 \leq j \leq s_i$, such that

$$X = \bigcup_{i-1}^{r} \bigcap_{j=1}^{s_i} \{x \in R^n : f_{ij}(x) = 0\} \cup \{x \in R^n : g_{ij}(x) > 0\}.$$

4: EXAMPLE R^n is semialgebraic.

5: REMARK An algebraic set in R^n defined using polynomials with coefficients in R_{alg} is semialgebraic.

<u>6:</u> LEMMA The semialgebraic sets are closed under the formation of finite unions and finite intersections and are closed under complementation.

<u>7:</u> LEMMA If $X \subset R^n$ is semialgebraic and $Y \subset R^m$ is semialgebraic, then $X \times Y \subset R^n \times R^m$ is semialgebraic.

<u>8:</u> EXAMPLE Let $\rho \in R_{alg}$, $\underline{x} = (x_1, \ldots, x_n) \in R^n_{alg}$ -- then

$$
\left[
\begin{array}{l}
\{(y_1, \ldots, y_n) \in R^n : \sum_{i=1}^{n} (y_i - x_i)^2 < \rho^2\} \\[2em]
\{(y_1, \ldots, y_n) \in R^n : \sum_{i=1}^{n} (y_k - x_i)^2 \leq \rho^2\}
\end{array}
\right.
$$

are semialgebraic as are

$$
\left[
\begin{array}{l}
\{(y_1, \ldots, y_n) \in R^n : \max_{i=1,\ldots,n} |y_i - x_i| < \rho\} \\[2em]
\{(y_1, \ldots, y_n) \in R^n : \max_{i=1,\ldots,n} |y_i - x_i| \leq \rho\}.
\end{array}
\right.
$$

<u>9:</u> EXAMPLE

$$\{(x,y) \in R^2 : \exists\, n \in N,\ y = nx\}$$

is not semialgebraic.

<u>10:</u> EXAMPLE

$$\{(x,y) \in R^2 : y = e^x\}$$

is not semialgebraic.

<u>11:</u> DEFINITION

- A <u>basic open</u> semialgebraic subset of R^n is a set of the form

$$\{x \in R^n: f_1(x) > 0,\ldots,f_r(x) > 0\},$$

where

$$f_1,\ldots,f_r \in R_{alg}[T_1,\ldots,T_n].$$

- A <u>basic closed</u> semialgebraic subset of R^n is a set of the form

$$\{x \in R^n: f_1(x) \geq 0,\ldots,f_r(x) \geq 0\},$$

where

$$f_1,\ldots,f_r \in R_{alg}[T_1,\ldots,T_n].$$

<u>12:</u> LEMMA Suppose that X is an open (closed) semialgebraic set -- then X is a finite union of basic open (basic closed) semialgebraic sets.

<u>13:</u> DEFINITION Let $X \subset R^n$, $Y \subset R^m$ be semialgebraic sets -- then a <u>semialgebraic map</u> $f:X \to Y$ is a continuous function such that the graph $\Gamma_f \subset X \times Y$ is a semialgebraic subset of $R^n \times R^m$.

<u>14:</u> SCHOLIUM Let

$$P_1,\ldots,P_m$$

be elements of $R_{alg}[T_1,\ldots,T_n]$ -- then the arrow $f:X \to Y$ defined by the prescription

$$(x_1,\ldots,x_n) \to (P_1(x_1,\ldots,x_n),\ldots,P_m(x_1,\ldots,x_n))$$

is a semialgebraic map.

[Note: One can replace $P_1,\ldots,P_m$ by elements

$$\frac{P_1}{Q_1},\ldots,\frac{P_m}{Q_m}$$

of $R_{alg}(T_1, \ldots, T_m)$ provided that none of the

$$Q_1, \ldots, Q_m$$

vanish at any point of X.]

__15:__ EXAMPLE The graph of $f(x) = \sqrt{x}$ is the semialgebraic set

$\{(x,y) \in R^2 : y^2 = x,\ y \geq 0\}$, so f is a semialgebraic map.

__16:__ EXAMPLE If X is semialgebraic, then the diagonal $\Delta : X \to X \times X$ is a semialgebraic map.

[The graph of Δ is the intersection of $X \times X \times X$ with the semialgebraic set

$$\{(x,y,z) : x = y = z\}.]$$

__17:__ EXAMPLE If $\emptyset \neq X \subset R^n$ is a semialgebraic set, then the distance function $x \to \mathrm{dist}(x,X)$ is a semialgebraic map.

__18:__ DEFINITION Let $X \subset R^n$ be an open semialgebraic subset --- then an analytic function $\varphi : X \to R$ is said to satisfy the __condition of Nash__ if there are elements

$$a_0, a_1, \ldots, a_d \neq 0$$

of $R_{alg}[T_1, \ldots, T_n]$ such that

$$a_0 + a_1\varphi + \cdots + a_d\varphi^d = 0,$$

i.e., φ is an analytic algebraic function. Such a function is necessarily a semialgebraic map.

__19:__ EXAMPLE Let $X = \,]-1,\ 1[\ \subset R$ and take $\varphi(x) = \sqrt{1 - x^2}$ --- then

$$\varphi(x)^2 - (1 - x^2) = 0,$$

so φ satisfies the condition of Nash.

20: Let $X \subset R^n$, $X' \subset R^{n'}$, $Y \subset R^m$, $Y' \subset R^{m'}$ be semialgebraic sets and let $f:X \to Y$, $f':X' \to Y'$ be semialgebraic maps — then

$$f \times f':X \times X' \to Y \times Y'$$

is a semialgebraic map.

PROOF The graph of $f \times f'$ is

$$\{(x,x',y,y'): y = f(x), y' = f'(x')\}$$

which is the intersection of

$$\{(x,x',y,y'): y = f(x)\} \simeq \Gamma_f \times R^{n'} \times R^{m'}$$

and

$$\{(x,x',y,y'): y' = f(x')\} \simeq \Gamma_{f'} \times R^n \times R^m\}.$$

21: THEOREM (Tarski-Seidenberg) If $f:R^n \to R^m$ is a polynomial function (cf. #14) and $X \subset R^n$ is a semialgebraic set, then $f(X)$ is semialgebraic.

22: SCHOLIUM Let $X \subset R^{n+m}$ be semialgebraic, $\Pi:R^{n+m} \to R^n$ the projection onto the space of the first n coordinates (or $\Pi:R^{n+m} \to R^m$ the projection onto the space of the second m coordinates) — then $\Pi(X)$ is a semialgebraic subset of R^n (or R^m).

23: LEMMA Let $X \subset R^n$, $Y \subset R^m$ be semialgebraic sets and let $f:X \to Y$ be a semialgebraic map — then $f(X)$ is a semialgebraic set.

PROOF Γ_f is a semialgebraic set by assumption. Now apply #22 to

$$\Gamma_f \subset R^n \times R^m \equiv R^{n+m} \to R^m.$$

<u>24:</u> APPLICATION If $X \subset R^n$, $Y \subset R^m$, $Z \subset R^\ell$ are semialgebraic sets and if $f:X \to Y$, $g:Y \to Z$ are semialgebraic maps, then $g \circ f$ is a semialgebraic map.

[In fact,

$$\Gamma_{g \circ f} = (1_X \times g)(\Gamma_f).]$$

<u>25:</u> <u>N.B.</u> It follows that there is a category whose objects are the semialgebraic sets and whose morphism are the semialgebraic maps.

<u>26:</u> REMARK Let S, X, Y be semialgebraic sets and let $f:X \to S$, $g:Y \to S$ be semialgebraic maps --- then the fiber product $X \times_S Y$ of sets is a semialgebraic set and is a fiber product in the category of semialgebraic sets and semialgebraic maps.

<u>27:</u> LEMMA The inverse image of a semialgebraic set under a semialgebraic map is a semialgebraic set.

PROOF Let $X \subset R^n$, $Y \subset R^m$, $Z \subset Y$ be semialgebraic sets and suppose that $f:X \to Y$ is a semialgebraic map. Write

$$f^{-1}(Z) = \Pi(\Gamma_f \cap (R^n \times Z))$$

and apply #22.

<u>28:</u> <u>N.B.</u> The converse of this lemma is false.

[The exponential function $\exp:R \to R$ is not semialgebraic but does have the property that the inverse image of a semialgebraic set is a semialgebraic set.]

<u>29:</u> THEOREM Suppose that X is a semialgebraic set --- then the semialgebraic maps $X \to R$ form a ring under pointwise addition and multiplication.

<u>30:</u> LEMMA If $X \subseteq R^n$ is a semialgebraic set, then the closure, the interior, and the frontier of X are semialgebraic.

<u>31:</u> EXAMPLE Take

$$X = \{x \in R : (x^2 - 1)(x-2)^2 < 0\}.$$

Then $\bar{X} = [-1,1]$ but

$$\{x \in R : (x^2-1)(x-2)^2 \leq 0\} = \bar{X} \cup \{2\}.$$

<u>Ref</u>: *Real Algebraic and Semialgebraic Sets*, R. Benedetti, J-J Risler, Hermann, 1990.

<u>Ref</u>: *Real Algebraic Geometry*, J. Bochnak, M. Coste, M-F Roy, Springer, 1998.

§2. *PERIODS: DEFINITION AND EXAMPLES*

1: NOTATION SA^n is the set of semialgebraic subsets of R^n with a non-empty interior.

2: DEFINITION A real number p is a <u>period</u> in the sense of Kontsevich-Zagier if

- $\exists\ n \in N$,

- $\exists\ X \in SA^n$,

- $\exists\ P, Q \in R_{alg}[T_1, \ldots, T_n]$ $(Q|X \neq 0)$

such that

$$p = I(X, \frac{P}{Q}) \equiv \int_X \frac{P(x_1, \ldots, x_n)}{Q(x_1, \ldots, x_n)}\, dx_1 \ldots dx_n$$

$$\equiv \int_X \frac{P}{Q}\, (\underline{x})\, d\underline{x}$$

is an absolutely convergent integral, $d\underline{x} = dx_1 \ldots dx_n$ being Lebesgue measure.

3: NOTATION Write

$$P_{KZ}$$

for the set of periods in the sense of Kontsevich-Zagier.

[Note: We shall work exclusively in the real domain, a period in the complex domain being a combination $p_1 + \sqrt{-1}\, p_2$, where $p_1, p_2 \in P_{KZ}$.]

4: LEMMA

$$R_{alg} \subset P_{KZ}.$$

[Take $\rho \in R_{alg}$, $\rho > 0$ — then

$$\rho = \int_{0 < x < \rho} 1\ dx.]$$

<u>5:</u> **EXAMPLE** Suppose that $\rho \in R_{alg}$, $\rho > 1$ — then

$$\ell n(\rho) = \int_1^\rho \frac{dt}{t} \in P_{KZ}.$$

[Note: Observe too that

$$\int_1^\rho \frac{dt}{t} = \int_{\substack{1 < x < \rho \\ 0 < xy < 1}} 1\ dxdy.]$$

<u>6:</u> **EXAMPLE** Consider

$$\int_{x^2 + y^2 \leq 1} 1\ dxdy$$

to see that $\pi \in P_{KZ}$.

[It is unknown whether $\frac{1}{\pi} \in P_{KZ}$ or not.]

<u>7:</u> **EXAMPLE**

$$\zeta(2) = \int_{1 > x_1 > x_2 > 0} \frac{dx_1}{x_1} \cdot \frac{dx_2}{1 - x_2}$$

$$\in P_{KZ}.$$

<u>8:</u> **EXAMPLE** Take for S the square

$$\left\lceil \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq 1. \end{array} \right.$$

Let

$$f(x,y) = \frac{xy}{(x^2 + y^2)^2}$$

if $x^2 + y^2 > 0$ and set $f(0,0) = 0$ — then f is not integrable. For if it were,

then the iterated integral

$$\int_0^1 dx \int_0^1 \frac{xy}{(x^2+y^2)^2} \, dy$$

would exist (Fubini). But for $x \neq 0$,

$$\int_0^1 \frac{xy}{(x^2+y^2)^2} \, dy = \frac{1}{2x} - \frac{x}{2(x^2+1)}$$

and this function is not integrable in $]0,1]$.

[Note: Replace

$$\frac{xy}{(x^2+y^2)^2} \quad \text{by} \quad \frac{x^2-y^2}{(x^2+y^2)^2} \; .$$

Then

$$\left|
\begin{array}{l}
\int_0^1 dx \int_0^1 f(x,y) dy = \frac{\pi}{4} \\[2em]
\int_0^1 dy \int_0^1 f(x,y) dx = -\frac{\pi}{4} \; .]
\end{array}
\right.$$

§3. PERIODS: ALGEBRAIC CONSIDERATIONS

It was pointed out in §2, #4 that

$$R_{alg} \subset P_{KZ}.$$

Of course R_{alg} is a countable field. And:

<u>1:</u> LEMMA P_{KZ} is countable.

PROOF $\forall\ n \in N$, $R_{alg}[T_1, \ldots, T_n]$ is countable, hence $R_{alg}(T_1, \ldots, T_n)$ is countable, as is SA^n, hence P_{KZ} is countable.

[Note: Consequently "most" real numbers are not periods. And if a real number is not a period, then it is transcendental.]

<u>2:</u> REMARK It is unknown if P_{KZ} is a field (but P_{KZ} is an R_{alg} algebra (see #8 below).

<u>3:</u> DISCUSSION Suppose that

$$p = I(X, \frac{P}{Q}) = \int_X \frac{P}{Q}\ (\underline{x})\,d\underline{x} \quad \text{(cf. §2, #2)}$$

and

$$\int_X \left| \frac{P}{Q}\ (\underline{x}) \right| d\underline{x} < \infty,$$

hence

$$\int_X \left| \frac{P}{Q}\ (\underline{x}) \right| d\underline{x} = 0$$

iff

$$\left| \frac{P}{Q}\ (x) \right| = 0$$

almost everywhere, thereby forcing $p = 0$.

<u>4</u>: NOTATION Given a measurable subset $X \subset R^n$, let

$$\mathrm{vol}_n(X) = \int_X 1 \, dx_1 \ldots dx_n$$

$$\equiv \int_X 1 \, d\underline{x}.$$

<u>5</u>: EXAMPLE Consider the n-simplex $\triangle_n$ -- then

$$\mathrm{vol}_n(\triangle_n) = \frac{1}{n!} \, .$$

<u>6</u>: EXAMPLE Consider the n-ball B^n -- then

$$\mathrm{vol}_n(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \, .$$

<u>7</u>: LEMMA Let $p \in R$ -- then $p \in P_{KZ}$ iff for some $d \in N$, there exist disjoint semialgebraic sets $X_1 \subset R^d$, $X_2 \subset R^d$ of finite volume such that

$$p = \mathrm{vol}_d(X_1) - \mathrm{vol}_d(X_2).$$

PROOF It can be assumed that $p \neq 0$.

● Given $p \in P_{KZ}$, put

$$\left[\begin{array}{l} X_+ = \{\underline{x} \in X : \mathrm{sgn} \, \frac{P}{Q}(\underline{x}) = + 1\} \\[2ex] X_- = \{\underline{x} \in X : \mathrm{sgn} \, \frac{P}{Q}(\underline{x}) = - 1\}, \end{array}\right.$$

a disjoint partition of X (to within a set of measure 0) (cf. #3) -- then

$$p = I(X, \frac{P}{Q}) = I(X_+, \frac{P}{Q}) - I(X_-, -\frac{P}{Q}).$$

Now introduce semialgebraic sets

$$X_1 = \{(\underline{x},t) \in X \times R : t > 0, \; t \le \frac{P}{Q}(\underline{x})\}$$

$$X_2 = \{(x,t) \in X \times R : t < 0, \; t \ge \frac{P}{Q}(x)\},$$

from which

$$I(X_+, \frac{P}{Q}) = \int_{X_1} 1 \; d\underline{x}dt$$

$$I(X_-, -\frac{P}{Q}) = \int_{X_2} 1 \; d\underline{x}dt.$$

Therefore

$$p = I(X, \frac{P}{Q}) = \int_{X_1} 1 \; d\underline{x}dt - \int_{X_2} 1 \; d\underline{x}dt$$

$$= vol_{n+1}(X_1) - vol_{n+1}(X_2).$$

Matters are thus settled with the choice $d = n + 1$.

● Suppose that

$$p = vol_d(X_1) - vol_d(X_2),$$

the claim being that $p \in P_{KZ}$. To see this, write

$$p = \int_{X_1} 1 \; dx_1 \ldots dx_d - \int_{X_2} 1 \; dx_1 \ldots dx_d$$

$$= \int_{X_1} (\int_0^1 2t dt) dx_1 \ldots dx_d + \int_{X_2} (\int_{-1}^0 2t dt) dx_1 \ldots dx_d$$

$$= \int_{Y_1 \cup Y_2} 2t dt dx_1 \ldots dx_d,$$

where

$$Y_1 = \;]0,1[\; \times X_1, \; Y_2 = \;]-1,0[\; \times X_2$$

are disjoint semialgebraic sets in $R^1 \times R^d$. Therefore $p \in P_{KZ}$.

$\underline{8:}$ THEOREM P_{KZ} is an R_{alg} algebra.

PROOF There are two issues:

$$\left[\begin{array}{l} \text{Stability of the product of two periods} \\ \text{Stability of the sum of two periods.} \end{array} \right.$$

● (Product) Given

$$p_1 = \int_{X_1} \frac{P_1}{Q_1} (\underline{x}_1) d\underline{x}_1, \quad p_2 = \int_{X_2} \frac{P_2}{Q_2} (\underline{x}_2) d\underline{x}_2,$$

write

$$\left[\begin{array}{l} \mathrm{pro}_1 : R^{d_1} \times R^{d_2} \to R^{d_1} \\ \\ \mathrm{pro}_2 : R^{d_1} \times R^{d_2} \to R^{d_2} \end{array} \right.$$

and define

$$F : R^{d_1} \times R^{d_2} \to R$$

by

$$F(\underline{x}_1, \underline{x}_2) = \frac{P_1}{Q_1} \circ \mathrm{pro}_1(x_1, x_2) \cdot \frac{P_2}{Q_2} \circ \mathrm{pro}_2(\underline{x}_1, x_2).$$

Then

$$\int_{X_1 \times X_2} F(\underline{x}_1, \underline{x}_2) d\underline{x}_1 d\underline{x}_2$$

$$= (\int_{X_1} \frac{P_1}{Q_1} (\underline{x}_1) d\underline{x}_1) (\int_{X_2} \frac{P_2}{Q_2} (\underline{x}_2) d\underline{x}_2)$$

$$= p_1 p_2.$$

- (Sum) Let $p_1, p_2 \in P_{KZ}$. Per #7, write

$$\left[\begin{array}{l} p_1 = \text{vol}_{d_1}(X_1) - \text{vol}_{d_1}(X_2) \\[2mm] p_2 = \text{vol}_{d_2}(Y_1) - \text{vol}_{d_2}(Y_2), \end{array} \right.$$

where

$$\left[\begin{array}{l} X_1 \subset R^{d_1}, \; X_2 \subset R^{d_1} \\[3mm] Y_1 \subset R^{d_2}, \; T_2 \subset R^{d_2} \end{array} \right.$$

are semialgebraic and

$$X_1 \cap X_2 = \emptyset, \; Y_1 \cap Y_2 = \emptyset.$$

There is no loss of generality in assuming that $d_1 = d_2 = d$ (if, e.g., $d_1 < d_2$,

let $k = d_2 - d_1$ and work with $X_1 \times [0,1]^k$, $X_2 \times [0,1]^k$). This said, it then follows

that

$$p_1 + p_2 = \text{vol}_d(X_1) + \text{vol}_d(Y_1) - \text{vol}_d(X_2) - \text{vol}_d(Y_2)$$

or still,

$$p_1 + p_2 = \text{vol}_d(X_1 \cup Y_1) + \text{vol}_d(X_1 \cap Y_1) - \text{vol}_d(X_2 \cup Y_2) - \text{vol}_d(X_2 \cap Y_2).$$

Put

$$\left[\begin{array}{l} W_1 = X_1 \cup Y_1, \; W_2 = X_2 \cup Y_2 \\[2mm] Z_1 = X_1 \cap Y_1, \; Z_2 = X_2 \cap Y_2 \end{array} \right.$$

and let $I_1, J_1, I_2, J_2 \subset R$ be disjoint open intervals of length 1 with endpoints

in R_{alg} -- then

$$p_1 + p_2 = \text{vol}_{d+1}((W_1 \times I_1) \cup (Z_1 \times J_1))$$

$$- \mathrm{vol}_{d+1}((W_2 \times I_2) \cup (Z_2 \times J_2)).$$

Here

$$((W_1 \times I_1) \cup (Z_1 \times J_1)) \cap ((W_2 \times I_2) \cup (Z_2 \times J_2))$$

$$= \emptyset.$$

In addition

$$\begin{bmatrix} (W_1 \times I_1) \cup (Z_1 \times J_1) \\ \\ (W_2 \times I_2) \cup (Z_2 \times J_2) \end{bmatrix}$$

are semialgebraic subsets of R^{d+1}, thus to finish it remains only to quote #7.

<u>9:</u> APPLICATION Let $x \in R^{\times}_{\mathrm{alg}}$, $p \in \mathcal{P}_{KZ}$ --- then

$$\begin{bmatrix} x + p \in \mathcal{P}_{KZ} \\ \\ xp \in \mathcal{P}_{KZ}. \end{bmatrix}$$

1.

§4. PERIODS: THEORETICAL CONSIDERATIONS

<u>1:</u> THEOREM Let

$$p = I(X, \frac{P}{Q}) \quad (X \in SA^n)$$

be a nonzero period — then there are compact semialgebraic sets $K_1, \ldots, K_m \in SA^n$, polynomials

$$\begin{aligned} P_1, \ldots, P_m &\in R_{alg}[T_1, \ldots, T_n] \\ Q_1, \ldots, Q_m &\in R_{alg}[T_1, \ldots, T_n] \end{aligned}$$

with $Q_1 | K_1 \neq 0, \ldots, Q_m | K_m \neq 0$ such that

$$I(X, \frac{P}{Q}) = \sum_{i=1}^{m} I(K_i, \frac{P_i}{Q_i}).$$

[This result is due to Juan Vin-Sos[†]. Its proof is difficult, depending, as it does, on Hironaka's rectilinearization of semialgebraic sets.]

<u>2:</u> N.B. The integrals

$$I(K_i, \frac{P_i}{Q_i})$$

are absolutely convergent.

<u>3:</u> LEMMA Let

$$p = I(X, \frac{P}{Q}) \quad (X \in SA^n)$$

be a nonzero period — then there are compact semialgebraic sets S and T in SA^{n+1}

[†] arXiv: 1509.01097 [math.NT].

such that

$$p = \text{vol}_{n+1}(S) - \text{vol}_{n+1}(T).$$

PROOF Proceed as in §3, #7 (necessity), thus

$$I(X, \frac{P}{Q}) = I(X_+, \frac{P}{Q}) - I(X_-, -\frac{P}{Q}).$$

Per #1, write

$$
\left[
\begin{array}{l}
I(X_+, \frac{P}{Q}) = \sum_{i=1}^{m_+} I(K_i^+, \frac{P_i^+}{Q_i^+}) \\[2em]
I(X_-, -\frac{P}{Q}) = \sum_{i=1}^{m_-} I(K_i^-, \frac{P_i^-}{Q_i^-})
\end{array}
\right.
$$

or still,

$$
\left[
\begin{array}{l}
I(X_+, \frac{P}{Q}) = \sum_{i=1}^{m_+} \int_{L_i^+} 1 \, \underline{dxdt} \\[2em]
I(X_-, -\frac{P}{Q}) = \sum_{i=1}^{m_-} \int_{L_i^-} 1 \, \underline{dxdt}
\end{array}
\right.
$$

or still,

$$
\left[
\begin{array}{l}
I(X_+, \frac{P}{Q}) = \sum_{i=1}^{m_+} \text{vol}_{n+1}(L_i^+) \\[2em]
I(X_-, -\frac{P}{Q}) = \sum_{i=1}^{m_-} \text{vol}_{n+1}(L_i^-),
\end{array}
\right.
$$

where $L_i^+, L_i^- \in SA^{n+1}$ are compact. Working with semialgebraic translations, it can be arranged that the L_i^+ are pairwise disjoint and the L_i^- are pairwise disjoint.

Put now

$$S = \bigcup_{i=1}^{m_+} L_i^+, \quad T = \bigcup_{i=1}^{m_-} L_i^- .$$

Then

$$p = I(X, \frac{P}{Q})$$

$$= \sum_{i=1}^{m_+} \text{vol}_{n+1}(L_i^+) - \sum_{i=1}^{m_-} \text{vol}_{n+1}(L_i^-)$$

$$= \text{vol}_{n+1}(\bigcup_{i=1}^{m_+} L_i^+) - \text{vol}_{n+1}(\bigcup_{i=1}^{m_-} L_i^-)$$

$$= \text{vol}_{n+1}(S) - \text{vol}_{n+1}(T).$$

__4:__ THEOREM Let $p \in P_{KZ}$ $(p \neq 0)$, say

$$p = I(X, \frac{P}{Q}) \quad (X \in SA^n).$$

Then there exists a compact $K \in SA^k$ $(0 < k \leq n + 1)$ such that

$$p = \text{sgn}(p) \cdot \text{vol}_k(K).$$

It can be assumed that p is positive and in the notation of #3,

$$0 < \text{vol}_{n+1}(T) < \text{vol}_{n+1}(S).$$

The point then is to construct a compact semialgebraic set K from S and T so as to arrive at

$$p = \text{vol}_k(K) \quad (0 < k \leq n + 1).$$

While "elementary", the details are tedious and will be omitted.

<u>5</u>: EXAMPLE π^2 is the 4-dimensional volume of the product of two copies of the unit disk and the 3-dimensional volume of the set

$$\{(x,y,z) \in R^3 : x^2 + y^2 \leq 1, \ 0 \leq z((x^2 + y^2)^2 + 1) \leq 4\}.$$

<u>6</u>: DEFINITION Let $p \in P_{KZ}$ ($p \neq 0$) -- then the <u>degree</u> of p, denoted $\deg(p)$, is the smallest positive integer k such that

$$|p| = \mathrm{vol}_k(K) \quad (K \in SA^k, \ K \text{ compact}) \quad (\text{cf. } \#4).$$

[Note: Take $\deg(0) = 0$ and in the complex domain, let

$$\deg(p_1 + \sqrt{-1} \ p_2) = \max\{\deg(p_1), \ \deg(p_2)\}.]$$

<u>7</u>: EXAMPLE

$$\deg(\pi) = 2.$$

[In view of §2, #6, $\deg(\pi) \leq 2$. On the other hand, the fact that π is transcendental rules out the possibility that

$$\deg(\pi) = 1 \quad (\text{cf. } \#9 \text{ infra}).]$$

<u>8</u>: REMARK It is conjectured that $\forall \ n \in N$,

$$\deg(\pi^n) = n + 1.$$

<u>9</u>: LEMMA $\deg(p) = 1$ iff $p \in R^{\times}_{alg}$.

[Since K is compact semialgebraic, it can be written as a finite disjoint union of points and open intervals. In the other direction, any nonzero $\rho \in R_{alg}$ can be written up to sign as the length of $[0,\rho]$ (cf. §2, #4).]

__10:__ APPLICATION A period $p \in P_{KZ}$ is transcendental iff $\deg(p) \geq 2$.

__11:__ LEMMA Let $p_1, p_2 \in P_{KZ}$ -- then

● $\deg(p_1 + p_2) \leq \max\{\deg(p_1), \deg(p_2)\}$

and

● $\deg(p_1 p_2) \leq \deg(p_1) + \deg(p_2)$.

__12:__ RAPPEL P_{KZ} is an R_{alg} algebra (cf. §3, #8).

__13:__ LEMMA Let $x \in R_{alg}^{\times}$, $p \in P_{KZ}$. Assume: $p \notin R_{alg}$ -- then

$$\deg(x + p) = \deg(p).$$

[In fact,

$$\deg(p) = \deg(- x + x + p)$$

$$\leq \max\{\deg(- x), \deg(x + p)\}$$

$$= \max 1, \{\deg(x + p)\}$$

$$= \deg(x + p)$$

$$\leq \max\{1, \deg(p)\}$$

$$= \deg(p).]$$

__14:__ LEMMA Let $x \in R_{alg}^{\times}$, $p \in P_{KZ}$. Assume: $p \notin R_{alg}$ -- then

$$\deg(xp) = \deg(p).$$

[Consider $\sqrt[k]{|x|} \in R_{alg}^{\times}$ $(k = \deg(p))$.]

Period theory leads to some transcendental conclusions.

<u>15:</u> THEOREM Let p_1, p_2 be transcendental periods. Assume:

$$\deg(p_1) \neq \deg(p_2).$$

Then $p_1 + p_2$ is a transcendental number.

PROOF Assume false, so

$$\deg(p_1 + p_2) = 0 \text{ or } \deg(p_1 + p_2) = 1.$$

But

$$\deg(p_1 + p_2) = 0 \Rightarrow p_1 + p_2 = 0 \Rightarrow p_1 = -p_2 \Rightarrow \deg(p_1) = \deg(p_2),$$

leaving

$$\deg(p_1 + p_2) = 1,$$

thus $p_1 + p_2 \in R_{alg}^{\times}$ (cf. #10), say $p_1 + p_2 = x \in R_{alg}^{\times}$, from which

$$\deg(p_1) = \deg(x - p_2)$$

$$= \deg((-1)p_2) \quad (\text{cf. } \#13)$$

$$= \deg(p_2) \quad (\text{cf. } \#14).$$

Contradiction.

<u>16:</u> EXAMPLE It is conjectured that $e + \pi$ is transcendental. Recalling that $\deg(\pi) = 2$ (cf. #7), suppose for sake of argument that $e \in P_{KZ}$ and $\deg(e) \geq 3$, hence $\deg(e) \neq \deg(\pi)$. Recalling that e and π are transcendental, it follows that $e + \pi$ is transcendental.

[Note: Nevertheless, the conjecture is that e is not a period.]

<u>17:</u> THEOREM Let p_1, p_2 be transcendental periods. Assume:

$$\deg(p_1) \neq \deg(p_2).$$

Then p_1/p_2 is a transcendental number.

[For $p_1/p_2 = x \in R^\times_{alg}$ implies that

$$\deg(p_1) = \deg(xp_2) = \deg(p_2).]$$

§5. PRIMITIVE RECURSIVE FUNCTIONS

1: <u>N.B.</u> In this § (and all subsequent ones), N will stand for $\{0,1,2,\ldots\}$ and not $\{1,2,\ldots\}$. Elements of N will be denoted by x,y,z or a,b,c or n,m,k,ℓ.

2: <u>N.B.</u> In what follows, it will be a question of functions $f:N^n \to N$ $(n = 0,1,2,\ldots)$.

3: DEFINITION

- The <u>zero function</u> $Z:N \to N$, $Z(x) = 0$.

- The <u>successor function</u> $S:N \to N$, $S(x) = x + 1$.

- The <u>projection functions</u> $P_i^n:N^n \to N$, $P_i^n(x_1,\ldots,x_n) = x_i$.

[Note: These functions are the so-called <u>initial functions</u>.]

4: EXAMPLE Constant functions are built up from S and Z:
$$S(Z(x)) = 1, \ S(S(Z(x))) = 2 \text{ etc.}$$

5: EXAMPLE The <u>addition function</u> add:$N^2 \to N$ is defined by
$$add(x,y) = x + y.$$
Here
$$add(0,y) = y, \ add(x + 1,y) = S(add(x,y)).$$
[Note:
$$S \Rightarrow add.$$
For
$$x + y = \underbrace{x + 1 + 1 + \cdots + 1}_{y} \ .]$$

6: EXAMPLE The <u>multiplication function</u> $mul:N^2 \to N$ is defined by

$$mul(x,y) = xy.$$

Here

$$mul(0,y) = 0, \; mul(x + 1,y) = add\;(mul(x,y),y).$$

[Note:

$$add \Rightarrow mul.$$

For

$$xy = \underbrace{x + x + x + \cdots + x}_{y}.]$$

7: DEFINITION The <u>modified subtraction function</u> $sub:N^2 \to N$ is the pre-scription

$$sub(x,y) = x \;\dot{-}\; y = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases}$$

8: EXAMPLE The <u>absolute value function</u> $AV:N^2 \to N$ is defined by

$$AV(x,y) = |x - y|$$

and

$$|x - y| = add(x \;\dot{-}\; y,\; y \;\dot{-}\; x).$$

9: EXAMPLE The $\begin{bmatrix} \text{minimum} \\ \text{maximum} \end{bmatrix}$ function $\begin{bmatrix} min:N^2 \to N \\ max:N^2 \to N \end{bmatrix}$

is defined by

$$\begin{bmatrix} min(x,y) = x \;\dot{-}\;(x \;\dot{-}\; y) = y \;\dot{-}\;(y \;\dot{-}\; x) \\ max(x,y) = y + (x \;\dot{-}\; y) = x + (y \;\dot{-}\; x). \end{bmatrix}$$

<u>10:</u> EXAMPLE The <u>identity function</u> id_N can be obtained from S and Z and sub:

$$S(x) \;\overset{\cdot}{-\!\!-}\; S \circ Z(x).$$

<u>11:</u> DEFINITION Given a function $g : N^m \to N$ and given functions $h_i : N^n \to N$ $(i = 1, \ldots, m)$, the <u>composition of g and the h_i</u> is the function $f : N^n \to N$, denoted by

$$g \circ (h_1, \ldots, h_m),$$

such that

$$f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n)).$$

<u>12:</u> EXAMPLE Take $m = 2$, $g : N^2 \to N$ and take $n = 2$, $h_1 = P_2^2$, $h_2 = P_1^2$ — then

$$f = g \circ (P_2^2, P_1^2)$$

is given by

$$f(x_1, x_2) = g(P_2^2(x_1, x_2), P_1^2(x_1, x_2))$$

$$= g(x_2, x_1).$$

<u>13:</u> DEFINITION Suppose that $g : N^{m-1} \to N$, $h : N^{m+1} \to N$ — then the function $f : N^m \to N$ is said to be obtained by <u>primitive recursion</u> from g and h if

$$f(0, x_2, \ldots, x_m) = g(x_2, \ldots, x_m)$$

and

$$f(x + 1, x_2, \ldots, x_m) = h(x, f(x, x_2, \ldots, x_m), x_2, \ldots, x_m)$$

for all $x, x_2, \ldots, x_m \in N$.

14: EXAMPLE Take m = 1 — then $f:N \rightarrow N$ is obtained from the constant $c \in N$ $(g(0) = c)$ and $h:N^2 \rightarrow N$ by primitive recursion of $f(0) = c$ and $f(x + 1) = h(x,f(x))$.

15: DEFINITION If F is a set of natural number functions and if Ω is a set of operators on natural number functions, then $clos(F,\Omega)$ is the _inductive closure_ of F with respect to Ω, i.e., the smallest set of natural number functions containing F and closed with respect to the operations of Ω.

16: DEFINITION The set of _primitive recursive functions_, denoted by $\mathfrak{PR}$, is the inductive closure of the initial functions with respect to the operations of composition and primitive recursion.

17: EXAMPLE The _factorial function_ is primitive recursive:

$$0! = 1, \quad (n + 1)! = mul \circ (S \circ P_1^2, P_2^2)(n,n!).$$

18: REMARK All the functions encountered in this § are primitive recursive (but see below).

APPENDIX

Define the _Ackermann function_ $A:N^2 \rightarrow N$ by the following equations:

$$A(0,n) = n + 1$$

$$A(m + 1,0) = A(m,1)$$

$$A(m + 1,n + 1) = A(m,A(m + 1,n)).$$

Then A is not primitive recursive.

§6. ELEMENTARY FUNCTIONS

1: N.B. All conventions introduced in §5 remain in force.

2: DEFINITION Let $f(t,x_1,\ldots,x_n)$ be a function with $n + 1$ arguments.

● bounded summation is defined by

$$\sum_{t\leq x} f(t,x_1,\ldots,x_n) = f(0,x_1,\ldots,x_n) + \cdots + f(x,x_1,\ldots,x_n).$$

● bounded product is defined by

$$\prod_{t\leq x} f(t,x_1,\ldots,x_n) = f(0,x_1,\ldots,x_n) \times \cdots \times f(x,x_1,\ldots,x_n).$$

3: DEFINITION The set of elementary functions, denoted by EL, is the inductive closure of the initial functions, the addition, the multiplication, and the modified subtraction with respect to the operations of composition, bounded summation, and bounded product.

4: THEOREM

$$EL \subset \dot{P}R.$$

[Note: The containment is strict. Consider, e.g.,

$$f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}} \quad .]$$

5: EXAMPLE

$$\text{sgn}(x) = \begin{cases} 1 \text{ if } x \neq 0 \\ \\ 0 \text{ if } x = 0 \end{cases} = 1 \mathbin{\dot{-}} (1 \mathbin{\dot{-}} x)$$

is elementary.

6: EXAMPLE

$$> (x,y) = \begin{cases} 1 \text{ if } x > y \\ 0 \text{ if } x \le y \end{cases} = \text{sgn}(x \mathbin{\dot{-}} y)$$

is elementary.

[Ditto for $\ge$, $<$, $\le$.]

7: EXAMPLE

$$\text{div}(x,y) = \left\lfloor \frac{x}{y+1} \right\rfloor$$

is elementary.

[In fact,

$$\text{div}(x,y) = \left(\sum_{i=0}^{x} \ge (x, i(y+1)) \right) \mathbin{\dot{-}} 1.]$$

8: EXAMPLE

$$\text{pow}(n,m) = n^m$$

is elementary.

9: NOTATION Given a set F of natural number functions, let

$$F^* = F \cup \{P_i^n\}$$

and put

$$\text{clos}(F) = \text{clos}(F^*, \{\text{composition}\}).$$

[Note: Here, in the notation of §5, #13, Ω is the operation of composition.]

10: THEOREM

$$\text{EL} = \text{clos}(s, \text{sub}, \text{div}, \text{pow}).$$

[This result is due to Stefano Mazzanti [†].]

11: DEFINITION The set of <u>lower elementary functions</u>, denoted by ℓEL, is the inductive closure of the initial functions, the addition, the multiplication, and the modified subtraction with respect to the operations of composition and bounded summation.

12: N.B. Obviously

$$\ell\text{EL} \subset \text{EL}.$$

[Note: The containment is strict. Consider, e.g.,

$$f(n) = 2^n.]$$

[†] *Mathematical Logic Quarterly* 48 (2002), pp. 93–104.

§7. HIERARCHIES

Here is an approach to $E^n (n \in \mathbb{N})$, the nth set in the Grzegorczyk hierarchy.
[Note: Recall that n runs through 0,1,2,... and not 1,2,... .]

<u>1:</u> NOTATION Put

$$f_0(x,y) = x + 1, \quad f_1(x,y) = x + y, \quad f_2(x,y) = xy,$$

$$f_{n+1}(x,0) = 1, \quad f_{n+1}(x,y+1) = f_n(x,f_{n+1}(x,y)) \quad (n \geq 2).$$

<u>2:</u> DEFINITION E^n is the inductive closure of the initial functions and the nth f_n with respect to the operations of composition and bounded primitive recursion.

<u>3:</u> <u>N.B.</u> Suppose that $g:\mathbb{N}^{m-1} \to \mathbb{N}$, $h:\mathbb{N}^{m+1} \to \mathbb{N}$, $j:\mathbb{N}^m \to \mathbb{N}$ are in E^n, that f is obtained by primitive recursion from g and h, and in addition,

$$f(x,x_2,\ldots,x_m) \leq j(x,x_2,\ldots,x_m).$$

Then f is in E^n as well and f is said to be obtained by <u>bounded primitive recursion</u> from g,h,j.

<u>4:</u> REMARK If g and h belong to E^n and f is obtained by primitive recursion from g and h, then f belongs to E^{n+1}.

<u>5:</u> THEOREM $\forall$ n $\in \mathbb{N}$, E^n is a proper subset of E^{n+1}.

<u>6:</u> THEOREM $\forall$ n $\in \mathbb{N}$, $E^n \subset \mathcal{PR}$ and

$$\underset{n}{\cup} E^n = \mathcal{PR}.$$

Therefore the sets

$$E^0, \; E^1 - E^0, \; E^2 - E^1, \ldots, \; E^n - E^{n-1}, \; \ldots$$

partition PR.

7: THEOREM

$$E^3 = EL.$$

8: THEOREM

$$E^2 \supset \ell EL.$$

[Note: It is not known whether all functions in E^2 are lower elementary.]

9: N.B. For the record, ℓEL is the inductive closure of the initial functions, the addition, the multiplication, and the modified subtraction with respect to the operations of composition and bounded summation.

10: N.B. For the record, E^2 is the inductive closure of the initial functions, the addition, the multiplication, and the modified subtraction with respect to the operations of composition and bounded primitive recursion.

11: LEMMA Bounded summation can be derived from bounded primitive recursion.

Consequently

$$E^2 \supset \ell EL.$$

§8. COMPUTABILITY

Let F be a set of natural number functions.

__1:__ DEFINITION F is said to satisfy the <u>standard conditions</u> if it contains the initial functions, the addition, the multiplication, and the modified subtraction and is closed under composition.

__2:__ __N.B.__ Both ℓEL and E^2 satisfy the standard conditions.

In what follows, it will be assumed that F satisfies the standard conditions.

[Note: One consequence of this is that F necessarily contains all polynomials with coefficients from N, in particular, F contains the constant functions.]

__3:__ DEFINITION An <u>F-sequence</u> is a function $A:N \rightarrow Q$ that has a representation of the form

$$A(x) = \frac{f(x) \,\dot{-}\, g(x)}{h(x) + 1} \quad (x = 0, 1, 2, \dots),$$

where $f, g, h: N \rightarrow N$ belong to F.

__4:__ EXAMPLE Every rational number q gives rise to an F-sequence $A:N \rightarrow Q$, viz. $\forall\, x,\ A(x) = q$.

[Suppose that $x = \dfrac{r}{s}$ and consider the situation when $r \geq 0$, $s \geq 1$.

- $\underline{s = 1}$ Set $f(x) = r$, $g(x) = 0$, $h(x) = 0$, hence

$$\frac{f(x) \,\dot{-}\, g(x)}{h(x) + 1} = r.$$

- $\underline{s > 1}$ Set $f(x) = r$, $g(x) = 0$, $h(x) = s - 1$, hence

$$\frac{f(x) - g(x)}{h(x) + 1} = \frac{r}{s} .]$$

5: SUBLEMMA If f,g are one-argument elements of F, then

$$f + g \in F, \quad f \cdot g \in F.$$

[Consider

$$\text{add} \ ^3 \ (f,g), \quad \text{mul} \circ \ (f,g),$$

bearing in mind that F is closed under composition.]

6: LEMMA If $A, B : N \to Q$ are F-sequences, then so are $A + B$, $A - B$, and $A \cdot B$.

7: SUBLEMMA If h is a one-argument element of F, then $|h| \in F$.

[The fact that the modified subtraction function belongs to F implies that the absolute value function belongs to F (cf. §5, #8).]

8: LEMMA If $A : N \to Q$ is an F-sequence and if $\forall x$, $A(x) \neq 0$, then $\frac{1}{A}$ belongs to F.

[In fact,

$$\frac{1}{A(x)} = \frac{(h(x) + 1)f(x) - (h(x) + 1)g(x)}{| \ |f(x) - g(x)|^2 - 1| + 1} .]$$

9: DEFINITION A real number α is said to be F-computable if there exists an F-sequence $A : N \to Q$ such that $\forall x$,

$$|A(x) - \alpha| \leq \frac{1}{x + 1} .$$

[Note: In practice, it can happen that the relation

$$|A(x) - \alpha| \leq \frac{1}{x + 1}$$

is valid only for $x \geq x_0$. To remedy this, let

$$A(0) = A(x_0), \ A(1) = A(x_0), \ \ldots, \ A(x_0 - 1) = A(x_0).$$

Then for $0 \leq x \leq x_0 - 1$,

$$|A(x) - \alpha| = |A(x_0) - \alpha| \leq \frac{1}{x_0 + 1} \leq \frac{1}{x + 1} \ .]$$

10: **NOTATION** Denote the set of all F-computable real numbers by the symbol R_F.

11: **N.B.** The constant functions from N to Q are F-computable, hence $Q \subset R_F$ (in particular, $-1 \in R_F$).

12: **EXAMPLE** Take for F the set $\mathfrak{PR}$ of primitive recursive functions — then the F-computable real numbers are the primitive recursive real numbers.

[To arrive at $R_F = R$, take instead for F the set of all natural number functions.]

13: **LEMMA** If A is an F-sequence and if ϕ is a one-argument element of F, then the assignment $x \to A(\phi(x))$ is an F-sequence.

[For

$$A(\phi(x)) = \frac{f(\phi(x)) - g(\phi(x))}{h(\phi(x)) + 1}$$

$$= \frac{(f \circ \phi)(x) - (g \circ \phi)(x)}{(h \circ \phi)(x) + 1}$$

and F is closed under composition.]

14: **APPLICATION** Suppose that A is an F-sequence and α is a real number. Assume: $x|A(x) - \alpha|$ is bounded — then $\alpha \in R_F$.

PROOF Choose a positive integer c such that $\forall\, x$,

$$x \left| A(x) - \alpha \right| \le c.$$

Then

$$\left| A(cx + c) - \alpha \right| \le \frac{c}{cx + c} = \frac{1}{x + 1} \, .$$

[Note: In #13, take $\phi(x) = cx + c$.]

15: THEOREM R_F is a field.

We shall break the proof up into two parts.

PART 1 Let $\alpha, \beta \in R_F$ -- then

$$\left| (A(x) + B(x)) - (\alpha + \beta) \right| \le \left| A(x) - \alpha \right| + \left| B(x) - \beta \right|$$

$$\le \frac{1}{x + 1} + \frac{1}{x + 1} = \frac{2}{x + 1}$$

$\Rightarrow$

$$x \left| (A(x) + B(x)) - (\alpha + \beta) \right| \le \frac{2x}{x + 1} = \frac{2}{1 + \frac{1}{x}} \le 2$$

$$\Rightarrow \alpha + \beta \in R_F$$

and

$$\left| A(x)B(x) - \alpha\beta \right| \le \left| A(x) - \alpha \right| \left| B(x) \right| + \left| \alpha \right| \left| B(x) - \beta \right|$$

$$\le \frac{\left| \beta \right| + 1 + \left| \alpha \right|}{x + 1}$$

$\Rightarrow$

$$x \left| A(x)B(x) - \alpha\beta \right| \le \frac{x}{x + 1} \left(\left| \beta \right| + 1 + \left| \alpha \right| \right)$$

$$\le \left| \beta \right| + 1 + \left| \alpha \right|$$

$$\Rightarrow \alpha\beta \in R_F.$$

PART 2 Let $\alpha \neq 0$ and choose $c \in \mathbb{N}$ such that $(c+1)|\alpha| \geq 2$ — then $\forall\, x \geq c$,

$$|A(x)| \geq |\alpha| - |\alpha - A(x)| \geq \frac{2}{c+1} - \frac{1}{x+1} \geq \frac{1}{c+1}$$

$$\Rightarrow A(x) \neq 0$$

and

$$\left| \frac{1}{A(x)} - \frac{1}{\alpha} \right| = \frac{|\alpha - A(x)|}{|A(x)||\alpha|} \leq \frac{a}{x+1}\,,$$

where $a = (c+1)^2/2$. Define now a function $C : \mathbb{N} \to \mathbb{Q}$ by the prescription

$$C(k) = \frac{1}{A(k+c)} \quad (k + c \geq c \Rightarrow A(k+c) \neq 0).$$

Then C is an F-sequence and

$$\left| C(k) - \frac{1}{\alpha} \right| \leq \frac{a}{k+c+1}$$

$$\Rightarrow$$

$$k\left| C(k) - \frac{1}{\alpha} \right| \leq \frac{ka}{k+c+1}$$

$$= \frac{a}{1 + \frac{c}{k} + \frac{1}{k}} \leq a$$

$$\Rightarrow \frac{1}{\alpha} \in R_F.$$

In summary: R_F is a field.

16: DEFINITION Suppose that $f : \mathbb{N}^{k+1} \to \mathbb{N}$ — then the minimizer μf of f is the function

$$(x_1, \ldots, x_k, x_{k+1}) \to \min\{j \in \mathbb{N} : f(x_1, \ldots, x_k, j) = 0 \ \lor \ j = x_{k+1}\}.$$

[Note: Spelled out,

$$\mu f(x_1,\ldots,x_k,x_{k+1})$$

is the least $j \leq x_{k+1}$ such that

$$f(x_1,\ldots,x_k,j) = 0$$

if such a j exists, otherwise

$$\mu f(x_1,\ldots,x_k,x_{k+1}) = x_{k+1}$$

if for every $j \leq x_{k+1}$,

$$f(x_1,\ldots,x_k,j) > 0.]$$

__17:__ __N.B.__ To say that F is closed under the minimizer operation simply means that

$$f \in F \Rightarrow \mu f \in F.$$

__18:__ THEOREM Suppose that F is closed under the minimizer operation. Let

$$\alpha_0 \neq 0, \ \alpha_1,\ldots,\alpha_{k-1}, \ \alpha_k \in R_F.$$

Then the real roots of the polynomial

$$P(X) = \alpha_0 X^k + \alpha_1 X^{k-1} + \cdots + \alpha_{k-1}X + \alpha_k$$

belong to R_F.

PROOF Let ζ be a real root of P and without loss of generality, assume that $P'(\zeta) \neq 0$. Choose rational numbers a,b,c,d such that $a < \zeta < b$, $0 < c < d$ subject to

$$c|X - \zeta| \leq |P(X)| \leq d|X - \zeta|$$

whenever $a \leq X \leq b$. Establish the notation:

$$A_0 \longleftrightarrow \alpha_0, \ A_1 \longleftrightarrow \alpha_1, \ \ldots, \ A_{k-1} \longleftrightarrow \alpha_{k-1}, \ A_k \longleftrightarrow \alpha_k$$

and introduce the polynomials

$$P_x(X) = A_0(x)X^k + A_1(x)X^{k-1} + \cdots + A_{k-1}(x)X + A_k(x) \quad (x = 0,1,2,\ldots).$$

Choose $q \in Q$:

$$|P_x(X) - P(X)| \leq \frac{q}{x+1} \quad (a \leq X \leq b).$$

Define now a function $A: N \to Q$ via the following procedure. Given any $x \in N$, divide $[a,b]$ into $x + 1$ equally long subintervals. Let M_x be the set of midpoints of these subintervals, there being at least one $X \in M_x$ such that

$$|P_x(X)| \leq \frac{d(b-a) + 2q}{2(x+1)}.$$

Proof: Choose $X \in M_x$:

$$|X - \zeta| \leq \frac{b-a}{2(x+1)}$$

$$\Rightarrow$$

$$|P_x(X)| \leq |P(X)| + \frac{q}{x+1}$$

$$\leq d\,\frac{b-a}{2(x+1)} + \frac{q}{x+1} = \frac{d(b-a) + 2q}{2(x+1)}.$$

Let $A(x) \in M_x$ be the left most element of M_x with the property that

$$|P_x(A(x))| \leq \frac{d(b-a) + 2q}{2(x+1)},$$

so

$$c|A(x) - \zeta| \leq |P(A(x))|$$

$$\leq |P_x(A(x))| + \frac{q}{x+1}$$

$$\leq \frac{d(b-a) + 4q}{2(x+1)}.$$

Therefore the product $x|A(x) \doteq \zeta|$ stays bounded as a function of x. Accordingly, recalling #14, it remains only to show that $A: N \to Q$ is an F-sequence. To this end, note that

$$A(x) = a + (b\text{-}a) \frac{2\phi(x) + 1}{2x + 2} ,$$

where $\phi(x)$ is the smallest $j \in \{0,1,2,\ldots,x\}$ such that

$$\left| P_x(a + (b\text{-}a) \frac{2j + 1}{2x + 2}) \right| \leq \frac{d(b\text{-}a) + 2q}{2(x + 1)}$$

or still,

$$\phi(x) = \min\{j \in \{0,1,2,\ldots,x\}:$$

$$\left| P_x(a + (b\text{-}a) \frac{2j + 1}{2x + 2}) \right| - \frac{q'}{x + 1} \leq 0\},$$

$q' = d(b\text{-}a)/2 + q.$ Since

$$P_x(a + (b\text{-}a) \frac{2j + 1}{2x + 2}) = \sum_{i=0}^{k} A_i(x) \, (a + (b\text{-}a) \frac{2j + 1}{2x + 2})^{k-i},$$

the function

$$x \to P_x(a + (b\text{-}a) \frac{2j + 1}{2x + 2})$$

defines an F-sequence for each j, hence can be represented in the form

$$x \to \frac{f(x,j) - g(x,j)}{h(x,j) + 1} .$$

Put

$$\Phi(x,j) = f(x,j) \mathbin{\dot{-}} g(x,j),$$

thus

$$\phi(x) = \min\{j \in N: \Phi(x,j) = 0 \vee j = x\},$$

so $\phi \in F$ and this implies that $A: N \to Q$ is an F-sequence.

19: APPLICATION

$$R_{alg} \subset R_F \; (\subset R).$$

[In other words, every real algebraic number is an F-computable real number.]

20: REMARK It is a fact that the minimizer operation can be derived from bounded summation, hence ℓEL and E^2 are closed under the minimizer operation (cf. #2).

21: THEOREM Suppose that F is closed under the minimizer operation — then R_F is a real closed field.

PROOF R_F is an ordered field. And:

● Every polynomial of odd degree with coefficients in R_F has at least one root in R_F.

[Since all data is real, on general grounds such a polynomial has at least one real root ζ. And, in view of #18, $\zeta \in R_F$.]

● If $\alpha > 0$ is an element of R_F, then α has a square root $\sqrt{\alpha}$ in R_F.

[As a positive real number, $\sqrt{\alpha}$ is a root of the polynomial $X^2 - \alpha$. But the coefficients of this polynomial are in R_F, hence by #18, $\sqrt{\alpha}$ belongs to R_F.]

§9. *THE SKORDEV CRITERION* [†]

Given a set F of natural number functions, assume as in §8 that F satisfies the standard conditions.

<u>1:</u> DEFINITION An F-2-sequence is a function $A: N^2 \to Q$ that has a representation of the form

$$A(x,n) = \frac{f(x,n) - g(x,n)}{h(x,n) + 1} \quad (x,n = 0,1,2,\ldots),$$

where $f,g,h: N^2 \to N$ belong to F.

<u>2:</u> DEFINITION A real valued function $\alpha: N \to R$ is said to be <u>F-computable</u> if there exists an F-2-sequence $A: N^2 \to Q$ such that $\forall\, x, \forall\, n$,

$$|A(x,n) - \alpha(n)| \leq \frac{1}{x+1}.$$

<u>3:</u> <u>N.B.</u> It is clear that $\forall\, n \in N$, the real number $\alpha(n)$ is F-computable (cf. §8, #9). On the other hand, if $\forall\, n \in N$, $\alpha(n)$ is F-computable, then there exists an F-sequence $A_n(x)$ such that $\forall\, x$,

$$|A_n(x) - \alpha(n)| \leq \frac{1}{x+1},$$

so setting

$$A(x,n) = A_n(x)$$

leads to the conclusion that α is F-computable.

<u>4:</u> METHODOLOGY Given an F-sequence $A: N \to Q$, view it as a function $\alpha: N \to R$ — then α is F-computable, i.e., there exists an F-2-sequence $A: N^2 \to Q$

such that

$$\left| A(x,n) - \alpha(n) \right| \le \frac{1}{x + 1} \, .$$

Thus let

$$A(x,n) = A(n)$$

to get

$$\left| A(x,n) - \alpha(n) \right| = \left| A(n) - A(n) \right|$$

$$= 0 \le \frac{1}{x + 1} \, .$$

<u>5:</u> LEMMA If $\alpha : N \to R$ is F-computable, then there exist functions $f : N^2 \to N$ and $g : N^2 \to N$ in F such that $\forall \, x, \, \forall \, n$,

$$\left| \frac{f(x,n) - g(x,n)}{x + 1} - \alpha(n) \right| \le \frac{1}{x + 1} \, .$$

PROOF Changing notation, start with a relation

$$\left| \frac{u(x,n) - v(x,n)}{w(x,n) + 1} - \alpha(n) \right| \le \frac{1}{x + 1}$$

per #2.   Introduce

$$f_0(x,n) = u(2x + 1, n)$$

$$g_0(x,n) = v(2x + 1, n)$$

$$h_0(x,n) = w(2x + 1, n) \, .$$

Then

$$\left| \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} - \alpha(n) \right|$$

$$= \left| \frac{u(2x + 1, n) - v(2x + 1, n)}{w(2x + 1, n) + 1} - \alpha(n) \right|$$

$$\leq \frac{1}{2x + 1 + 1} = \frac{1}{2(x+1)} \; .$$

Define $C: \mathbb{N}^2 \to \mathbb{N}$ by the rule

$$C(i,j) = \left\lceil \frac{i}{j + 1} + \frac{1}{2} \right\rceil \; ,$$

thus $C \in F$ and

$$\left| C(i,j) - \frac{i}{j + 1} \right| \leq \frac{1}{2} \; .$$

Put now

$$f(x,n) = C((x + 1)(f_0(x,n) \mathbin{\dot{-}} g_0(x,n)), \; h_0(x,n))$$

and

$$g(x,n) = C((x + 1)(g_0(x,n) \mathbin{\dot{-}} f_0(x,n)), \; h_0(x,n)) .$$

Then $f, g \in F$ and (details below)

$$\left| f(x,n) - g(x,n) - (x + 1) \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} \right| \leq \frac{1}{2} \; .$$

Multiply this by $\frac{1}{x + 1}$ to get

$$\left| \frac{f(x,n) - g(x,n)}{x + 1} - \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} \right| \leq \frac{1}{2(x + 1)} \; .$$

Therefore

$$\left| \frac{f(x,n) - g(x,n)}{x + 1} - \alpha(n) \right|$$

$$= \left| \frac{f(x,n) - g(x,n)}{x + 1} - \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} \right.$$

$$\left. + \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} - \alpha(n) \right|$$

$$\leq \left| \frac{f(x,n) - g(x,n)}{x + 1} - \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} \right|$$

$$+ \left| \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} - \alpha(n) \right|$$

$$\leq \frac{1}{2(x+1)} + \frac{1}{2(x+1)} = \frac{1}{x+1} \; .$$

6: DETAILS The claim is that

$$\left| f(x,n) - g(x,n) - (x+1) \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} \right| \leq \frac{1}{2} \; .$$

● Suppose that

$$f_0(x,n) < g_0(x,n) .$$

By definition,

$$f_0(x,n) \overset{\cdot}{-} g_0(x,n) = \begin{cases} f_0(x,n) - g_0(x,n) \text{ if } f_0(x,n) \geq g_0(x,n) \\ \\ 0 \text{ if } f_0(x,n) < g_0(x,n) . \end{cases}$$

Accordingly

$$f_0(x,n) \overset{\cdot}{-} g_0(x,n)$$

$$= 0 .$$

Therefore

$$f(x,n) = C(0, h_0(x,n))$$

$$= \left\lfloor \frac{0}{h_0(x,n) + 1} + \frac{1}{2} \right\rfloor$$

$$= \left\lfloor \frac{1}{2} \right\rfloor = 0 .$$

So consider

$$\left| - g(x,n) - (x+1) \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} \right|$$

or still,

$$\left| \ g(x,n) \ - \ (x + 1) \ \frac{g_0(x,n) \ - \ f_0(x,n)}{h_0(x,n) \ + \ 1} \ \right| \ .$$

By definition,

$$g(x,n) \ = \ C((x + 1)(g_0(x,n) \ \dot{-} \ f_0(x,n)), \ h_0(x,n)).$$

But here

$$g_0(x,n) \ \dot{-} \ f_0(x,n) \ = \ g_0(x,n) \ - \ f_0(x,n)$$

since

$$f_0(x,n) \ < \ g_0(x,n).$$

Recalling that

$$\left| \ C(i,j) \ - \ \frac{i}{j + 1} \ \right| \ \leq \ \frac{1}{2} \ ,$$

specialize and take

$$\left[\begin{array}{l} i \ = \ (x + 1)(g_0(x,n) \ - \ f_0(x,n)) \\ \\ j \ = \ h_0(x,n). \end{array} \right.$$

Then

$$\left| \ g(x,n) \ - \ (x + 1) \ \frac{g_0(x,n) \ - \ f_0(x,n)}{h_0(x,n) \ + \ 1} \ \right|$$

$$= \ \left| \ C((x + 1)(g_0(x,n) \ - \ f_0(x,n)), \ h_0(x,n)) \right.$$

$$\left. - \ (x + 1) \ \frac{g_0(x,n) \ - \ f_0(x,n)}{h_0(x,n) \ + \ 1} \ \right| \ \leq \ \frac{1}{2} \ .$$

- Suppose that

$$f_0(x,n) \ \geq \ g_0(x,n).$$

By definition,

$$g_0(x,n) \mathbin{\dot{-}} f_0(x,n) = \begin{cases} g_0(x,n) - f_0(x,n) & \text{if } g_0(x,n) \geq f_0(x,n) \\ 0 & \text{if } g_0(x,n) < f_0(x,n). \end{cases}$$

Accordingly

$$\begin{cases} g_0(x,n) < f_0(x,n) \Rightarrow g_0(x,n) \mathbin{\dot{-}} f_0(x,n) = 0 \\ g_0(x,n) = f_0(x,n) \Rightarrow g_0(x,n) \mathbin{\dot{-}} f_0(x,n) \\ \qquad\qquad = g_0(x,n) - f_0(x,n) = 0. \end{cases}$$

Therefore

$$g(x,n) = C(0, h_0(x,n + 1))$$

$$= \left\lceil \frac{0}{h_0(x,n) + 1} + \frac{1}{2} \right\rceil$$

$$= \left\lceil \frac{1}{2} \right\rceil = 0.$$

So consider

$$\left| f(x,n) - (x + 1) \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} \right|.$$

By definition,

$$f(x,n) = C((x + 1)(f_0(x,n) \mathbin{\dot{-}} g_0(x,n)), h_0(x,n)).$$

But here

$$f_0(x,n) \mathbin{\dot{-}} g_0(x,n) = f_0(x,n) - g_0(x,n)$$

since

$$f_0(x,n) \geq g_0(x,n).$$

Recalling that

$$\left| C(i,j) - \frac{i}{j+1} \right| \le \frac{1}{2} \, ,$$

specialize and take

$$\begin{cases} i = (x+1)(f_0(x,n) - g_0(x,n)) \\ j = h_0(x,n). \end{cases}$$

Then

$$\left| f(x,n) - (x+1) \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} \right|$$

$$= \left| C((x+1)(f_0(x,n) - g_0(x,n)), h_0(x,n)) \right.$$

$$\left. - (x+1) \frac{f_0(x,n) - g_0(x,n)}{h_0(x,n) + 1} \right| \le \frac{1}{2} \, .$$

7: N.B. The upshot is that in the definition of F-computability, one can take $h(x,n) = x$.

Thus far the only conditions imposed on F are the standard ones but to proceed to the main result it will be assumed henceforth that F is closed under bounded summation (which, of course, is the case if $F = \ell EL$ or E^2).

8: THEOREM Let $\alpha : N \to R$ be F-computable, assume that the series $\sum_{n=0}^{\infty} \alpha(n)$ is convergent, and let Ξ be its sum. Suppose there exists a function $\xi : N \to N$ in F such that $\forall \, x \in N$,

$$\left| \sum_{n=\xi(x)+1}^{\infty} \alpha(n) \right| \le \frac{1}{x+1} \, .$$

Then Ξ is F-computable.

9: LEMMA Let $\alpha : N \to R$ be F-computable. Define $\alpha^\Sigma : N \to R$ by setting

$$\alpha^\Sigma (m) = \sum_{n=0}^{m} \alpha(n).$$

Then α^Σ is F-computable.

PROOF Per #5, write

$$\left| \frac{f(x,n) - g(x,n)}{x + 1} - \alpha(n) \right| \leq \frac{1}{x + 1}$$

and define functions

$$f^\Sigma : N^2 \to N, \; g^\Sigma : N^2 \to N$$

in F by stipulating that

$$\left[\begin{array}{l} f^\Sigma (x,m) = \sum_{n=0}^{m} f(xm + x + m, n) \\[2mm] g^\Sigma (x,m) = \sum_{n=0}^{m} g(xm + x + m, n). \end{array} \right.$$

Then

$$\left| \frac{f(xm + x + m, n) - g(xm + x + m, n)}{xm + x + m + 1} - \alpha(n) \right| \leq \frac{1}{xm + x + m + 1}$$

$$\Rightarrow$$

$$\left| \frac{f^\Sigma (x,m) - g^\Sigma (x,m)}{xm + x + m + 1} - \alpha^\Sigma (m) \right| \leq \frac{1}{x + 1} .$$

I.e.:

$$\left| \frac{f^\Sigma (x,m) - g^\Sigma (x,m)}{h(x,m) + 1} - \alpha^\Sigma (m) \right| \leq \frac{1}{x + 1}$$

if

$$h(x,m) = xm + x + m,$$

thus α^Σ is F-computable.

Turning now to the proof of the theorem, per #9, determine functions

$$F,G,H:N^2 \to N$$

in F such that $\forall x$, $\forall n$,

$$\left| \frac{F(x,n) - G(x,n)}{H(x,n) + 1} - \alpha^{\Sigma}(n) \right| \leq \frac{1}{x + 1} .$$

Introduce

$$
\begin{aligned}
&u(x) = F(2x + 1, \xi(2x + 1)) \\
&v(x) = G(2x + 1, \xi(2x + 1)) \\
&w(x) = H(2x + 1, \xi(2x + 1)).
\end{aligned}
$$

Then $u,v,w \in F$ and

$$\left| \frac{u(x) - v(x)}{w(x) + 1} - \Xi \right|$$

$$= \left| \frac{u(x) - v(x)}{w(x) + 1} - \sum_{n=0}^{\infty} \alpha(n) \right|$$

$$= \left| \frac{u(x) - v(x)}{w(x) + 1} - \sum_{n=0}^{\xi(2x+1)} \alpha(n) - \sum_{n=\xi(2x+1) + 1}^{\infty} \alpha(n) \right|$$

$$= \left| \frac{F(2x + 1, \xi(2x + 1)) - G(2x + 1, \xi(2x + 1))}{H(2x + 1, \xi(2x + 1))} - \alpha^{\Sigma}(\xi(2x + 1)) \right.$$

$$\left. - \sum_{n=\xi(2x+1) + 1}^{\infty} \alpha(n) \right|$$

$$\leq \left| \frac{F(2x + 1, \xi(2x + 1)) - G(2x + 1, \xi(2x + 1))}{H(2x + 1, \xi(2x + 1))} - \alpha^{\Sigma}(\xi(2x + 1)) \right|$$

$$+ \left| \sum_{n=\xi(2x + 1) + 1}^{\infty} \alpha(n) \right|$$

$$\leq \frac{1}{2x + 1 + 1} + \left| \sum_{n=\xi(2x + 1) + 1}^{\infty} \alpha(n) \right|$$

$$\leq \frac{1}{2x + 1 + 1} + \frac{1}{2x + 1 + 1} = \frac{1}{2x + 2} + \frac{1}{2x + 2}$$

$$= \frac{2}{2x + 2} = \frac{1}{x + 1}.$$

<u>10</u>: <u>N.B.</u> Assuming that the series $\sum_{n=0}^{\infty} \alpha(n)$ is convergent (in practice, this is invariably a non-issue), there are then two points.

    1. Establish that α is F-computable (or that $\forall$ n, $\alpha(n)$ is F-computable).

    2. Find ξ and deal with the speed of convergence.

<u>11</u>: EXAMPLE e is E^2-computable, i.e., $e \in R_{E^2}$. Thus write

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

and let $\alpha(n) = \frac{1}{n!}$, hence $\alpha:N \to Q \subset R$, the claim being that α is E^2-computable.

To see this, let

$$f(x,n) = \left\lceil \frac{x}{n!} \right\rceil .$$

Then $f \in E^2$:

$$f(x,0) = x, \quad f(x,n + 1) = \left\lceil \frac{f(x,n)}{n + 1} \right\rceil , \quad f(x,n) \leq x$$

and one can quote bounded recursion. In addition

$$\left| \frac{f(x + 1,n)}{x + 1} - \alpha(n) \right| = \left| \frac{f(x + 1,n)}{x + 1} - \frac{1}{n!} \right|$$

$$= \frac{1}{x + 1} \left| f(x + 1,n) - \frac{x + 1}{n!} \right|$$

$$= \frac{1}{x + 1} \left| \left[\frac{x + 1}{n!} \right] - \frac{x + 1}{n!} \right|$$

$$\leq \frac{1}{x + 1} .$$

Therefore α is F-computable. It remains to define $\xi : N \to N$ and consider

$$\sum_{n = \xi(x) + 1}^{\infty} \alpha(n) = \sum_{n = \xi(x) + 1}^{\infty} \frac{1}{n!} .$$

So put

$$\xi(0) = 1, \quad \xi(1) = 2, \quad \xi(x) = x \ (x \geq 2).$$

Then

$$(x = 0) \quad \sum_{n = \xi(0) + 1}^{\infty} \frac{1}{n!} = \sum_{n = 2}^{\infty} \frac{1}{n!} = e - 2 \leq 1 = \frac{1}{0 + 1}$$

$$(x = 1) \quad \sum_{n = \xi(1) + 1}^{\infty} \frac{1}{n!} = \sum_{n = 3}^{\infty} \frac{1}{n!} = e - 2.5 \sim 2.7 - 2.5 = .2 \leq \frac{1}{1 + 1}$$

$$(x > 1) \quad \sum_{n = \xi(x) + 1}^{\infty} \frac{1}{n!} = \sum_{n = x + 1}^{\infty} \frac{1}{n!}$$

$$= \frac{1}{(x + 1)!} + \frac{1}{(x + 2)!} + \cdots$$

$$= \frac{1}{x!} \left[\frac{1}{x + 1} + \frac{1}{(x + 1)(x + 2)} + \cdots \right.$$

$$< \frac{1}{x!} \left[\frac{1}{x + 1} + \frac{1}{(x + 1)^2} + \cdots \right.$$

$$= \frac{1}{x!} \left(\frac{\frac{1}{x + 1}}{1 - \frac{1}{x + 1}} \right)$$

$$= \frac{1}{x!} \frac{1}{x} \leq \frac{1}{x + 1} .$$

Conclusion:

$$\Xi = e$$

is E^2-computable.

[Note: It turns out that e is actually ℓEL-computable (cf. §10, #6).]

11: REMARK Recall that $R_{alg} \subset R_{E^2}$ (cf. §8, #19) and since e is trans-

cendental, it follows that the containment is proper.

§10. *TECHNICALITIES*

<u>1:</u> DEFINITION A relation R $\subset N^n$ is said to be <u>lower elementary</u> if its characteristic function belongs to ℓEL.

<u>2:</u> LEMMA Suppose that $f:N \to N$ is in ℓEL. Define a function $\varphi:N \to N$ by the formula

$$\varphi(n) = \prod_{k=0}^{n} f(k).$$

Then the graph of φ is lower elementary.

[Note: Recall that ℓEL is closed under bounded summation.]

<u>3:</u> EXAMPLE Fix a positive natural number N and define $f:N \to N$ by stipulating that $\forall$ k, $f(k) = N$ --- then $f \in \ell EL$. Moreover $\varphi(n) = N^{n+1}$ and the graph of φ is lower elementary.

<u>4:</u> LEMMA Suppose that $\varphi:N \to N$ has the property that $\forall$ n, $\varphi(n) \neq 0$. Assume further that the graph of φ is lower elementary --- then the function

$$n \to \frac{1}{\varphi(n)}$$

is ℓEL-computable.

<u>5:</u> EXAMPLE For every positve natural number N, the function

$$n \to \frac{1}{N^{n+1}}$$

is ℓEL-computable.

<u>6:</u> REMARK It was shown in §9, #11 that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

is E^2-computable. However more is true: e is ℓEL-computable. To see this, consider $f: N \to N$, where

$$f(0) = 1, \quad f(k) = k \quad (k > 0).$$

Then

$$\varphi(n) = \prod_{k=0}^{n} f(k) = f(0)f(1)f(2) \cdots f(n)$$

$$= n!.$$

Therefore the function

$$n \to \frac{1}{n!}$$

is ℓEL-computable (and the argument proceeds...).

[Note: It is clear that f is in ℓEL.]

<u>7:</u> LEMMA If $\alpha: N \to R$, $\beta: N \to R$ are ℓEL-computable bounded functions, then the product $\alpha\beta: N \to R$ is also ℓEL-computable.

<u>8:</u> EXAMPLE The function

$$n \to \frac{(-1)^n}{N^{n+1}}$$

is ℓEL-computable.

[Note:

$$(-1)^n = (n + 1) \bmod 2 - n \bmod 2$$

$$= \text{"} f(n) \dot{-} g(n) \text{"},$$

hence $(-1)^n$ is an ℓEL-sequence, hence is ℓEL-computable (cf. §9, #4).]

§11. *NUMERICAL EXAMPLES*

The basis for the calculations infra is §9, #8, which will not be quoted over and over, as well as the generalities in §10, which will also be taken without attribution.

<u>1:</u> EXAMPLE π is ℓEL-computable, i.e., $\pi \in R_{\ell EL}$. Thus write

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} \cdot$$

Then the function

$$n \rightarrow \frac{(-1)^n}{2n + 1}$$

is ℓEL-computable. As for convergence, the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1}$$

is alternating, so $\forall\, x \in N \,(\xi(x) = x)$,

$$\left| \sum_{n=x+1}^{\infty} \frac{(-1)^n}{2n + 1} \right| \leq \frac{1}{2(x+1) + 1}$$

$$= \frac{1}{2x + 3} \leq \frac{1}{x + 1} \cdot$$

These considerations establish the ℓEL-computability of $\frac{\pi}{4}$, thus that of $\pi = 4(\frac{\pi}{4})$.

<u>2:</u> EXAMPLE $\ell n(N)$ $(N = 1,2,\ldots)$ is ℓEL-computable, i.e., $\ell n(N) \in R_{\ell EL}$. Thus write

$$\ell n(1 + \frac{1}{N}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)N^{n+1}}$$

=>

$$\ln(N + 1) = \ln(N) + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)N^{n+1}} .$$

Proceed by induction on N. When $N = 1$, $\ln(1) = 0$, which is obviously ℓEL-computable. So take $N > 1$ and suppose that $\ln(N)$ is ℓEL-computable. Since $R_{\ell EL}$ is a field, it need only be shown that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)N^{n+1}} \in R_{\ell EL} .$$

But the function

$$n \to \frac{(-1)^n}{(n+1)N^{n+1}}$$

is ℓEL-computable. And $\forall x \in \mathbb{N}\ (\xi(x) = x)$,

$$\left| \sum_{n=x+1}^{\infty} \frac{(-1)^n}{(n+1)N^{n+1}} \right|$$

$$\leq \frac{1}{(x+2)N^{x+2}} \leq \frac{1}{x+1} .$$

3: **EXAMPLE** Catalan's constant

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

is ℓEL-computable, i.e., $G \in R_{\ell EL}$.

4: **EXAMPLE** Euler's constant γ is ℓEL-computable, i.e., $\gamma \in R_{\ell EL}$. Thus write

$$\gamma = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \ell\left(1 + \frac{1}{n+1}\right) \right) .$$

Then

$$\frac{1}{n+1} - \ell n(1 + \frac{1}{n+1})$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+2)(n+1)^{m+2}} \cdot$$

Per m,

$$\alpha_n(m) = \frac{(-1)^m}{(m+2)(n+1)^{m+2}}$$

is a product of three bounded ℓEL-computable functions, thus $\alpha_n(m)$ is ℓEL-computable. As for convergence, $\forall\, m \in \mathbb{N}\ (\xi(x) = x)$,

$$\left| \sum_{m=x+1}^{\infty} \alpha_n(m) \right| \leq \frac{1}{(x+3)(n+1)^{x+3}} \leq \frac{1}{x+3} \leq \frac{1}{x+1} \cdot$$

Therefore per n the sum

$$\Xi_n = \sum_{m=0}^{\infty} \alpha_n(m)$$

is ℓEL-computable. And finally

$$\left| \sum_{n=x+1}^{\infty} \Xi_n \right| \leq \sum_{n=x+1}^{\infty} \left| \Xi_n \right|$$

$$\leq \sum_{n=x+1}^{\infty} \left| \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+2)(n+1)^{m+2}} \right|$$

$$\leq \sum_{n=x+1}^{\infty} \frac{1}{2(n+1)^2}$$

$$= \frac{1}{2} \sum_{n=x+1}^{\infty} \frac{1}{(n+1)^2}$$

$$\leq \frac{1}{2} \sum_{n=x+1}^{\infty} \frac{1}{n(n+1)}$$

$$= \frac{1}{2} \sum_{n=x+1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\leq \frac{1}{2} \frac{1}{x+1} \leq \frac{1}{x+1} \cdot$$

Therefore

$$\Xi = \sum_{n=0}^{\infty} \Xi_n = \gamma$$

is ℓEL-computable.

<u>5:</u> EXAMPLE Liouville's number

$$L = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$$

is ℓEL-computable, i.e., $L \in R_{\ell EL}$.

[As regards convergence, write

$$L = \sum_{n=0}^{\infty} \frac{1}{10^{(n+1)!}}$$

and note that $\forall\, n \in N\ (\xi(x) = x)$,

$$\sum_{n=x+1}^{\infty} \frac{1}{10^{(n+1)!}}$$

$$\leq \sum_{n=x+1}^{\infty} \frac{1}{(n+1)!} = \frac{1}{(x+2)!} + \frac{1}{(x+3)!} + \frac{1}{(x+4)!} + \cdots$$

$$= \frac{1}{(x+2)!} \left(1 + \frac{1}{x+3} + \frac{1}{(x+3)(x+4)} + \cdots\right)$$

$$\leq \frac{1}{(x+2)!} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) = \frac{2}{(x+2)!}$$

$$\le \frac{2}{2^{x+1}} \le \frac{1}{x+1} \cdot]$$

6: **EXAMPLE** Let

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \quad (x > 1)$$

and define

$$f_R : N \to R$$

by

$$f_R(k) = \zeta(k+2) \quad (k = 0,1,\ldots).$$

Then f_R is an ℓEL-computable function. Consequently $\forall k$,

$$\zeta(k+2) \in R_{\ell EL}.$$

In particular: $\zeta(3)$ is an ℓEL-computable real number.

[Note: Put $\ell = k + 2$, thus $\ell = 2,3,\ldots$ — then

$$\zeta(\ell) = \int_{1 > t_1 > \cdots > t_\ell > 0} \frac{dt_1}{t_1} \cdots \frac{dt_{\ell-1}}{t_{\ell-1}} \cdot \frac{dt_\ell}{1-t_\ell}$$

is a period, so

$$\zeta(\ell) \in R_{\ell EL} \quad (cf. \ \S12, \ \#9),$$

which is another way of looking at matters.]

7: **FACTS**

- $\forall k$,

$$f_R(k+1) < f_R(k).$$

[$f_R(k) - f_R(k+1)$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{k+2}} - \sum_{n=0}^{\infty} \frac{1}{(n+1)^{k+3}}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{(n+1)^{k+2}} - \frac{1}{(n+1)^{k+3}} \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{k+2}} \left(1 - \frac{1}{n+1} \right)$$

$$= \sum_{n=0}^{\infty} \frac{n}{(n+1)^{k+3}} > 0.]$$

● $\forall \, k,$

$$1 < f_R(k) < 2.$$

$[\, f_R(0) = \frac{\pi^2}{6} < 2.$ On the other hand,

$$f_R(k) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{k+2}} > 1.$$

Therefore

$$2 > f_R(0) \geq f_R(k) > 1.]$$

<u>8:</u>   EXAMPLE $\ell n(\pi)$ is ℓEL-computable, i.e., $\ell n(\pi) \in R_{\ell EL}.$ Thus write

$$\ell n(\pi) - \ell n(2) = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n) 2^{2n-1}}$$

$$= \sum_{n=0}^{\infty} \frac{\zeta(2(n+1))}{2(n+1) 2^{2(n+1)-1}}$$

$$= \sum_{n=0}^{\infty} \frac{f_R(2n)}{2(n+1) 2^{2n+1}} \, .$$

Then

$$\ell n(\pi) = \ell n(2) + \sum_{n=0}^{\infty} \frac{f_R(2n)}{2(n+1) 2^{2n+1}} \, .$$

Since $\ln(2) \in R_{\ell EL}$ (cf. #2), it suffices to examine the series

$$\sum_{n=0}^{\infty} \frac{f_R(2n)}{2(n+1)2^{2n+1}} \; .$$

But

(i) $f_R(n)$ is a bounded ℓEL-computable function, hence so is $f_R(2n)$.

(ii) $\frac{1}{2(n+1)} = \frac{1}{2n+1 + 1}$ is an ℓEL-computable function.

(iii) $\frac{1}{2^{2n+1}}$ is an ℓEL-computable function.

Conclusion:

$$\frac{f_R(2n)}{2(n+1)2^{2n+1}}$$

is an ℓEL-computable function.

To handle the convergence, $\forall \; x \in N \; (\xi(x) = x)$,

$$\sum_{n=x+1}^{\infty} \frac{f_R(2n)}{2(n+1)2^{2n+1}}$$

$$\leq \sum_{n=x+1}^{\infty} \frac{2}{2(n+1)2^{2n+1}}$$

$$\leq \sum_{n=x+1}^{\infty} \frac{1}{2^{2n+1}}$$

$$= \frac{1}{2^{2x+3}} \cdot \frac{1}{1 - \frac{1}{4}}$$

$$= \frac{4}{3 \cdot 2^{2x+3}}$$

$$= \frac{1}{3 \cdot 2^{2x+1}}$$

$$\leq \frac{1}{3(2x+1)}$$

$$= \frac{1}{6x+3}$$

$$\leq \frac{1}{x+1} \cdot$$

APPENDIX

FACT If $\xi \in R_{\ell EL}$, then $e^{\xi} \in R_{\ell EL}$ and if $\xi \in R_{\ell EL}$ is > 0, then $\ell n(\xi) \in R_{\ell EL}$.

FACT If $\xi \in R_{\ell EL}$, then

$$\left[\begin{array}{c} \sin \xi \\ \\ \cos \xi \end{array} \right. \in R_{\ell EL} \cdot$$

FACT If $\xi \in R_{\ell EL}$ and if $|\xi| \leq 1$, then

$$\left[\begin{array}{c} \text{Arc sin } \xi \\ \\ \text{Arc cos } \xi \end{array} \right. \in R_{\ell EL} \cdot$$

FACT If $\xi \in R_{\ell EL}$, then

$$\text{Arc Tan } \xi \in R_{\ell EL} \cdot$$

§12. R_{E^3} versus R_{E^2}

Before getting down to business, there is a preliminary fact, frankly technical, which will be needed below.

<u>1:</u> RAPPEL (cf. §9, #5) If $\alpha \in R$ is F-computable, then there exist functions $f:N \to N$ and $g:N \to N$ in F such that $\forall$ x,

$$\left| \frac{f(x) - g(x)}{x + 1} - \alpha \right| \leq \frac{1}{x + 1} .$$

[Note: Here it is assumed that F satisfies the standard conditions.]

<u>2:</u> SUBLEMMA If $\alpha \in R$ is F-computable, then there exist functions f_0, g_0, h_0 in F such that $\forall$ x,

$$\left| \frac{f_0(x) - g_0(x)}{h_0(x) + 1} - \alpha \right| < \frac{1}{2(x+1)} .$$

PROOF Start with u,v,w in F such that $\forall$ x,

$$\left| \frac{u(x) - v(x)}{w(x) + 1} - \alpha \right| \leq \frac{1}{x + 1} .$$

Introduce

$$\begin{array}{l} f_0(x) = u(2(x+1) + 1) \\[2mm] g_0(x) = v(2(x+1) + 1) \\[2mm] h_0(x) = w(2(x+1) + 1) . \end{array}$$

Then

$$\left| \frac{f_0(x) - g_0(x)}{h_0(x) + 1} - \alpha \right|$$

$$= \left| \frac{u(2(x+1) + 1) - v(2(x+1) + 1)}{h_0(2(x+1) + 1)} - \alpha \right|$$

$$\leq \frac{1}{2(x+1) + 1 + 1}$$

$$= \frac{1}{2x + 4} = \frac{1}{2(x+2)}$$

$$< \frac{1}{2(x+1)} \, .$$

<u>3:</u> LEMMA If $\alpha \in R$ is F-computable, then there exist functions $f : N \to N$ and $g : N \to N$ in F such that $\forall\ x$,

$$\left| \frac{f(x) - g(x)}{x + 1} - \alpha \right| < \frac{1}{x + 1} \, .$$

PROOF Proceed as in §9, #5, taking $f, g \in F$ per f_0, g_0, h_0 to arrive at

$$\left| \frac{f(x) - g(x)}{x + 1} - \alpha \right|$$

$$\leq \left| \frac{f(x) - g(x)}{x + 1} - \frac{f_0(x) - g_0(x)}{h_0(x) + 1} \right|$$

$$+ \left| \frac{f_0(x) - g_0(x)}{h_0(x) + 1} - \alpha \right|$$

$$\leq \frac{1}{2(x+1)} + \left| \frac{f_0(x) - g_0(x)}{h_0(x) + 1} - \alpha \right|$$

$$< \frac{1}{2(x+1)} + \frac{1}{2(x+1)} = \frac{1}{x+1} \ .$$

4: N.B. The point, of course, is that in §9, #5, $\leq$ can be replaced by <.

5: LEMMA There exists a two-ary function h in E^3 which is universal for the one-ary functions in E^2.

[Note: I.e., the functions

$$h \to h(X,n) \quad (X = 0,1,2,\ldots)$$

exhaust the one-ary functions in E^2.]

6: RAPPEL The natural number function

$$pow(x,y) = x^y$$

is elementary (cf. §6, #8).

On general grounds,

$$R_{E^2} \subset R_{E^3}$$

and the claim is that this containment is strict.

In detail: Start by defining a one-ary function g as follows: g(0) = 0 and $\forall \ k \in N$,

$$g(k+1) = \left[\begin{array}{l} 3g(k) \text{ if } 6g(k) + 3 \leq h(k, \ 2 \cdot 3^{k+1} - 1) \\ \\ 3g(k) + 2 \text{ otherwise,} \end{array} \right.$$

thus

$$g(k+1) - 3g(k) \in \{0,2\}.$$

7: LEMMA $g \in E^3$.

[Use the inequality $g(k) \leq 3^k - 1$.]

Put

$$\alpha = \sum_{k=0}^{\infty} \frac{g(k+1) - 3g(k)}{3^{k+1}} \,.$$

Then for any $K \in \mathbb{N}$,

$$\sum_{k=0}^{K-1} \frac{g(k+1) - 3g(k)}{3^{k+1}} = \frac{g(K)}{3^K}$$

$\Rightarrow$

$$0 \leq \alpha - \frac{g(K)}{3^K}$$

$$= \sum_{k=K}^{\infty} \frac{g(k+1) - 3g(k)}{3^{k+1}}$$

$$\leq 2 \sum_{k=K}^{\infty} \frac{1}{3^{k+1}}$$

$$= 2 \left(\frac{1}{3^{K+1}} + \frac{1}{3^{K+2}} + \frac{1}{3^{K+3}} + \cdots \right)$$

$$= \frac{2}{3^{K+1}} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \right)$$

$$= \frac{2}{3^{K+1}} \cdot \frac{3}{2} = \frac{1}{3^K}$$

$\Rightarrow$

$$0 \leq \alpha - \frac{g(K)}{3^K}$$

$$= \alpha - \frac{g(K)}{(3^K - 1) + 1}$$

$$\leq \frac{1}{3^K} \leq \frac{1}{K+1} \,.$$

Therefore the real number α is E^3-computable:

$$\alpha \in R_{E^3}.$$

Still

$$\alpha \notin R_{E^2}.$$

To establish this, proceed by contradiction and assume that $\forall\, x \in N$,

$$\left| \frac{f(x) - g(x)}{x + 1} - \alpha \right| < \frac{1}{x + 1} \quad \text{(cf. \#3)}.$$

Here f and g belong to E^2, as does $|f - g|$. And, since α is nonnegative, $\forall\, x \in N$,

$$\left| \frac{|f(x) - g(x)|}{x + 1} - \alpha \right| < \frac{1}{x + 1}.$$

Proof: For all real S and T,

$$\left|\, |S| - |T| \,\right| \leq |S - T|$$

$$\Rightarrow$$

$$\left| \frac{|f(x) - g(x)|}{|x + 1|} - |\alpha| \right|$$

$$= \left| \frac{|f(x) - g(x)|}{x + 1} - \alpha \right|$$

$$\leq \left| \frac{f(x) - g(x)}{x + 1} - \alpha \right| < \frac{1}{x + 1}.$$

Choose now per \#5 a natural number X such that $\forall\, x \in N$,

$$|f(x) - g(x)| = h(X, x),$$

hence

$$\left| \frac{h(X, x)}{x + 1} - \alpha \right| < \frac{1}{x + 1}.$$

In particular:

$$\left| \frac{h(X, 2 \cdot 3^{X+1} - 1)}{2 \cdot 3^{X+1}} - \alpha \right| < \frac{1}{2 \cdot 3^{X+1}} \ .$$

There are then two possibilities, both of which lead to a contradiction.

●

$$6g(X) + 3 \leq h(X, 2 \cdot 3^{X+1} - 1)$$

=>

$$\frac{6g(X) + 3}{2 \cdot 3^{X+1}} \leq \frac{h(X, 2 \cdot 3^{X+1} - 1)}{2 \cdot 3^{X+1}}$$

=>

$$\frac{3g(X)}{3^{X+1}} + \frac{1}{2 \cdot 3^{X}} \leq \frac{h(X, 2 \cdot 3^{X+1} - 1)}{2 \cdot 3^{X+1}}$$

=>

$$\frac{g(X)}{3^{X}} + \frac{1}{2 \cdot 3^{X}} \leq \frac{h(X, 2 \cdot 3^{X+1} - 1}{2 \cdot 3^{X+1}}$$

$$< \alpha + \frac{1}{2 \cdot 3^{X+1}} \ .$$

But

$$0 \leq \alpha - \frac{g(X+1)}{3^{X+1}} \leq \frac{1}{3^{X+1}}$$

=>

$$\alpha \leq \frac{g(X+1)}{3^{X+1}} + \frac{1}{3^{X+1}}$$

$$= \frac{3g(X)}{3^{X+1}}$$

$$= \frac{g(X)}{3^{X}} + \frac{1}{3^{X+1}}$$

$\Rightarrow$

$$\cdots \; < \; \alpha + \frac{1}{2 \cdot 3^{X+1}}$$

$$\leq \; \frac{g(X)}{3^X} + \frac{1}{3^{X+1}} + \frac{1}{2 \cdot 3^{X+1}}$$

$$= \; \frac{g(X)}{3^X} + \frac{1}{3^{X+1}} \; \left(1 + \frac{1}{2}\right)$$

$$= \; \frac{g(X)}{3^X} + \frac{1}{3^{X+1}} \; \left(\frac{3}{2}\right)$$

$$= \; \frac{g(X)}{3^X} + \frac{1}{2 \cdot 3^X} \; .$$

Contradiction.

●

$$6g(X) + 3 > h(X, 2 \cdot 3^{X+1} - 1)$$

$\Rightarrow$

$$\frac{g(X)}{3^X} + \frac{1}{2 \cdot 3^X} > \frac{h(X, 2 \cdot 3^{X+1} - 1)}{2 \cdot 3^{X+1}}$$

$$> \; \alpha - \frac{1}{2 \cdot 3^{X+1}} \; .$$

But

$$\alpha \; \geqslant \; \frac{g(X+1)}{3^{X+1}}$$

$\Rightarrow$

$$\alpha = \frac{3g(X) + 2}{3^{X+1}} = \frac{g(X)}{3^X} + \frac{2}{3^{X+1}}$$

$\Rightarrow$

$$\cdots \; > \; \alpha - \frac{1}{2 \cdot 3^{X+1}}$$

$$\geq \frac{g(X)}{3^X} + \frac{2}{3^{X+1}} - \frac{1}{2 \cdot 3^{X+1}}$$

$$= \frac{g(X)}{3^X} + \frac{1}{3^{X+1}} \left(2 - \frac{1}{2}\right)$$

$$= \frac{g(X)}{3^X} + \frac{1}{3^{X+1}} \left(\frac{3}{2}\right)$$

$$= \frac{g(X)}{3^X} + \frac{1}{2 \cdot 3^X} .$$

Contradiction.

$\underline{8:}$ REMARK Suppose that $n \geq 2$ -- then it can be shown that #5 remains

valid if E^3 is replaced by E^{n+1} and E^2 is replaced by E^n. This said, the argument

above goes through without change, the conclusion being that

$$\exists \ \alpha \in R_{E^{n+1}} : \alpha \notin R_{E^n} .$$

$\underline{9:}$ THEOREM

$$P_{KZ} \subset R_{\ell EL} .$$

[For the details, see Katvin Tent and Martin Ziegler[†] .]

So there is a chain

$$P_{KZ} \subset R_{\ell EL} \subset R_{E^2} \subset R_{E^3} \equiv R_{EL} .$$

And in view of what has been said above, the containment

$$R_{E^2} \subset R_{E^3}$$

is strict.

[†] *Münster Journal of Mathematics* 3 (2010), pp. 43-66.

10: <u>N.B.</u> It is unknown whether $\ell\mathrm{EL}$ equals E^2 or not.

11: EXAMPLE $\pi \in R_{\ell\mathrm{EL}}$ (cf. §11, #1) but actually $\pi \in P_{\mathrm{KZ}}$ (cf. §2, #6).
[Note: $e \in R_{\ell\mathrm{EL}}$ but it is not known if $e \in P_{\mathrm{KZ}}$.]

12: EXAMPLE $e^{\pi} \in R_{\ell\mathrm{EL}}$ (see the Appendix to §11).

§13. RECURSIVE FUNCTIONS

<u>1:</u> DEFINITION The set of <u>recursive functions</u>, denoted by R, is the inductive closure of the initial functions with respect to the operations of composition, primitive recursion, and minimization.

<u>2:</u> N.B. Obviously

$$\mathfrak{PR} \subset \mathfrak{R}.$$

[Note: The containment is proper (the Ackermann function figuring in the Appendix to §5 is recursive but not primitive recursive).]

<u>3:</u> REMARK An important property of $\mathfrak{PR}$ is that it is a recursively enumerable subset of $\mathfrak{R}$, i.e., there is a two-way function $u(m,n)$ that enumerates the primitive recursive functions in the sense that

- $\forall\ f\ \in\ \mathfrak{PR},\ \exists\ m{:}\forall n,\ f(n)\ =\ u(m,n)$.

- $\forall\ m,\ u(m,\text{--})\ \in\ \mathfrak{PR}$.

[Note: On the other hand, $\mathfrak{R}$ itself is not recursively enumerable.]

In the theory developed in §8, take $F = \mathfrak{R}$ (the standard conditions are then obviously in force). So an $\mathfrak{R}$-sequence is a function $A{:}\mathbb{N} \to \mathbb{Q}$ that has a representation of the form

$$A(x)\ =\ \frac{f(x)\ -\ g(x)}{h(x)\ +\ 1}\ (x\ =\ 0,1,2,\ldots),$$

where $f,g,h{:}\mathbb{N} \to \mathbb{N}$ belong to $\mathfrak{R}$ and a real number α is said to be <u>$\mathfrak{R}$-computable</u> if there exists an $\mathfrak{R}$-sequence $A{:}\mathbb{N} \to \mathbb{Q}$ such that $\forall\ x$,

$$\left|A(x)\ -\ \alpha\right|\ \leqq\ \frac{1}{x\ +\ 1}\ .$$

<u>4:</u> <u>N.B.</u> Rather than working with $(x+1)^{-1}$ one can work instead with 2^{-x}, either definition leading to the same set of $\mathfrak{R}$-computable real numbers.

[Note: This is not always permissible. E.g.: Take $F = E^2$ — then the use of 2^{-x} would imply that the E^2-computables are precisely the rationals, which is untenable. However, the switch to 2^{-x} is permissible if $F = E^n$ ($n \geq 3$), in particular if $n = 3$ ($\Rightarrow E^3 = EL$) or if $F = \mathfrak{PR}$.]

<u>5:</u> NOTATION Denote the set of all $\mathfrak{R}$-computable real numbers by the symbol $R_{\mathfrak{R}}$ (cf. §8, #10).

<u>6:</u> THEOREM $R_{\mathfrak{R}}$ is a real closed field (cf. §8, #21).

[Note: In addition, $R_{\mathfrak{R}}$ is countable.]

<u>7:</u> <u>N.B.</u> It is customary to refer to the elements of $R_{\mathfrak{R}}$ as simply the <u>computable reals</u>.

<u>8:</u> EXAMPLE

$$P_{KZ} \subset R_{\mathcal{l}EL} \text{ (cf. §12, #9)} \subset R_{\mathfrak{PR}} \subset R_{\mathfrak{R}}.$$

Therefore periods are computable.

<u>9:</u> EXAMPLE Chaitins constant(s) Ω is(are) not computable.

While the very definition of "computable real" involves recursive funtions, matters can also be formulated in terms of primitive recursive functions.

<u>10:</u> DEFINITION Let α be a real number — then a <u>primitive recursive approximation</u> of α is a pair (A,E) of $\mathfrak{PR}$-sequences $A,E:N \to Q$ such that E is

monotonically decreasing to 0 and such that $\forall$ x,

$$|A(x) - \alpha| \leq E(x).$$

[Note: In general, E depends on α.]

<u>11</u>: EXAMPLE Suppose that $\alpha \in R_{PR}$, so there exists a PR-sequence $A: N \to Q$ such that $\forall$ x,

$$|A(x) - \alpha| \leq \frac{1}{x + 1}.$$

Then the pair

$$(A(x), \frac{1}{x + 1})$$

is a primitive recursive approximation to α.

<u>12</u>: THEOREM A real number α is computable iff it has a primitive recursive approximation.

One direction is straightforward. Thus consider a real number α with the stated property. Define $s: N \to N$ by the rule

$$s(x) = \min\{n:E(n) \leq \frac{1}{x + 1}\}.$$

Then s is recursive, hence $A \circ s$ is an R-sequence (cf. §8, #13) and $\forall$ x,

$$|A(s(x)) - \alpha| \leq \frac{1}{x + 1}.$$

Therefore α is R-computable.

In the other direction:

<u>13</u>: LEMMA Suppose that $\alpha \in R_R$ — then there exists a pair (A,E) of PR-sequences $A,E: N \to Q$ with the property that there are elements of E(N) which are arbitrarily close to 0 and such that $\forall$ x,

$$|A(x) - \alpha| \leq E(x).$$

PROOF The assumption on α implies that there exists an $\mathfrak{R}$-sequence A' such that $\forall\, x$,

$$\left|A'(x) - \alpha\right| \leq \frac{1}{x + 1} \, .$$

This said, choose a surjective primitive recursive function $f: \mathbb{N} \to \mathbb{N}$ such that

$$A(x) = A'(f(x))$$

is a $\mathfrak{PR}$-sequence. Put

$$E(x) = \frac{1}{f(x) + 1} \, .$$

Then

$$\left|A(x) - \alpha\right| = \left|A'(f(x)) - \alpha\right|$$

$$\leq \frac{1}{f(x) + 1} = E(x) \, .$$

To finish the proof of #12, one has only to take the data supplied by #13 and transform it into that required of #10. Using primes, put

$$\left\lceil \begin{array}{l} E'(n) = \min\{E(i): 0 \leq i \leq n\} \\[2mm] k(n) \;= \min\{i: 0 \leq i \leq n, \; E(i) = E'(n)\} \\[2mm] A'(n) = A(k(n)) \, . \end{array} \right.$$

Then the pair (A',E') is a primitive recursive approximation of α.

<u>14:</u> RAPPEL To say that a real number is computable means that there exists an $\mathfrak{R}$-sequence $A: \mathbb{N} \to \mathbb{Q}$ such that $\forall\, x$,

$$\left|A(x) - \alpha\right| \leq \frac{1}{x + 1} \, .$$

Question: Can one instead utilize a $\mathfrak{PR}$-sequence? The answer in general is "no".

15: EXAMPLE Let $f: N \to \{0,1\}$ be recursive and put

$$\alpha = \sum_{n=0}^{\infty} \frac{f(n)}{4^n} .$$

Then $\alpha \in R_{\Re}$ and the claim is that there is no $\Re$-sequence $A: N \to Q$ with the property that $\forall x$,

$$|A(x) - \alpha| \leq \frac{1}{x+1} .$$

[The initial observation is that if $k \in N$, $q \in Q$, and

$$|q - \alpha| \leq \frac{1}{4^{k+1}} ,$$

then

$$f(k) = \lfloor \frac{[2 \cdot 4^k q + 1/2] \bmod 4}{2} \rfloor \quad (cf. \; infra).$$

Granted this, consider a $\Re$-sequence $A: N \to Q$ subject to $\forall x$,

$$|A(x) - \alpha| \leq \frac{1}{x+1} ,$$

and let

$$q = A(4^{k+1} - 1),$$

thus

$$|q - \alpha| \leq \frac{1}{4^{k+1} - 1 + 1} = \frac{1}{4^{k+1}} ,$$

and the formula for $f(k)$ implies that f is primitive recursive. Accordingly, if α is constructed by using a function $f: N \to \{0,1\}$ that is recursive but not primitive recursive, i.e., if

$$f \in \Re \text{ but } f \notin \Re,$$

then there will be no $\Re$-sequence $A: N \to Q$ per supra.]

[Details:

$$2\alpha = \sum_{n=0}^{\infty} \frac{2f(n)}{4^n}$$

=>

$$[2 \cdot 4^k\alpha] \bmod 4 = 2f(k).$$

And

$$2 \cdot 4^k\alpha \le 2 \cdot 4^k q + 1/2 \le 2 \cdot 4^k\alpha + 1$$

=>

$$[2 \cdot 4^k q + 1/2] = [2 \cdot 4^k\alpha] + d \quad (d = 0 \text{ or } d = 1)$$

=>

$$[2 \cdot 4^k q + 1/2] \bmod 4 = 2f(k) + d.]$$

16: DEFINITION Let α be a real number — then α has a <u>primitive recursive nested interval representation</u> if there are PR-sequences $f,g:N \to Q$ such that $\forall x$,

$$f(x) \le f(x + 1) \le \alpha \le g(x + 1) \le g(x)$$

and

$$\lim_{x \to \infty} (g(x) - f(x)) = 0.$$

17: LEMMA A real number α has a primitive recursive nested interval representation iff it admits a primitive recursive approximation.

PROOF

=> Given f,g, let

$$A(x) = \frac{g(x) + f(x)}{2} \quad \text{and} \quad E(x) = \frac{g(x) - f(x)}{2}.$$

Then

$$\left|A(x) - \alpha\right| = \left|\frac{g(x) + f(x)}{2} - \alpha\right|$$

$$= \left|\frac{g(x)}{2} - \frac{\alpha}{2} + \frac{f(x)}{2} - \frac{\alpha}{2}\right|$$

$$\leq \left|\frac{g(x)}{2} - \frac{\alpha}{2}\right| + \left|\frac{f(x)}{2} - \frac{\alpha}{2}\right|$$

$$= \frac{g(x)}{2} - \frac{\alpha}{2} + \frac{\alpha}{2} - \frac{f(x)}{2}$$

$$= \frac{g(x) - f(x)}{2} = E(x).$$

And

$$E(x + 1) \leq E(x)$$

iff

$$g(x + 1) - f(x + 1) \leq g(x) - f(x)$$

iff

$$g(x + 1) - g(x) \leq f(x + 1) - f(x)$$

iff

$$g(x) - g(x + 1) \geq f(x) - f(x + 1).$$

But

$$\left[\begin{array}{l} g(x) - g(x + 1) \geq 0 \\ f(x) - f(x + 1) \leq 0. \end{array}\right.$$

$\Leftarrow$ Given A,E, let

$$\left[\begin{array}{l} f(x) = \max\{A(n) - E(n): n \leq x\} \\ g(x) = \min\{A(n) + E(n): n \leq x\}. \end{array}\right.$$

18: SCHOLIUM A real number α is computable iff it has a primitive recursive nested interval representation.

19: EXAMPLE

$$Q \subseteq R_{\mathbb{R}}.$$

[Given $q \in Q$, let

$$
\left\lfloor
\begin{array}{l}
f(x) = q - 2^{-x} \\
g(x) = q + 2^{x}.
\end{array}
\right.
$$
]

§14. EXPANSION THEORY

Let $b > 1$ be a natural number and let α be a nonnegative real number -- then α has a b-adic representation if there exists a recursive function $f : N \to \{0, 1, \ldots, b-1\}$ such that

$$\alpha = \sum_{n=0}^{\infty} \frac{f(n)}{b^n} .$$

1: LEMMA If α has a b-adic representation, then $\alpha \in R_{\mathbb{R}}$.

PROOF Put

$$A(x) = \sum_{n=0}^{x+1} \frac{f(n)}{b^n} \quad (x = 0, 1, 2, \ldots).$$

Then $A : N \to Q$ is an R-sequence and

$$|A(x) - \alpha| = \left| \sum_{n=0}^{x+1} \frac{f(n)}{b^n} - \sum_{n=0}^{\infty} \frac{f(n)}{b^n} \right|$$

$$= \sum_{n=x+2}^{\infty} \frac{f(n)}{b^n}$$

$$\leq \sum_{n=x+2}^{\infty} \frac{b-1}{b^n}$$

$$\leq b \sum_{n=x+2}^{\infty} \frac{1}{b^n}$$

$$= b \left(\frac{1}{b^{x+2}} + \frac{1}{b^{x+3}} + \cdots \right)$$

$$= \frac{b}{b^{x+2}} \left(1 + \frac{1}{b} + \cdots \right)$$

$$= \frac{b}{b^{x+2}} \; \frac{1}{1 - \frac{1}{b}}$$

$$= \frac{b}{b^{x+2}} \; \frac{b}{b - 1}$$

$$= \frac{b^2}{b^{x+2}} \; \frac{1}{b - 1}$$

$$= \frac{1}{b^x} \; \frac{1}{b - 1}$$

$$\leq \frac{1}{b^x} \quad (2 \leq b \Rightarrow 1 \leq b - 1 \Rightarrow \frac{1}{b - 1} \leq 1)$$

$$= b^{-x} \leq 2^{-x}.$$

2: __NOTATION__ $R_{\mathcal{R}}^b$ is the set of nonnegative real numbers which admit a b-adic representation.

Therefore

$$R_{\mathcal{R}}^b \subset R_{\mathcal{R}} \cap [0,\infty[.$$

3: __DEFINITION__ A subset $A \subset \mathbb{N}$ is __recursive__ if its characteristic function χ_A is recursive, i.e., if $\chi_A \in \mathcal{R}$.

4: __EXAMPLE__ Suppose that $\alpha \in R_{\mathcal{R}}^2$, hence

$$\alpha = \sum_{n=0}^{\infty} \frac{f(n)}{2^n} \quad (f(n) \in \{0,1\}).$$

Let

$$A = \{n : f(n) = 1\}.$$

Then

$$\alpha = \sum_{n \in A} 2^{-n}$$

is computable and $\chi_A = f$, so A is recursive (this being the case of f).

<u>5:</u> SUBLEMMA Suppose that α is an irrational computable real number, thus there exists an R-sequence $A: \mathbb{N} \to \mathbb{Q}$ such that $\forall x$,

$$|A(x) - \alpha| \leq 2^{-x} \quad (cf. \ \S13, \ \#4).$$

Let $f: \mathbb{N} \to \mathbb{Q}$ be an R-sequence -- then $\forall n \ \exists x$ such that

$$|A(x) - f(n)| > 2^{-x}.$$

And

$$\begin{cases} A(x) - f(n) > 2^{-x} \Rightarrow x > f(n) \\ A(x) - f(n) < 2^{-x} \Rightarrow x < f(n). \end{cases}$$

<u>6:</u> THEOREM

$$R_{\mathbb{R}}^b = R_{\mathbb{R}} \cap [0, \infty[.$$

PROOF Take an α in $R_{\mathbb{R}} \cap [0, \infty[$. If it is rational, say $\frac{p}{q}$, work relative to the base b and carry out the long division of p by q. Assume, therefore, that α is irrational and without loss of generality take α in $R_{\mathbb{R}} \cap]0, 1[$. Suppose now that we have found natural numbers $n_0 = 0, \ldots, n_k$ such that $0 \leq n_j \leq b - 1$ $(1 \leq j \leq k)$ and such that

$$\sum_{j=0}^{k} n_j b^{-j} < \alpha < \sum_{j=0}^{k-1} n_j b^{-j} + (n_k + 1)b^{-k}.$$

Applying #5, there is a unique $x \in \{0, 1, \ldots, b-1\}$ for which

$$\sum_{j=0}^{k} n_j b^{-j} + x b^{-k-1} < \alpha < \sum_{j=0}^{k} n_j b^{-j} + (x + 1) b^{-k-1}.$$

Definition: $n_{k+1} = x$. With this agreement, set $f(k) = n_k$ -- then $f:N \to N$ is a recursive function, $f(k) \in \{0,1,\ldots,b-1\}$, and

$$\alpha = \sum_{k=0}^{\infty} \frac{f(k)}{b^k}.$$

7: **N.B.** It is to be emphasized that the realization

$$R_{\mathbb{R}}^{b} = R_{\mathbb{R}} \cap [0,\infty[$$

is valid for all b.

8: **EXAMPLE** Given a subset $A \subset N$, put

$$\alpha_A = \sum_{n \in A} 2^{-n}.$$

Then α_A is computable iff A is recursive.

PROOF If α_A is computable, then $\alpha_A \in R_{\mathbb{R}}^{2}$ (cf. #6), so A is recursive (cf. #4). On the other hand, if A is recursive, let

$$s_x = \sum_{n \leq x, n \in A} 2^{-n}.$$

Take

$$\begin{cases} f(x) = s_x \\ g(x) = s_x + 2^{-x}. \end{cases}$$

Then

$$s_x \leq \alpha_A = \sum_{n \in A} 2^{-n}$$

$$= s_x + \sum_{n>x, n\in A} 2^{-n}$$

$$\leq s_x + 2^{-(x+1)} + 2^{-(x+2)} + \cdots$$

$$= s_x + 2^{-(x+1)}(1 + 2^{-1} + \cdots)$$

$$= s_x + 2^{-(x+1)}(2)$$

$$= s_x + 2^{-x}.$$

Now quote §13, #18.

$\underline{9:}$ REMARK Assume that A is not recursive -- then for certain A, it is possible to find an injective $f \in \mathcal{R}$ such that A equals the range of f, hence

$$\alpha_A = \sum_{n\in N} 2^{-f(n)}.$$

Put

$$q_x = \sum_{n\leq x} 2^{-f(n)} \quad (x = 0,1,2,\ldots).$$

Then $q_0, q_1, q_2, \ldots$ is a bounded increasing sequence of rational numbers whose limit α_A is not computable.

We have worked thus far with

$$R_{\mathcal{R}} \cap [0, \infty[.$$

Replace now $R_{\mathcal{R}}$ by $P_{\mathcal{PR}}$ --- then it is still the case that

$$R^b_{\mathcal{PR}} \subset R_{\mathcal{PR}} \cap [0, \infty[$$

but it is no longer true that

$$R^b_{\mathcal{PR}} = R_{\mathcal{PR}} \cap [0, \infty[.$$

10: THEOREM R_{PR} is a field (cf. §8, #15).

11: LEMMA[†] For each b > 1, there are $x, y \in R_{PR}^b$ such that $x + y \notin R_{PR}^b$.

Consequently the containment

$$R_{PR}^b \subseteq R_{PR} \cap [0, \infty[$$

is proper.

One can also consider the relationship among the R_{PR}^b for different b.

12: THEOREM Let b,d > 1 — then

$$R_{PR}^b \subseteq R_{PR}^d$$

iff d divides a power of b, i.e., iff there exist $k, s \in N$ such that $b^k = sd$.

[For details and references, see the paper of Qingliang Chen et al.]

[†] Qingliang Chen et al., *Mathematical Logic Quarterly* 53 (2007), pp. 365–380.