Analysis 101:

Curves and Length
ABSTRACT

In addition to providing a systematic account of the classical theorems of Jordan and Tonelli, I have also provided an introduction to the theory of the Weierstrass integral which in its definitive form is due to Cesari.
§1. FUNDAMENTALS
§2. ESTIMATES
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§10. LINE INTEGRALS (bis)
\[ \frac{3\text{. EXAMPLE}}{3} \] Every function \( f: [a, b] \to \mathbb{R} \) gives rise to a curve \( C \) in \( \mathbb{R}^2 \), viz. the arrow \( x \mapsto (x, f(x)) \).

\[ \frac{4\text{. DEFINITION}}{4} \] The **graph** of \( C \), denoted \([C]\), is the range of \( f \).

\[ \frac{5\text{. EXAMPLE}}{5} \] Take \( M = 2 \), let \( k = 1, 2, \ldots \), and put
\[
\bar{f}_k(x) = (\sin^2(kx), 0) \quad (0 \leq x \leq \frac{\pi}{2}).
\]
Then the \( \bar{f}_k \) all have the same range, i.e., \([C_1] = [C_2] = \cdots \) if \( C_k \leftrightarrow \bar{f}_k \) but the \( C_k \) are different curves.

\[ \frac{6\text{. REMARK}}{6} \] If \( C \) is a continuous curve, then its graph \([C]\) is closed, bounded, connected, and uniformly locally connected. Owing to a theorem of Hahn
2.

and Mazurkiewicz, these properties are characteristic: Any such set is the graph of a continuous curve. So, e.g., a square in $\mathbb{R}^2$ is the graph of a continuous curve, a cube in $\mathbb{R}^3$ is the graph of a continuous curve etc.

7: DEFINITION The **length** of a curve $C$, denoted $l(C)$, is

$$T_f[a,b] \equiv \sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^{n} ||f(x_i) - f(x_{i-1})||,$$

$C$ being termed **rectifiable** if $l(C) < +\infty$.

(Note: If $C$ is continuous and rectifiable, then $\forall \varepsilon > 0, \exists \delta > 0$:

$$||P|| < \delta \Rightarrow \forall (f,P) \equiv \sum_{i=1}^{n} ||f(x_i) - f(x_{i-1})|| > l(C) - \varepsilon.$$ )

8: LEMMA Given a curve $C$,

$$T_f[a,b] \leq l(C) \leq T_f[a,b] + \cdots + T_f[a,b]_{1 \leq m \leq M}. $$

9: SCHOLIUM $C$ is rectifiable iff

$$f_1 \in BV[a,b], \ldots, f_M \in BV[a,b].$$

10: THEOREM Let

$$C_n \leftrightarrow f_n : [a,b] \rightarrow \mathbb{R}^M$$

and assume that $f_n$ converges pointwise to $f$ -- then

$$l(C) \leq \lim \inf_{n \rightarrow \infty} l(C_n).$$
A continuous curve

\[ \Gamma \leftrightarrow \gamma : [a,b] \to \mathbb{R}^M \]

is said to be a **polygonal line** (and \( \gamma \) quasi linear in \([a,b]\)) if there exists a \( P \in \mathbb{P}[a,b] \) in each segment of which \( \gamma \) is linear or a constant.

11: **DEFINITION** The **elementary length** \( \ell_e(\Gamma) \) of \( \Gamma \) is the sum of the lengths of these segments, hence \( \ell_e(\Gamma) = \ell(\Gamma) \).

12: **NOTATION** Given a continuous curve \( C \), denote by \( \Gamma(C) \) the set of all sequences

\[ \Gamma_n \leftrightarrow \gamma_n : [a,b] \to \mathbb{R}^M \]

of polygonal lines such that

\[ \gamma_n \rightharpoonup f \quad (n \to \infty) \]

uniformly in \([a,b]\).

Therefore

\[ \ell(C) \leq \liminf_{n \to \infty} \ell(\Gamma_n) = \liminf_{n \to \infty} \ell_e(\Gamma_n). \]

On the other hand, by definition, there is some \( \{\Gamma_n\} \in \Gamma(C) \) such that

\[ \ell_e(\Gamma_n) \to \ell(C) \quad (n \to \infty). \]

13: **SCHOLIUM** If \( C \) is a continuous curve, then

\[ \ell(C) = \inf_{\{\Gamma_n\} \in \Gamma(C)} \liminf_{n \to \infty} \ell_e(\Gamma_n). \]

14: **REMARK** Let

\[ C \leftrightarrow f : [a,b] \to \mathbb{R}^M. \]
Assume: C is continuous and rectifiable -- then f can be decomposed as a sum
\[ f = f_{AC} + f_C, \]
where \( f_{AC} \) is absolutely continuous and \( f_C \) is continuous and singular.

Therefore
\[ l(C) = \int_{AC} f \, [a,b] + \int_C f \, [a,b]. \]
§2. ESTIMATES

1: NOTATION Write
\[ T_f[a,b] \]
in place of \( \ell(C) \).

2: DEFINITION Assume that \( C \) is rectifiable — then the **arc length function**
\[ s: [a,b] \to \mathbb{R} \]
is defined by the prescription
\[ s(x) = T_f[a,x] \quad (a \leq x \leq b). \]

Obviously
\[ s(a) = 0, \quad s(b) = \ell(C), \]
and \( s \) is an increasing function.

3: LEMMA If \( C \) is continuous and rectifiable, then \( s \) is continuous as are the \( T_f[a,-] \quad (m = 1,\ldots,M). \)

4: LEMMA If \( C \) is continuous and rectifiable, then \( s \) is absolutely continuous iff all the \( T_f[a,-] \quad (m = 1,\ldots,M) \) are absolutely continuous, hence iff all the \( f'_m \quad (m = 1,\ldots,M) \) are absolutely continuous.

If \( C \) is continuous and rectifiable, then the \( f'_m \in BV[a,b] \), thus the derivatives \( f'_m \) exist almost everywhere in \([a,b]\) and are Lebesgue integrable. On the other hand, \( s \) is an increasing function, thus it too is differentiable almost everywhere in \([a,b]\) and is Lebesgue integrable.
5: SUBLEMMA The connection between $f'$ and $s'$ is given by the relation
\[ \|f'\| \leq s' \]
almost everywhere in $[a,b]$.

[For any subinterval $[\alpha, \beta] \subset [a,b]$,
\[ |f(\beta) - f(\alpha)| \leq s(\beta) - s(\alpha). \]

6: LEMMA

\[ \ell(C) = s(b) - s(a) \geq \int_{a}^{b} s' \geq \int_{a}^{b} \|f'\|. \]

I.e.: Under the assumption that $C$ is continuous and rectifiable,
\[ \ell(C) \geq \int_{a}^{b} \|f'\|. \]

7: THEOREM

\[ \ell(C) = \int_{a}^{b} \|f'\| \]

iff all the $f_{m}$ ($m = 1, \ldots, M$) are absolutely continuous.

This is established in the discussion to follow.

- Suppose that the equality sign obtains, hence
\[ s(b) - s(a) = \int_{a}^{b} s'. \]

But also
\[ s(x) - s(a) \geq \int_{a}^{x} s', \quad s(b) - s(x) \geq \int_{x}^{b} s'. \]

If
\[ s(x) - s(a) > \int_{a}^{x} s', \quad s(b) - s(x) > \int_{x}^{b} s', \]

then
\[ s(b) - s(a) > \int_{a}^{b} s'. \]
a contradiction. Therefore

\[ s(x) - s(a) = \int_a^x s' \]

\[ \Rightarrow s \in AC[a,b] \Rightarrow f_m \in AC[a,b] \quad (m = 1, \ldots, M). \]

Consider the other direction, i.e., assume that the \( f_m \in AC[a,b] \), the claim being that

\[ \ell(C) = \int_a^b ||f'||. \]

Given \( P \in P[a,b] \), write

\[
\begin{align*}
\sum_{i=1}^n \sum_{m=1}^M \left| f'(x_i) - f'(x_{i-1}) \right| \\
= \sum_{i=1}^n \sum_{m=1}^M \left( \int_{x_{i-1}}^{x_i} f_m' \right)^2 \left( \sum_{m=1}^M \left( f_m' \right)^2 \right)^{1/2} \\
\leq \sum_{i=1}^n \sum_{m=1}^M \left( \int_{x_{i-1}}^{x_i} f_m' \right)^2 \left( \sum_{m=1}^M \left( f_m' \right)^2 \right)^{1/2} \\
= \int_a^b ||f'||. 
\end{align*}
\]

Taking the sup of the first term over all \( P \) then gives

\[ \ell(C) \leq \int_a^b ||f'|| \quad (\leq \ell(C)) \]

\[ \Rightarrow \]

\[ \ell(C) = \int_a^b ||f'||. \]

8: N.B. Under canonical assumptions,

\[ \left( (\int_{x_1} f_1')^2 + \cdots + (\int_{x_n} f_n')^2 \right)^{1/2} \]
4. 

\[ \leq \int_X (\phi_1^2 + \cdots + \phi_n^2)^{1/2}. \]

9: **RAPPEL** Suppose that \( f \in BV[a,b] \) -- then for almost all \( x \in [a,b] \),

\[ |f'(x)| = T^f_{[a,x]} \]

10: **LEMMA** Suppose that \( C \) is continuous and rectifiable -- then

\[ s' = |f'| \]

almost everywhere in \([a,b]\).

**PROOF** Since

\[ ||f'|| \leq s', \]

it suffices to show that

\[ s' \leq ||f'||. \]

Let \( E_0 \subset [a,b] \) be the set of \( x \) such that \( f \) and \( s \) are differentiable at \( x \) and \( s'(x) > ||f'(x)|| \) and for \( k = 1, 2, \ldots \), let \( E_k \) be the set of \( x \in E_0 \) such that

\[ \frac{s(t_2) - s(t_1)}{t_2 - t_1} \geq \frac{||f(t_2) - f(t_1)||}{t_2 - t_1} + \frac{1}{k} \]

for all intervals \([t_1,t_2]\) such that \( x \in [t_1,t_2] \) and \( 0 < t_2 - t_1 \leq \frac{1}{k} \). So, by construction,

\[ E_0 = \bigcup_{k=1}^{\infty} E_k \]

and matters reduce to establishing that \( \forall k, \lambda(E_k) = 0 \). To this end, let \( \varepsilon > 0 \) and choose \( P \in P[a,b] : \)

\[ \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| > T^f_{[a,b]} - \varepsilon. \]

Expanding \( P \) if necessary, it can be assumed without loss of generality that
5.

\[ 0 < x_i - x_{i-1} \leq \frac{1}{k} \quad (i = 1, \ldots, n). \]

For each \( i \), either \([x_{i-1}, x_i] \cap E_k \neq \emptyset\) and then

\[ s(x_i) - s(x_{i-1}) \geq \| \overline{f(x_i)} - \overline{f(x_{i-1})} \| + \frac{x_i - x_{i-1}}{k}, \]

or \([x_{i-1}, x_i] \cap E_k = \emptyset\) and then

\[ s(x_i) - s(x_{i-1}) \geq \| \overline{f(x_i)} - \overline{f(x_{i-1})} \|. \]

Consequently

\[ T_f[a, b] = s(b) = s(x_n) \]

\[ = \sum_{i=1}^{n} (s(x_i) - s(x_{i-1})) \quad (s(x_0) = s(a) = 0) \]

\[ \geq \sum_{i=1}^{n} \| \overline{f(x_i)} - \overline{f(x_{i-1})} \| + \frac{1}{k} \lambda^*(E_k) \]

\[ \geq T_f[a, b] - \epsilon + \frac{1}{k} \lambda^*(E_k) \]

\[ = \]

\[ \lambda^*(E_k) \leq k \epsilon \Rightarrow \lambda(E_k) = 0 \quad (\epsilon \downarrow 0). \]

\textbf{11: THEOREM} Suppose that \( C \) is continuous and rectifiable. Assume: \( M > 1 \) --
then the \( M \)-dimensional Lebesgue measure of \([C]\) is equal to 0.

\textbf{12: NOTATION} Let

\[ C \leftrightarrow f: [a, b] \to \mathbb{R}^M \]

be a continuous curve. Given \( x \in [C] \), let \( N(f; x) \) be the number of points \( x \in [a, b] \)
(finite or infinite) such that \( f(x) = x \) and let \( N(f; \rightarrow) = 0 \) in the complement \( \mathbb{R}^M - [C] \) of \([C]\).
13: **THEOREM**

\[ \ell(C) = \int_{\mathbb{R}^M} N(f; \cdot) \, dH^1. \]

[Note: \( H^1 \) is the 1-dimensional Hausdorff outer measure in \( \mathbb{R}^M \) and]

\[ H^1([C]) = \int_{\mathbb{R}^M} \chi_{[C]} \, dH^1 \leq \int_{\mathbb{R}^M} N(f; \cdot) \, dH^1, \]

i.e.,

\[ H^1([C]) \leq \ell(C) \]

and it can happen that

\[ H^1([C]) < \ell(C). \]

14: **N.B.** If \( f \) is one-to-one, then

\[ N(f; \cdot) = \chi_{[C]} \]

and when this is so,

\[ H^1([C]) = \ell(C). \]
§3. EQUVALENCEs

In what follows, by interval we shall understand a finite closed interval \( c \subseteq \mathbb{R} \).

[Note: If \( I, J \) are intervals and if \( \exists I = \{a, b\}, \exists J = \{c, d\} \), then the agreement is that a homeomorphism \( \phi: I \to J \) is sense preserving, i.e., sends \( a \) to \( c \) and \( b \) to \( d \).]

1: DEFINITION Suppose given intervals \( I, J \), and curves \( f: I \to \mathbb{R}^M \), \( g: J \to \mathbb{R}^M \) -- then \( f \) and \( g \) are said to be Lebesgue equivalent if there exists a homeomorphism \( \phi: I \to J \) such that \( f = g \circ \phi \).

2: LEMMA If

\[
\begin{align*}
\quad & f: [a, b] \to \mathbb{R}^M \\
\quad & g: [a, b] \to \mathbb{R}^M
\end{align*}
\]

are Lebesgue equivalent and if

\[
\begin{align*}
\quad & C \leftrightarrow f \\
\quad & D \leftrightarrow g
\end{align*}
\]

then

\[ \ell(C) = \ell(D). \]

PROOF The homeomorphism \( \phi: [a, b] \to [c, d] \) induces a bijection

\[
\begin{align*}
\quad & P[a, b] \to P[c, d] \\
\quad & P \to Q
\end{align*}
\]

Therefore

\[ \ell(C) = \sup_{P \in P[a, b]} \sum_{i=1}^{n} ||f(x_i) - f(x_{i-1})|| \]
2. 

\[ n = \sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^{n} \left| g(\phi(x_{i})) - g(\phi(x_{i-1})) \right| \]

\[ = \sup_{Q \in \mathcal{Q}[c,d]} \sum_{i=1}^{n} \left| g(y_{i}) - g(y_{i-1}) \right| \]

\[ = \ell(D). \]

3: DEFINITION Suppose given intervals I, J and curves \( f : I \rightarrow \mathbb{R}^{M}, g : J \rightarrow \mathbb{R}^{M} \) -- then \( f \) and \( g \) are said to be Fréchet equivalent if for every \( \varepsilon > 0 \) there exists a homeomorphism \( \phi : I \rightarrow J \) such that

\[ \left| \| f(x) - g(\phi(x)) \| \right| < \varepsilon \quad (x \in I). \]

4: REMARK It is clear that two Lebesgue equivalent curves are Fréchet equivalent but two Fréchet equivalent curves need not be Lebesgue equivalent.

5: LEMMA If

\[ f : [a, b] \rightarrow \mathbb{R}^{M}, \quad g : [a, b] \rightarrow \mathbb{R}^{M} \]

are Fréchet equivalent and if

\[ C \leftrightarrow f, \quad D \leftrightarrow g, \]

then

\[ \ell(C) = \ell(D). \]

PROOF For each \( n = 1, 2, \ldots, \) there is a homeomorphism \( \phi_{n} : [a, b] \rightarrow [c, d] \) such that \( \forall x \in [a, b], \)

\[ \left| \| f(x) - g(\phi_{n}(x)) \| \right| < \frac{1}{n}. \]
3.

Put \( f_n = g \circ \phi_n \), hence \( f_n \) is Lebesgue equivalent to \( g \) (viz. \( g \circ \phi_n = g \circ \phi_n \ldots \)),
thus if
\[
C_n \leftrightarrow f_n, \quad D \leftrightarrow g,
\]
then from the above
\[
\ell(C_n) = \ell(D).
\]
But \( \forall x \in [a,b] \),
\[
\left| f(x) - f_n(x) \right| < \frac{1}{n},
\]
i.e., \( f_n \to f \) pointwise, so
\[
\ell(C) \leq \liminf_{n \to \infty} \ell(C_n)
\]
\[
= \liminf_{n \to \infty} \ell(D)
\]
\[
= \ell(D).
\]
Analogously
\[
\ell(D) \leq \ell(C).
\]
Therefore
\[
\ell(C) = \ell(D).
\]
§4. Fréchet Distance

Let

\[ C \leftrightarrow f: [a, b] \rightarrow \mathbb{R}^M \]
\[ D \leftrightarrow g: [a, b] \rightarrow \mathbb{R}^M \]

be two continuous curves.

1. NOTATION \( H \) is the set of all homeomorphisms \( \phi: [a, b] \rightarrow [c, d] \) (\( \phi(a) = c \), \( \phi(b) = d \)).

Given \( \phi \in H \), the expression

\[ ||f(x) - g(\phi(x))|| \quad (a \leq x \leq b) \]

has an absolute maximum \( M(f, g; \phi) \).

2. DEFINITION The Fréchet distance between \( C \) and \( D \), denoted \( ||C, D|| \), is

\[ \inf_{\phi \in H} M(f, g; \phi). \]

[Note: In other words, \( ||C, D|| \) is the infimum of all numbers \( \varepsilon \geq 0 \) with the property that there exists a homeomorphism \( \phi \in H \) such that

\[ ||f(x) - g(\phi(x))|| \leq \varepsilon \]

for all \( x \in [a, b] \).]

3. N.B. If \( ||C, D|| < \varepsilon \), then there exists a \( \phi \in H \) such that

\[ M(f, g; \phi) < \varepsilon. \]

4. LEMMA Let \( C, D, C_0 \) be continuous curves -- then

(i) \( ||C, D|| \geq 0; \)
2.

(ii) $|C, D| = |D, C|$;

(iii) $|C, D| \leq |C, C_0| + |C_0, D|$;

(iv) $|C, D| = 0$ iff C and D are Fréchet equivalent.

Therefore the Fréchet distance is a premetric on the set of all continuous curves with values in $R^M$.

5: THEOREM Let

$$\begin{align*}
C_n & \leftrightarrow f_n^{-1}: [a_n, b_n] \to R^M \quad (n = 1, 2, \ldots) \\
C & \leftrightarrow f: [a, b] \to R^M
\end{align*}$$

be continuous curves. Assume:

$$|C_n, C| \to 0 \quad (n \to \infty).$$

Then

$$\ell(C) \leq \liminf_{n \to \infty} \ell(C_n).$$

PROOF For every $n$, there is a homeomorphism

$$\phi_n: [a, b] \to [a_n, b_n] \quad (\phi_n(a) = a_n, \quad \phi_n(b) = b_n)$$

such that for all $x \in [a, b],

$$||f(x) - f_n(\phi_n(x))|| < |C_n, C| + \frac{1}{n}.$$ 

Let

$$D_n \leftrightarrow f_n \circ \phi_n: [a, b] \to R^M.$$ 

Then pointwise

$$f_n \circ \phi_n + f$$

=>
3.

\[ \ell(C) \leq \liminf_{n \to \infty} \ell(D_n). \]

But \( \ell(D_n) = \ell(C_n) \), hence

\[ \ell(C) \leq \liminf_{n \to \infty} \ell(C_n). \]

In the set of continuous curves, introduce an equivalence relation by stipulating that \( C \) and \( D \) are equivalent provided \( C \) and \( D \) are Fréchet equivalent. The resulting set \( E_F \) of equivalence classes is then a metric space: If

\[
\begin{align*}
\{C\} & \in E_F \\
\{D\} & \in E_F,
\end{align*}
\]

then

\[ ||\{C\},\{D\}|| = ||C,D||. \]

6: N.B. If \( C, C' \) are Fréchet equivalent and if \( D, D' \) are Fréchet equivalent, then

\[
\begin{align*}
||C,D|| & \leq ||C,C'|| + ||C',D|| \\
& \leq ||C',D'|| \leq ||C',D'|| + ||D',D'|| \\
& = ||C',D'||
\end{align*}
\]

and in reverse

\[ ||C',D'|| \leq ||C,D||. \]

So

\[ ||C,D|| = ||C',D'||. \]
§5. THE REPRESENTATION THEOREM

Assume:

\[ C \leftrightarrow f : [a,b] \to \mathbb{R}^M \]

is a curve which is continuous and rectifiable.

1: THEOREM There exists a continuous curve

\[ D \leftrightarrow g : [c,d] \to \mathbb{R}^M \]

with the property that

\[ \ell(D) = \ell(C) \quad (< + \infty) \]

and

\[ \ell(D) = \int_C ||g'||. \]

where \( g_1, \ldots, g_M \) are absolutely continuous and in addition \( f \) and \( g \) are Fréchet equivalent.

Take \( \ell(C) > 0 \) and define \( g \) via the following procedure. In the first place, the domain \([c,d]\) of \( g \) is going to be the interval \([0, \ell(C)]\). This said, note that \( s(x) \) is constant in an interval \([a,b]\) iff \( f(x) \) is constant there as well. Next, for each point \( s_0 \) \((0 \leq s_0 \leq \ell(C))\) there is a maximal interval \( a \leq x \leq b \) \((a \leq a \leq b \leq b)\) with \( s(x) = s_0 \). Definition: \( g(s_0) = f(x) \) \((a \leq x \leq b)\).

2: LEMMA

\[ g(s_0^-) = g(s_0) \quad (0 < s_0 < \ell(C)) \]

\[ g(s_0^+) = g(s_0) \quad (0 \leq s_0 < \ell(C)). \]
Therefore
\[ g: [c,d] \to \mathbb{R}^M \]
is a continuous curve.

3: **SUBLEMMA** Suppose that \( \phi_n: [A,B] \to [C,D] \) \((n = 1,2,\ldots)\) converges uniformly to \( \phi: [A,B] \to [C,D] \). Let \( \phi: [C,D] \to \mathbb{R}^M \) be a continuous function -- then \( \phi \circ \phi_n \) converges uniformly to \( \phi \circ \phi \).

**PROOF** Since \( \phi \) is uniformly continuous, given \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that \[ |u - v| < \delta \Rightarrow ||\phi(u) - \phi(v)|| < \varepsilon \ (u,v \in [C,D]). \]

Choose \( N \):
\[ n \geq N \Rightarrow |\phi_n(x) - \phi(x)| < \delta \ (x \in [A,B]). \]

Then
\[ ||\phi(\phi_n(x)) - \phi(\phi(x))|| < \varepsilon. \]

4: **LEMMA** \( f \) and \( g \) are Fréchet equivalent.

**PROOF** Approximate \( s \) by quasilinear, strictly increasing functions \( s_n(x) \) \((a \leq x \leq b)\) with \( s_n(a) = 0, s_n(b) = \ell(C) \) and
\[ |s_n(x) - s(x)| < \frac{1}{n} \ (n = 1,2,\ldots). \]

Then
\[ s_n: [a,b] \to [0,\ell(C)] \]
converges uniformly to
\[ s: [a,b] \to [0,\ell(C)] \]
and
\[ g: [0,\ell(C)] \to \mathbb{R}^M \]
is continuous, so
3.

\[ g \circ s_n \to g \circ s \]

uniformly in \([a,b]\), thus \(\forall \varepsilon > 0, \exists N \ni n \geq N \)

\[ \Rightarrow |g(s_n(x)) - g(s(x))| < \varepsilon \quad (a \leq x \leq b) \]

or still,

\[ |f(x) - g(s_n(x))| < \varepsilon \quad (a \leq x \leq b). \]

Since the \(s_n\) are homeomorphisms, it follows that \(f\) and \(g\) are Fréchet equivalent.

5: LEMMA

\[ 0 \leq u < v \leq \ell(C) \]

\[ \Rightarrow |g(v) - g(u)| = v - u \]

\[ \Rightarrow \quad g_m(v) - g_m(u) \leq v - u \quad (1 \leq m \leq M). \]

Consequently \(g_1, \ldots, g_M\) are absolutely continuous (in fact, Lipschitz).

6: LEMMA

\[ \ell(C) = \ell(D) = \int_0^{\ell(D)} |g'| \]

where \(||g'|| \leq 1\).

So

\[ 0 = \ell(D) - \int_0^{\ell(D)} ||g'|| \]

\[ = \int_0^{\ell(D)} 1 - \int_0^{\ell(D)} ||g'|| \]

\[ = \int_0^{\ell(D)} (1 - ||g'||) \]

implying thereby that \(||g'|| = 1\) almost everywhere.
1. NOTATION \( \mathcal{B}[a,b] \) is the set of Borel subsets of \([a,b]\).

Let

\[
C \leftrightarrow f : [a,b] \to \mathbb{R}^M
\]

be a curve, continuous and rectifiable.

2. LEMMA The interval function defined by the rule

\[
[c,d] \to s(d) - s(c) \quad ([c,d] \subset [a,b])
\]

can be extended to a measure \( \mu_C \) on \( \mathcal{B}[a,b] \).

3. LEMMA For \( m = 1, \ldots, M \), the interval function defined by the rule

\[
[c,d] \to T^+_f [c,d] \quad ([c,d] \subset [a,b])
\]

can be extended to a measure \( \mu_m \) on \( \mathcal{B}[a,b] \).

4. FACT Given \( S \in \mathcal{B}[a,b] \),

\[
\mu_m(S) \leq \mu_C(S) \leq \mu_1(S) + \cdots + \mu_M(S).
\]

5. LEMMA For \( m = 1, \ldots, M \), the interval functions defined by the rule

\[
[c,d] \to T^+_f [c,d] \\
[c,d] \to T^-_f [c,d]
\]

\(([c,d] \subset [a,b])\)

can be extended to measures

\[
\mu^+_m \\
\mu^-_m
\]

on \( \mathcal{B}[a,b] \).
2. NOTATION Put

\[ \nu_m = \mu_m^+ - \mu_m^- \quad (m = 1, \ldots, M). \]

[Thus \( \nu_m \) is a countably additive, totally finite set function on \( \mathcal{B}[a,b] \).]

7. RECOVERY PRINCIPLE For any \( S \in \mathcal{B}[a,b] \),

\[ \mu_c(S) = \sup_{\{P\}} \sum_{m=1}^{M} \left( \sum_{E \in P} \nu_m^2(E) \right)^{1/2}, \]

where the supremum is taken over all partitions \( P \) of \( S \) into disjoint Borel measurable sets \( E \).

8. FACT The set functions \( \mu_m^+, \mu_m^-, \mu_m^-, \nu_m \) are absolutely continuous w.r.t. \( \mu_c \).

9. NOTATION The corresponding Radon-Nikodym derivatives are denoted by

\[ \begin{align*}
\beta^+_m &= \frac{d\mu_m^+}{d\mu_c} \\
\beta^-_m &= \frac{d\mu_m^-}{d\mu_c} \\
\gamma_m &= \frac{d\nu_m}{d\mu_c} \\
\end{align*} \]

10. CONVENTION The term almost everywhere (or measure 0) will refer to the measure space

\(([a,b], \mathcal{B}[a,b], \mu_c)\).
11: FACT

\[ \beta_m = \beta_m^+ + \beta_m^- \]

and \( (m = 1, \ldots, M) \)

\[ \Theta_m = \beta_m^+ - \beta_m^- \]

almost everywhere.

12: NOTATION Let

\[ \Theta = (\theta_1, \ldots, \theta_M). \]

[Note: By definition,

\[ ||\Theta(x)|| = (\theta_1(x)^2 + \cdots + \theta_M(x)^2)^{1/2}. \]

13: NOTATION Given a linear orthogonal transformation \( \lambda: \mathbb{R}^M \rightarrow \mathbb{R}^M \), let \( \overline{C} = \lambda C. \)

14: N.B.

\[ \frac{\mu_\overline{C}}{C} = \mu_{\overline{C}}. \]

15: LEMMA

\( (\overline{v}_1, \ldots, \overline{v}_M) = \lambda (v_1, \ldots, v_M). \)

16: APPLICATION

\( (\overline{\Theta}_1, \ldots, \overline{\Theta}_M) = \lambda (\Theta_1, \ldots, \Theta_M) \)

almost everywhere.

[Differentiate the preceding relation w.r.t. \( \frac{\mu}{\overline{C}} = \mu_{\overline{C}}. \)]
17: **Lemma**

\[ |\Theta_m| \leq 1 \text{ (} m = 1, \ldots, M \text{)} \]

almost everywhere, so

\[ |\Theta| \leq M^{1/2} \]

almost everywhere.

18: **Theorem**

\[ |\Theta| = 1 \]

almost everywhere.

**Proof** Let \( 0 < \delta < 1 \) and let

\[ S = \{ x : |\Theta(x)| < 1 - \delta \} \]

Then

\[ \mu_C(S) = \sup_{\{P\}} \sum_{m=1}^{M} \frac{\nu_m(E)^2}{\mu_C(E)}^{1/2} \]

But

\[ \nu_m(E) = \int_E \frac{d\mu_m}{d\mu_C} d\mu_C \]

\[ = \int_E \Theta_m d\mu_C \]

Therefore

\[ \left\{ \sum_{m=1}^{M} \nu_m(E)^2 \right\}^{1/2} \]

\[ = \left\{ \sum_{m=1}^{M} (\int_E \Theta_m d\mu_C)^2 \right\}^{1/2} \]

\[ \leq \int_E \left\{ \sum_{m=1}^{M} \Theta_m^2 \right\}^{1/2} d\mu_C \]
5.

\[ S = \bigcup E, \]

it follows that

\[ \sum_{E \in \mathcal{P}} \left\{ \sum_{m=1}^{M} \mu_m(E)^2 \right\}^{1/2} \leq (1 - \delta) \mu_c(S). \]

Taking the supremum over the \( P \) then implies that

\[ \mu_c(S) \leq (1 - \delta) \mu_c(S), \]

thus \( \mu_c(S) = 0 \) and \( \|\Theta(x)\| \geq 1 \) almost everywhere (let \( \delta = \frac{1}{2}, \frac{1}{3}, \ldots \)). To derive a contradiction, take \( M \geq 2 \) and suppose that \( \|\Theta(x)\| \geq 1 + \delta > 1 \) on some set \( T \) such that \( \mu_c(T) > 0 \) — then for some vector

\[ \xi = (\xi_1, \ldots, \xi_M) \in \mathbb{R}^M \quad (\|\xi\| = 1), \]

the set

\[ T(\xi) = \{x \in T: \|\frac{\Theta(x)}{\|\Theta(x)\|} - \xi\| < \frac{\delta}{M^2}\} \]

has measure \( \mu_c(T(\xi)) > 0 \) (see below). Let

\[ \lambda_j = (\lambda_{j1}, \ldots, \lambda_{jM}) \quad (j = 2, \ldots, M) \]

be unit vectors such that

\[ = \int_E \|\Theta(x)\| \, d\mu_c \]

\[ \leq (1 - \delta) \int_E \, d\mu_c \]

\[ = (1 - \delta) \mu_c(E). \]
6.

\[
\lambda = \begin{pmatrix}
\xi_1, \ldots, \xi_M \\
\lambda_1, \ldots, \lambda_{2M} \\
\vdots \\
\lambda_M, \ldots, \lambda_{MM}
\end{pmatrix}
\]

is an orthogonal matrix. Viewing \( \lambda \) as a linear orthogonal transformation, form as above \( \overline{C} = \lambda C \), hence

\[
(\overline{\xi}_1, \ldots, \overline{\xi}_M) = \lambda (\xi_1, \ldots, \xi_M).
\]

On \( T(\xi) \),

\[
|\overline{\xi}_j| = |\lambda_j \xi_1 + \cdots + \lambda_{2M} \xi_M| \\
\leq |\lambda_j| \frac{\delta}{M^2} \\
\leq M^{1/2} \frac{\delta}{M^2} \leq M \frac{\delta}{M^2} = \frac{\delta}{M},
\]

while

\[
|\bar{\xi}| \leq |\overline{\xi}_1| + \cdots + |\overline{\xi}_M| \\
\Rightarrow |\overline{\xi}_1| \geq |\bar{\xi}| - |\overline{\xi}_2| - \cdots - |\overline{\xi}_M| \\
\geq (1 + \delta) - (M - 1) \frac{\delta}{M} = 1 + \frac{\delta}{M}.
\]

However

\[
|\overline{\xi}_1| \leq 1,
\]

so we have a contradiction.
Let \( \xi_n \) be a dense subset of the unit sphere \( U(M) \) in \( \mathbb{R}^M \) (thus \( \forall n, \| \xi_n \| = 1 \)). Given a point \( x \in T \), pass to \( \Theta(x) \in U(M) \).

Then there exists a \( \xi_n^x \):

\[
\| \Theta(x) \|_{\mathbb{R}^M} - \xi_n^x < \frac{\delta}{M^2},
\]

a point in the \( \frac{\delta}{M^2} \) - neighborhood of \( \Theta(x) \) in \( U(M) \).

Therefore

\[
T = \bigcup_{n=1}^{\infty} T(\xi_n^x)
\]

\[
\Rightarrow
\]

\[
0 < \mu_C(T) \leq \sum_{n=1}^{\infty} \mu_C(T(\xi_n^x))
\]

\[
\Rightarrow \exists n:
\]

\[
\mu_C(T(\xi_n^x)) > 0.
\]
§7. TWO THEOREMS

Let

$$C \leftrightarrow f: [a, b] \rightarrow \mathbb{R}^m$$

be a curve, continuous and rectifiable.

Let \( P \in P[a, b] \), say

\[ P: a = x_0 < x_1 < \cdots < x_n = b. \]

1. **DEFINITION** Let \( i = 1, \ldots, n \) and for \( m = 1, \ldots, M \) let

\[ \eta_m(x; P) = \frac{f_m(x_i) - f_m(x_{i-1})}{\mu_C([x_{i-1}, x_i])}, \]

where \( x_{i-1} < x < x_i \) if \( \mu_C([x_{i-1}, x_i]) \neq 0 \) and let

\[ \eta_m(x; P) = 0, \]

where \( x_{i-1} < x < x_i \) if \( \mu_C([x_{i-1}, x_i]) = 0. \)

2. **NOTATION**

\[ \eta(x; P) = (\eta_1(x; P), \ldots, \eta_M(x; P)). \]

3. **THEOREM**

\[ \int_a^b \| \Theta(x) - \eta(x; P) \|^2 \, d\mu_C \leq 2[\ell(C) - \sum_{i=1}^n \| f(x_i) - f(x_{i-1}) \|]. \]

**PROOF** Given \( P \in P[a, b] \), let \( \Sigma' \) denote a sum over intervals \([x_{i-1}, x_i]\), where

\[ \| \eta(x; P) \|^2 \neq 0 \]

and let \( \Sigma'' \) denote a sum over what remains. Now compute:
\[
\int_a^b \left| \tilde{\phi}(x) - \bar{n}(x; P) \right|^2 \, d\mu_c
\]

\[
= \Sigma' \int_{x_{i-1}}^{x_i} \left| \tilde{\phi}(x) - \bar{n}(x; P) \right|^2 \, d\mu_c
\]

\[
+ \Sigma'' \int_{x_{i-1}}^{x_i} \left| \tilde{\phi}(x) \right|^2 \, d\mu_c
\]

\[
= \Sigma' \int_{x_{i-1}}^{x_i} \left[ \left| \tilde{\phi}(x) \right|^2 + \left| \bar{n}(x; P) \right|^2 - 2 \tilde{\phi}(x) \cdot \bar{n}(x; P) \right] \, d\mu_c
\]

\[
+ \Sigma'' \int_{x_{i-1}}^{x_i} \left| \tilde{\phi}(x) \right|^2 \, d\mu_c
\]

\[
= \Sigma' \int_{x_{i-1}}^{x_i} \left[ 1 + \left| \bar{n}(x; P) \right|^2 - 2 \tilde{\phi}(x) \cdot \bar{n}(x; P) \right] \, d\mu_c
\]

\[
+ \Sigma'' \int_{x_{i-1}}^{x_i} \, d\mu_c
\]

\[
= \Sigma' \left[ \mu_c([x_{i-1}, x_i]) \right] + \left[ \frac{\left| \tilde{f}(x_i) - \tilde{f}(x_{i-1}) \right|}{\mu_c([x_{i-1}, x_i])} \right] \left| \tilde{f}(x_i) - \tilde{f}(x_{i-1}) \right|^2 \mu_c([x_{i-1}, x_i])
\]

\[
- 2 \frac{\left| \tilde{f}(x_i) - \tilde{f}(x_{i-1}) \right|}{\mu_c([x_{i-1}, x_i])} + \Sigma'' \mu_c([x_{i-1}, x_i])
\]

\[
\leq \ell(c) - \Sigma' \frac{\left| \tilde{f}(x_i) - \tilde{f}(x_{i-1}) \right|}{\mu_c([x_{i-1}, x_i])} \left| \tilde{f}(x_i) - \tilde{f}(x_{i-1}) \right|^2
\]

\[
\leq \ell(c) - \Sigma' \left| \tilde{f}(x_i) - \tilde{f}(x_{i-1}) \right|
\]
3. 

\[ + \Sigma' \left| f(x_i) - f(x_{i-1}) \right| \left(1 - \frac{\left| f(x_i) - f(x_{i-1}) \right|}{\mu_C([x_{i-1}, x_i])} \right) \]

\[ \leq \ell(C) - \Sigma' \left| f(x_i) - f(x_{i-1}) \right| \]

\[ + \Sigma' \mu_C([x_{i-1}, x_i]) \left(1 - \frac{\left| f(x_i) - f(x_{i-1}) \right|}{\mu_C([x_{i-1}, x_i])} \right) \]

\[ \leq \ell(C) - \Sigma' \left| f(x_i) - f(x_{i-1}) \right| \]

\[ + \Sigma' \mu_C([x_{i-1}, x_i]) \]

\[ \leq \ell(C) - \Sigma' \left| f(x_i) - f(x_{i-1}) \right| \]

\[ + \Sigma' \mu_C([x_{i-1}, x_i]) \]

\[ \leq \ell(C) - \Sigma' \left| f(x_i) - f(x_{i-1}) \right| \]

\[ + \Sigma' \mu_C([x_{i-1}, x_i]) \]

\[ = \ell(C) + \Sigma' \mu_C([x_{i-1}, x_i]) - 2 \Sigma' \left| f(x_i) - f(x_{i-1}) \right| \]

\[ = \ell(C) + \ell(C) - 2 \Sigma' \left| f(x_i) - f(x_{i-1}) \right| \]

\[ = 2[\ell(C) - \Sigma' \left| f(x_i) - f(x_{i-1}) \right|] \]

\[ = 2[\ell(C) - \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|] . \]

4: N.B. By definition, \( \mu_C([x_{i-1}, x_i]) \) is the length of the restriction of \( C \) to \( [x_{i-1}, x_i] \), i.e.,

\[ \mu_C([x_{i-1}, x_i]) = s(x_i) - s(x_{i-1}) . \]

Moreover

\[ \left| f(x_i) - f(x_{i-1}) \right| \leq s(x_i) - s(x_{i-1}) . \]
So, if \( \mu_c([x_{i-1},x_i]) = 0 \), then
\[
\sum' \left| f(x_i) - f(x_{i-1}) \right| = 0 \Rightarrow f(x_i) = f(x_{i-1})
\]

=>
\[
\Sigma' \left| f(x_i) - f(x_{i-1}) \right| = 0
\]
\[
= \Sigma' \left| f(x_i) - f(x_{i-1}) \right| + \Sigma'' \left| f(x_i) - f(x_{i-1}) \right|
\]
\[
= \Sigma \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|.
\]

Abbreviate
\[L^2([a,b], BO([a,b]), \mu_c)\]
to
\[L^2(\mu_c).
\]

5: **APPLICATION** in \( L^2(\mu_c) \),
\[
\lim_{||P|| \to 0} \eta(P) = 0.
\]

6: **SETUP**

- \( C_0 \leftrightarrow f_0 : [a,b] \to R^M \)
is a curve, continuous and rectifiable.

- \( C_k \leftrightarrow f_k : [a,b] \to R^M \ (k = 1,2,...) \)
is a sequence of curves, continuous and rectifiable.

Assumption: \( f_k \) converges uniformly to \( f_0 \) in \([a,b]\) and
\[
\lim_{k \to \infty} \ell(C_k) = \ell(C_0),
\]
7: THEOREM

\[
\lim_{|Q| \to 0} \frac{b}{a} \int f_k(x) \, dx = \mathcal{L}(C_k) \quad (Q \in \mathcal{P}[a,b])
\]

uniformly in \(k\), i.e., \(\forall \varepsilon > 0, \exists \delta > 0\) such that

\[
|Q| < \delta \Rightarrow |\frac{b}{a} \int f_k(x) \, dx - \mathcal{L}(C_k)| < \varepsilon
\]

for all \(k = 1, 2, \ldots\), or still,

\[
|Q| < \delta \Rightarrow \mathcal{L}(C_k) - |\frac{b}{a} \int f_k(x) \, dx| < \varepsilon
\]

for all \(k = 1, 2, \ldots\).

The proof will emerge in the lines to follow. Start the process by choosing \(\delta_0 > 0\) such that

\[
\mathcal{L}(C_0) - |\frac{b}{a} \int f_0(x) \, dx| < \frac{\varepsilon}{4}
\]

provided \(|\mathcal{P}_0| < \delta_0\). Consider a \(P \in \mathcal{P}[a,b]\):

\[
a = x_0 < x_1 < \cdots < x_n = b
\]

with \(|P| < \delta_0\). Choose \(\rho > 0\) such that

\[
|f_k(c) - f_k(d)| < \frac{\varepsilon}{4n} \quad ([c,d] \subset [a,b])
\]

for all \(k = 0, 1, 2, \ldots\), so long as \(|c - d| < \rho\) (equi-continuity). Take a partition \(Q \in \mathcal{P}[a,b]\):

\[
a = y_0 < y_1 < \cdots < y_m = b
\]

subject to

\[
|Q| < \gamma \equiv \min_{i=1,\ldots,n} \left\{ \delta, \frac{x_i - x_{i-1}}{2} \right\} \quad (\Rightarrow |Q| < \delta_0).
\]
6.

Put
\[ \alpha_k = \sup_{a \leq x \leq b} \| f_k(x) - f_0(x) \| \]

and let \( k_0 \) be such that
\[ k > k_0 \implies \alpha_k < \frac{\varepsilon}{4n} \quad \text{and} \quad |\ell(C_k) - \ell(C_0)| < \frac{\varepsilon}{4}. \]

The preparations complete, to minimize technicalities we shall suppose that each \( I_j = [y_{j-1}, y_j] \) is contained in just one \( I_i = [x_{i-1}, x_i] \) and write \( \Sigma^{(i)} \) for a sum over all such \( I_j \) — then
\[
\begin{align*}
V_{k}(f_k; Q) &= \sum_{j=1}^{m} V(f_k; I_j) \\
&= \sum_{j=1}^{m} \| f_k(y_j) - f_k(y_{j-1}) \| \\
&= \sum_{i=1}^{n} \Sigma^{(i)} \| f_k(y_i) - f_k(y_{i-1}) \| \\
&\geq \sum_{i=1}^{n} \| f_k(x_i) - f_k(x_{i-1}) \|.
\end{align*}
\]

8: SUBLEMMA Let \( A, B, C, D \in \mathbb{R}^M \) — then
\[ ||C - D|| > ||A - B|| - ||A - C|| - ||B - D||. \]

[In fact,
\[ ||A - B|| = ||A - C + C - D + D - B|| \]
\[ \leq ||A - C|| + ||C - D|| + ||B - D||. \]

Take
\[
\begin{bmatrix}
C = f_k(x_i) \\
D = f_k(x_{i-1})
\end{bmatrix}, \quad \begin{bmatrix}
A = f_0(x_i) \\
B = f_0(x_{i-1})
\end{bmatrix}.
\]
Then
\[
\left|\left| \frac{f_k(x_i)}{n^k} - \frac{f_{k-1}(x_{i-1})}{n^{k-1}} \right|\right| \\
\geq \left|\left| \frac{f_0(x_i)}{n^k} - \frac{f_0(x_{i-1})}{n^{k-1}} \right|\right| \\
- \left|\left| \frac{f_0(x_i)}{n^k} - \frac{f_k(x_i)}{n^k} \right|\right| - \left|\left| \frac{f_0(x_{i-1})}{n^{k-1}} - \frac{f_k(x_{i-1})}{n^{k-1}} \right|\right|,
\]
thus
\[
\sum_{i=1}^{n} \left|\left| \frac{f_k(x_i)}{n^k} - \frac{f_k(x_{i-1})}{n^k} \right|\right| \\
\geq \ell(C_0) - \frac{\varepsilon}{4} - n_0 \alpha_k - n_0 \alpha_k \\
\geq \ell(C_0) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} \\
= \ell(C_0) - \frac{3\varepsilon}{4}.
\]
But
\[
k > k_0 \Rightarrow \left|\ell(C_k) - \ell(C_0) \right| < \frac{\varepsilon}{4} \\
\Rightarrow \ell(C_k) - \frac{\varepsilon}{4} < \ell(C_0).
\]
Therefore
\[
\ell(C_0) - \frac{3\varepsilon}{4} > \ell(C_k) - \frac{\varepsilon}{4} - \frac{3\varepsilon}{4} \\
= \ell(C_k) - \varepsilon.
\]
Thus: \(\forall k > k_0\),
\[
\ell(C_k) - \frac{b}{a} \frac{(f_k;Q)}{\alpha_k} \leq \varepsilon \quad (\left|\left|Q\right|\right| < \gamma).
\]
Finally, for $k \leq k_0$, let $\gamma_k$ be chosen so as to ensure that

$$\ell(C_k) - \mathcal{V}(f_k; Q) < \varepsilon$$

for all partitions $Q$ with $|Q| < \gamma_k$. Put now

$$\delta = \min_{1, \ldots, k_0, \gamma} \{ \gamma_1, \ldots, \gamma_{k_0}, \gamma \}.$$

Then

$$|Q| < \delta \Rightarrow \ell(C_k) - \mathcal{V}(f_k; Q) < \varepsilon$$

for all $k = 1, 2, \ldots$.

Changing the notation (replace $Q$ by $P$), $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|P| < \delta \Rightarrow \ell(C_k) - \mathcal{V}(f_k; P) < \varepsilon$$

for all $k = 1, 2, \ldots$. Consequently

$$\int_a^b \left| \Theta_k(x) - \eta_k(x; P) \right|^2 \, d\mu_{C_k}$$

$$\leq 2[\ell(C_k) - \sum_{i=1}^n \left| f_k(x_i) - f_k(x_{i-1}) \right|]$$

$$= 2[\ell(C_k) - \mathcal{V}(f_k; P)]$$

$$< 2\varepsilon.$$
1. 

**§8. LINE INTEGRALS**

Let

$$C \leftrightarrow f: [a,b] \rightarrow \mathbb{R}^M$$

be a curve, continuous and rectifiable.

Suppose that

$$F: [C] \times \mathbb{R}^M \rightarrow \mathbb{R},$$

say

$$F(x,t) \ (x \in [C], t \in \mathbb{R}^M).$$

1: **DEFINITION** \(F\) is a **parametric integrand** if \(F\) is continuous in \((x,t)\) and \(\forall K \geq 0,\)

$$F(x,Kt) = KF(x,t).$$

2: **EXAMPLE** Let

$$F(x,t) = (t_1^2 + \cdots + t_M^2)^{1/2}. $$

3: **EXAMPLE** \((M = 2)\) Let

$$F(x_1, x_2, t_1, t_2) = x_1 t_2 - x_2 t_1.$$ 

4: **N.B.** If \(F\) is a parametric integrand, then \(\forall x,\)

$$F(x,0) = 0.$$ 

5: **RAPPHEL**

$$||\mathbf{0}|| = 1$$

almost everywhere.
6: Lemma Suppose that $F$ is a parametric integrand -- then the integral

$$I(C) = \int_{a}^{b} F(\overline{f}(x), \overline{g}(x)) \, d\mu_C$$

exists.

Proof $[C] \times U(M)$ is a compact set on which $F$ is bounded. Since $(\overline{f}(x), \overline{g}(x)) \in [C] \times U(M)$ almost everywhere, the function $F(\overline{f}(x), \overline{g}(x))$ is Borel measurable and essentially bounded w.r.t. the measure $\mu_C$. Therefore

$$I(C) = \int_{a}^{b} F(\overline{f}(x), \overline{g}(x)) \, d\mu_C$$

exists.

[Note: The requirement "homogeneous of degree 1" in $t$ plays no role in the course of establishing the existence of $I(C)$. It will, however, be decisive in the considerations to follow.]

Let $P \in P[a,b]$ and let $\xi_i$ be a point in $[x_{i-1}, x_i]$ ($i = 1, \ldots, n$).

7: Theorem If $F$ is a parametric integrand, then

$$\lim_{||P|| \to 0} \sum_{i=1}^{n} \frac{F(\overline{f}(\xi_i), \overline{f}(x_i) - \overline{f}(x_{i-1}))}{||P||}$$

exists and equals $I(C)$, denote it by the symbol

$$\int_C F,$$

and call it the line integral of $F$ along $C$.

Proof Fix $\varepsilon > 0$ and let $B(M)$ be the unit ball in $\mathbb{R}^M$. Put

$$M_F = \sup_{[C] \times B(M)} |P|.$$
Choose \( \gamma > 0 \):

\[
\begin{align*}
\| x_1 - x_2 \| &< \gamma \quad (x_1, x_2 \in \mathcal{C}) \\
\| t_1 - t_2 \| &< \gamma \quad (t_1, t_2 \in \mathcal{B}(M))
\end{align*}
\]

\[
\Rightarrow \\
|F(x_1, t_1) - F(x_2, t_2)| < \frac{\varepsilon}{3\ell(C)}.
\]

Introduce \( \eta(x; P) \) and set

\[
g(x; P) = F(f(x; \eta), \eta(x; P))
\]

if \( x_{i-1} < x < x_i \) then

\[
\int_a^b g(x; P) \, d\mu_C = \sum_{i=1}^n F(f(x_i), \frac{f(x_i) - f(x_{i-1})}{\mu_C([x_{i-1}, x_i])}) \mu_C([x_{i-1}, x_i])
\]

\[
= \sum_{i=1}^n F(f(x_i), f(x_i) - f(x_{i-1}))
\]

modulo the usual convention if \( \mu_C([x_{i-1}, x_i]) = 0 \). Recall now that in \( L^2(\mu_C) \),

\[
\lim_{||P|| \to 0} \eta(\rightarrow; P) = \Theta,
\]

hence \( \eta(\rightarrow; P) \) converges in measure to \( \Theta \), so there is a \( \rho > 0 \) such that for all \( P \) with \( ||P|| < \rho \),

\[
||\Theta(x) - \eta(x; P)|| < \gamma
\]

except on a set \( S_P \) of measure

\[
\mu_C(S_P) < \frac{\varepsilon}{3M_F}.
\]

Define \( \sigma \):

\[
|t_1 - t_2| < \sigma \Rightarrow |f(t_1) - f(t_2)| < \gamma.
\]
Let $\delta = \min(\sigma, \rho)$ and let $P$ be any partition with $||P|| < \delta$ — then

$$I(C) = \sum_{i=1}^{n} F(\xi_{i}, \xi_{i} - \xi_{i-1})$$

$$= \int_{a}^{b} F(\xi(x), \xi(x)) d\mu_{C} - \int_{a}^{b} g(x; P) d\mu_{C}$$

$$= \int_{a}^{b} [F(\xi(x), \xi(x)) - g(x; P)] d\mu_{C}.$$ 

By definition, $\delta \leq \rho$, hence

$$||\xi(x) - \eta(x; P)|| < \gamma$$

except in $S_{P}$, and

$$||\xi(x) - \xi(x)|| < \gamma$$

since

$$|x - \xi| < \gamma \quad (x_{i-1} \leq x \leq x_{i}).$$

To complete the argument, take absolute values:

$$|I(C) - \sum_{i=1}^{n} F(\xi_{i}, \xi_{i} - \xi_{i-1})|$$

$$\leq \int_{a}^{b} |F(\xi(x), \xi(x)) - g(x; P)| d\mu_{C}$$

$$= \int_{[a,b]-S_{P}} \ldots d\mu_{C} + \int_{S_{P}} \ldots d\mu_{C}.$$ 

- On $[a,b] - S_{P}$ at an index $i$,

$$|F(\xi(x), \xi(x)) - g(x; P)|$$

$$= |F(\xi(x), \xi(x)) - F(\xi_{i}, \eta(x; P))|$$

$$\leq \frac{\varepsilon}{3F(C)}.$$
Here, of course, up to a set of measure 0,

$$\Theta(x) \in B(M) \text{ and } \eta(x;P) \in B(M).$$

Therefore

$$\int_{[a,b]-S_P} \cdots \, d\mu_C \leq \frac{\varepsilon}{3\ell(C)} \ell(C) = \frac{\varepsilon}{3}.$$  

- On $S_P$,

$$\left| F(f(x), \Theta(x)) \right| \leq M_P$$

$$\left| F(f(\xi), \eta(x;P)) \right| \leq M_P.$$  

Therefore

$$\int_{S_P} \cdots \, d\mu_C \leq 2M_P \int_{S_P} 1 \, d\mu_C$$

$$= 2M_P \mu_C(S_P)$$

$$< 2M_P \frac{\varepsilon}{3M_P} = \frac{2\varepsilon}{3}.$$  

So in conclusion,

$$\int_{[a,b]-S_P} \cdots \, d\mu_C + \int_{S_P} \cdots \, d\mu_C$$

$$< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon \quad (||P|| < \delta)$$

and

$$I(C) = \int_C F.$$  

8: N.B. The end result is independent of the choice of the $\xi_i$.  

9: THEOREM If $f_1, \ldots, f_M \in AC[a,b]$, then for any parametric integrand $F$,
\[ \int_{C} F = \int_{a}^{b} F(f_1(x), \ldots, f_M(x), f'_1(x), \ldots, f'_M(x)) \, dx, \]

the integral on the right being in the sense of Lebesgue.

**Proof** The absolute continuity of the \( f_m \) implies that

\[ \mu_C([c,d]) = \int_{c}^{d} |f'| \, dx \]

for every subinterval \([c,d] \subset [a,b]\), hence \( \mu_C \) is absolutely continuous w.r.t. Lebesgue measure. It is also true that \( \nu_m \) is absolutely continuous w.r.t. Lebesgue measure. This said, write

\[ f'_m = \frac{df_m}{dx} = \frac{d\nu_m}{dx} = \frac{d\nu_m}{d\mu_C} \frac{d\mu_C}{dx} = \Theta_m \frac{d\mu_C}{dx}. \]

Then

\[ I(C) = \int_{a}^{b} F(f(x), \Theta(x)) \, d\mu_C \]

\[ = \int_{a}^{b} F(f(x), \Theta(x)) \frac{d\mu_C}{dx} \, dx \]

\[ = \int_{a}^{b} F(f(x), \Theta(x) \frac{d\mu_C}{dx}) \, dx, \]

where

\[ \frac{d\mu_C}{dx} = ||f'|| \geq 0. \]

Continuing

\[ I(C) = \int_{a}^{b} F(f_1(x), \ldots, f_M(x), \Theta_1(x) \frac{d\mu_C}{dx}, \ldots, \Theta_M(x) \frac{d\mu_C}{dx}) \, dx \]

\[ = \int_{a}^{b} F(f_1(x), \ldots, f_M(x), f'_1(x), \ldots, f'_M(x)) \, dx, \]

the integrals being in the sense of Lebesgue.
Let

\[
\begin{align*}
C & \leftrightarrow f : [a,b] \to \mathbb{R} \\
D & \leftrightarrow g : [c,d] \to \mathbb{R}
\end{align*}
\]

be curves, continuous and rectifiable.

**10: RAPPEL** If \( C \) and \( D \) are Fréchet equivalent, then

\[ [C] = [D] \quad \text{and} \quad \ell(C) = \ell(D). \]

**11: THEOREM** If \( C \) and \( D \) are Fréchet equivalent and if \( F \) is a parametric integrand, then

\[ \int_C F = \int_D F. \]

**PROOF** Fix \( \varepsilon > 0 \) and choose \( \delta > 0 : \)

- \( P \in P[a,b] \) & \( |P| < \delta \Rightarrow \)

\[ |I(C) - \sum_{i=1}^{n} F(f(\xi_i), f(x_i) - f(x_{i-1}))| < \frac{\varepsilon}{3}. \]

- \( Q \in P[c,d] \) & \( |Q| < \delta \Rightarrow \)

\[ |I(D) - \sum_{j=1}^{m} F(f(\xi_j), f(y_j) - f(y_{j-1}))| < \frac{\varepsilon}{3}. \]

Fix \( P \) and \( Q \) satisfying these conditions and let \( k \) be the number of intervals in \( P \) and let \( \ell \) be the number of intervals in \( Q \). Fix \( \gamma > 0 \) such that

\[ |F(x_1, t_1) - F(x_2, t_2)| < \frac{\varepsilon}{3(k+\ell)} \]

when

\[ |x_1 - x_2| < \gamma \quad (x_1, x_2 \in [C] = [D]) \]

and

\[ |t_1 - t_2| < 2\gamma \quad (|t_1| \leq \ell(C), \ |t_2| \leq \ell(D)). \]
Let $\phi : [a, b] \to [c, d]$ be a homeomorphism ($\phi(a) = c$, $\phi(b) = d$) such that

$$|f(x) - g(\phi(x))| < \gamma \ (x \in [a, b]).$$

Let

$$P^* : a = x_0^* < x_1^* < \ldots < x_r^* = b$$

be the partition obtained from $P$ by adjoining the images under $\phi^{-1}$ of the partition points of $Q$. Let

$$Q^* : c = y_0^* < y_1^* < \ldots < y_s^* = d$$

be the partition obtained from $Q$ by adjoining the images under $\phi$ of the partition points of $P$. So, by construction, $r = s$, either one is $k + \ell$, and $y_p^* = \phi(x_p^*)$ ($p = 0, 1, \ldots, q$). Choose a point $\xi_p \in [x_p^*, x_{p+1}^*]$ and work with

$f(\xi_p)$ and $g(\phi(\xi_p))$.

Then

$$|I(C) - I(D)|$$

$$\leq |I(C) - \sum_{p=1}^{q} F(f(\xi_p), f(x_p^*) - f(x_{p-1}^*))|$$

$$+ \sum_{p=1}^{q} |F(f(\xi_p), f(x_p^*) - f(x_{p-1}^*)) - F(g(\phi(\xi_p)), g(y_p^*) - g(y_{p-1}^*))|$$

$$+ \sum_{p=1}^{q} |F(g(\phi(\xi_p)), g(y_p^*) - g(y_{p-1}^*)) - I(D)|.$$

Since

$$|P^*| \leq |P| < \delta$$

$$|Q^*| \leq |Q| < \delta,$$
the first and third terms are each \(< \frac{\varepsilon}{3}\). As for the middle term,

\[ |f(\xi_p) - g(\phi(\xi_p))| < \gamma \]

and

\[ |f(x^*_p) - f(x^*_{p-1}) - g(y^*_p) + g(y^*_{p-1})| \leq |f(x^*_p) - g(y^*_p)| + |f(x^*_{p-1}) - g(y^*_{p-1})| \]

\[ = |f(x^*_p) - g(\phi(x^*_p))| + |f(x^*_{p-1}) - g(\phi(x^*_{p-1}))| \]

\[ < \gamma + \gamma = 2\gamma. \]

Therefore the middle term is

\[ < q \frac{\varepsilon}{3(k + \ell)} = \frac{q}{k + \ell} \frac{\varepsilon}{3} < \frac{\varepsilon}{3}. \]

And finally

\[ |I(C) - I(D)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \]

\[ = \varepsilon \]

\[ = \varepsilon \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \int_C F = \int_D F. \]

12: SETUP

- \( C_0 \leftrightarrow f_0 : [a, b] \rightarrow R^M \)

is a curve, continuous and rectifiable.

- \( C_k \leftrightarrow f_k : [a, b] \rightarrow R^M \) \( (k = 1, 2, \ldots) \)

is a sequence of curves, continuous and rectifiable.

Assumption: \( f_k \) converges uniformly to \( f_0 \) in \([a, b]\) and
10. \[ \lim_{k \to \infty} \ell(C_k) = \ell(C_0). \]

13: THEOREM

\[ \lim_{k \to \infty} I(C_k) = I(C_0) \]

or still,

\[ \lim_{k \to \infty} \int_{C_k} F = \int_{C_0} F. \]
1. DATA  \( A \) is a nonempty set, \( I = \{I\} \) is a nonempty collection of subsets of \( A \), \( \mathcal{D} = \{D\} \) is a nonempty collection of nonempty finite collections \( D = [I] \) of sets \( I \in I \), and \( \delta \) is a real valued function defined on \( \mathcal{D} \).

2. DEFINITIONS The sets \( I \in I \) are called intervals, the collections \( D \in \mathcal{D} \) are called systems, and the function \( \delta \) is called a mesh.

3. ASSUMPTIONS \( A \) is a nonempty topological space, each interval \( I \) has a nonempty interior, the intervals of each system \( D \) are nonoverlapping: \( I_1, I_2 \in D, I_1 \neq I_2 \)
\[
\implies \begin{align*}
\text{int} \ I_1 \cap \text{cl} \ I_2 &= \emptyset \\
\text{cl} \ I_1 \cap \text{int} \ I_2 &= \emptyset.
\end{align*}
\]

4. ASSUMPTION For each system \( D \), \( 0 < \delta(D) < + \infty \), and each \( \epsilon > 0 \), there are systems with \( \delta(D) < \epsilon \).

5. REMARK In the presence of \( \delta \), one is able to convert \( \mathcal{D} \) into a directed set with direction "\( > > \)" by defining \( D_2 > > D_1 \) iff \( \delta(D_2) < \delta(D_1) \).

6. EXAMPLE Take \( A = [a,b] \) and let \( I = \{I\} \) be the collection of all closed subintervals of \([a,b] \). Take for \( \mathcal{D} \) the class of all partitions \( D \) of \([a,b] \), i.e., \( \mathcal{D} = \mathcal{P}[a,b] \), and let \( \delta(D) \) be the norm of \( D \).

[Note: Strictly speaking, an element of \( \mathcal{P}[a,b] \) is a finite set \( P = \{x_0, ..., x_n\} \), where
\[
a = x_0 < x_1 < ... < x_n = b,
\]
the associated element $D$ in $\mathcal{D}$ being the set

$$[x_{i-1}, x_i] \quad (i = 1, \ldots, n).$$

7: DEFINITION An interval function is a function $\phi: I \to \mathbb{R}^M$.

[Note: Associated with $\phi$ are the interval functions $||\phi||$, as well as

$$\begin{bmatrix}
\phi_m^+ \\
\phi_m^- \\
\phi_m'
\end{bmatrix}
\quad (m = 1, \ldots, M).]$$

8: NOTATION Given an interval function $\phi$, a subset $S \subset A$, and a system $D = [I]$, put

$$\Sigma[\phi, S, D] = \Sigma_{I \in D} s(I, S) \phi(I),$$

where $I$ ranges over all $I \in D$ and $s(I, S) = 1$ or 0 depending on whether $I \subset S$ or $I \not\subset S$.

[Note: Take for $S$ the empty set $\emptyset$ -- then $I \subset \emptyset$ is inadmissible (I has a nonempty interior) and $I \not\subset \emptyset$ gives rise to zero. Therefore

$$\Sigma[\phi, \emptyset, D] = 0.]$$

9: N.B. The absolute situation is when $S = A$, thus in this case,

$$\Sigma[\phi, A, D] \equiv \Sigma[\phi, D] = \Sigma_{I \in D} \phi(I).$$

10: DEFINITION Given an interval function $\phi$ and a subset $S \subset A$, the BC-integral of $\phi$ over $S$ is

$$\lim_{\delta(D) \to 0} \Sigma[\phi, S, D].$$
provided the limit exists in $\mathbb{R}^M$.

[Note: $B = \text{Burkill}$ and $C = \text{Cesari}.$]

11: NOTATION The BC-integral of $\phi$ over $S$ is denoted by

$$BC \int_S \phi.$$ 

12: EXAMPLE

$$BC \int_0 \phi = 0 \quad (\in \mathbb{R}^M).$$

13: DEFINITION An interval function $\phi$ is quasi additive on $S$ if for each $\epsilon > 0$ there exists $\eta(\epsilon, S) > 0$ such that if $D_0 = \{I_0\}$ is any system subject to $\delta(D_0) < \eta(\epsilon, S)$ there also exists $\lambda(\epsilon, S, D_0) > 0$ such that for every system $D = \{I\}$ with $\delta(D) < \lambda(\epsilon, S, D_0)$, the relations

$$(\text{qa}_1 - S) \sum_{I \in S} s(I_0, S) \sum_{I} s(I, I_0) \phi(I) - \phi(I_0) \leq \epsilon$$

$$(\text{qa}_2 - S) \sum_{I \in S} s(I, S) [1 - \sum_{I \in S} s(I, I_0) s(I_0, S)] || \phi(I) || \leq \epsilon$$

obtain.

14: N.B. In the absolute situation, matters read as follows: An interval function $\phi$ is quasi additive if for each $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ such that if $D_0 = \{I_0\}$ is any system subject to $\delta(D_0) < \eta(\epsilon)$ there exists $\lambda(\epsilon, D_0) > 0$ such that for every system $D = \{I\}$ with $\delta(D) < \lambda(\epsilon, D_0)$, the relations

$$(\text{qa}_1 - A) \sum_{I \in S} || \sum_{I \in S} \phi(I) - \phi(I_0) || \leq \epsilon$$
4.

\[(q_{a_2}-A) \sum_{I \not\in I_0} ||\phi(I)|| < \varepsilon\]

obtain.

[Note: The sum \(\sum_{I \not\in I_0} ||\phi(I)||\) is over all \(I \in D, I \not\in I_0\) for any \(I_0 \in D_0\).]

So, under the preceding conditions,

\[
\sum_{I} \phi(I) - \sum_{I_0} \phi(I_0)
\]

= \[
\sum_{I_0} \left[ \sum_{I \in I_0} \phi(I) - \phi(I_0) \right] + \sum_{I \not\in I_0} \phi(I)'
\]

=>

\[
||\sum_{I} \phi(I) - \sum_{I_0} \phi(I_0)|| < 2\varepsilon.
\]

15: THEOREM If \(\phi\) is quasi additive on \(S\), then

\[BC \int_{S} \phi\]

exists.

PROOF To simplify the combinatorics, take \(S = A\). Given \(\varepsilon > 0\), let \(\gamma(\varepsilon), D_0, \lambda(\varepsilon, D_0)\) be per \(qa_1-A, qa_2-A\) and suppose that \(D_1, D_2 \in D\), where

\[
\begin{cases}
\delta(D_1) < \lambda(\varepsilon, D_0) \\
\delta(D_2) < \lambda(\varepsilon, D_0).
\end{cases}
\]
Then
\[
\begin{align*}
&\|\sum_{I_1} \phi(I_1) - \sum_{I_0} \phi(I_0)\| < 2\varepsilon \\
&\|\sum_{I_2} \phi(I_2) - \sum_{I_0} \phi(I_0)\| < 2\varepsilon
\end{align*}
\]

\[\Rightarrow \quad \|\sum_{I_1} \phi(I_1) - \sum_{I_2} \phi(I_2)\| < 4\varepsilon.\]

Therefore \(BC f_A \phi\) exists.

16: REMARK

- If the \(\phi_m (m = 1, \ldots, M)\) are quasi additive, then \(\phi\) is quasi additive.
- If the \(|\phi_m| (m = 1, \ldots, M)\) are quasi additive, then \(||\phi||\) is quasi additive.

17: DEFINITION A real valued interval function \(\psi\) is quasi subadditive on \(S\) if for each \(\varepsilon > 0\) there exists \(\eta(\varepsilon, S) > 0\) such that if \(D_0 = [I_0]\) is any system subject to \(\delta(D_0) < \eta(\varepsilon, S)\) there also exists \(\lambda(\varepsilon, S, D_0) > 0\) such that for every system \(D = [I]\) with \(\delta(D) < \lambda(\varepsilon, S, D_0)\) the relation

\[(qsa - S) \sum_{I_0} s(I_0, S) \left[ \sum_{I} s(I, I_0) \psi(I) - \psi(I_0) \right] < \varepsilon \]

obtains.

18: N.B. In the absolute situation, matters read as follows: ...

\[(qsa - A) \sum_{I_0} \left[ \sum_{I \subseteq I_0} \psi(I) - \psi(I_0) \right] < \varepsilon.\]

19: LEMMA If \(\psi: D + \mathbb{R}_{\geq 0}\) is nonnegative and quasi subadditive on \(S\), then
exists (+∞ is a permissible value).

20: THEOREM If \( \psi: \mathcal{I} \to \mathbb{R}_{\geq 0} \) is nonnegative and quasi subadditive on \( \mathcal{S} \) and if

\[ \text{BC } f_{\mathcal{S}} \psi \]

is finite, then \( \psi \) is quasi additive on \( \mathcal{S} \).

PROOF To simplify the combinatorics, take \( \mathcal{S} = \mathcal{A} \). Since

\[ \text{BC } f_{\mathcal{A}} \psi \]

exists and is finite, given \( \varepsilon > 0 \) there is a number \( \mu(\varepsilon) > 0 \) such that for any \( D_0 = [I_0] \in \mathcal{D} \) with \( \delta(D_0) < \mu(\varepsilon) \), we have

\[ |\text{BC } f_{\mathcal{A}} \psi - \sum_{I_0} \psi(I_0)| < \frac{\varepsilon}{3}, \]

where \( \sum \) is a sum ranging over all \( I_0 \in D_0 \). Now choose \( D_0 \) in such a way that

\[ \delta(D_0) < \min\{\mu(\varepsilon), \eta(\varepsilon/6)\}, \]

take

\[ \lambda'(\varepsilon) = \min\{\mu(\varepsilon), \lambda(\varepsilon/6, D_0)\}, \]

and consider any system \( \mathcal{D} = [I] \) with \( \delta(D) < \lambda' \). Since \( \psi \) is quasi subadditive,

\[ \sum_{I_0} \left[ \sum_{I \subset I_0} \psi(I) - \psi(I_0) \right] < \frac{\varepsilon}{6}. \]

On the other hand,

\[ |\text{BC } f_{\mathcal{A}} \psi - \sum_{I} \psi(I)| < \frac{\varepsilon}{3}. \]

Denote by \( \Sigma' \) a sum over all \( I \in \mathcal{D} \) with \( I \neq I_0 \) for any \( I_0 \in D_0 \) -- then

\[ 0 \leq \sum_{I_0} \left[ \sum_{I \subset I_0} \psi(I) - \psi(I_0) \right] + \Sigma' \psi(I) \]
The requirements for quasi additivity are thus met.

21: THEOREM Suppose that \( \phi: I \rightarrow \mathbb{R}^M \) is quasi additive on \( S \) -- then \( ||\phi||: I \rightarrow \mathbb{R}_{\geq 0} \) is quasi subadditive on \( S \).

PROOF Fix \( \varepsilon > 0 \), take \( S = A \), and in the notation above, introduce \( n(\varepsilon) \), \( D_0 = [I_0], \lambda(\varepsilon,D_0), D = [I] -- then the objective is to show that

\[
\sum_{I_0} \left( \sum_{I \subseteq I_0} ||\phi(I)|| - ||\phi(I_0)|| \right)^- < \varepsilon.
\]

To this end, let

\[
\phi(I_0') = \sum_{I \subseteq I_0} \phi(I) - \phi(I_0).
\]

Then

\[
||\phi(I_0) + \phi(I_0')|| = \sum_{I \subseteq I_0} \phi(I)
\]
Meanwhile

\[ \phi(I_0) = [\phi(I_0) + \phi(I_0)] + [-\phi(I_0)] \]

\[ \geq ||\phi(I_0) + \phi(I_0)|| - ||\phi(I_0)|| \]

\[ \geq -||\phi(I_0)|| \]

\[ \rightarrow \]

\[ \geq [\sum \sum \frac{M}{m=1} \delta m(I) \delta 1/2 \]

\[ \leq \sum \sum \frac{M}{m=1} \delta m(I) \delta 1/2 \]

\[ = \sum ||\phi(I)||. \]

\[ \rightarrow \]

\[ \sum \sum ||\phi(I)|| - ||\phi(I_0)|| ]^- \leq ||\phi(I_0)|| \]

\[ \rightarrow \]

\[ \sum \sum ||\phi(I)|| - ||\phi(I_0)|| ]^- \leq \sum ||\phi(I_0)|| \]

\[ = \sum ||\sum \phi(I) - \phi(I_0)|| \]

\[ < \varepsilon, \]

\[ \phi \text{ being quasi additive.} \]

22: APPLICATION If \( \phi: I \rightarrow R^M \) is quasi additive, then the interval functions
9.

\[ I \rightarrow |\phi_m(I)| \quad (m = 1, \ldots, M) \]

are quasi subadditive.

[In fact, the quasi additivity of \( \phi \) implies the quasi additivity of the \( \phi_m \) and
\[ ||\phi_m|| = ||\phi_m||. \]

[Note: It is also true that \( \phi^+_m, \phi^-_m \) are quasi subadditive.]

23: Lemma If \( \phi: I \rightarrow \mathbb{R}^M \) is quasi additive on \( S \) and if
\[ BC \int_S ||\phi|| < +\infty, \]
then \( \phi \) is quasi additive on every subset \( S' \subset S \).

Proof First of all, \( ||\phi|| \) is quasi subadditive on \( S \), hence also on \( S' \).

Therefore
\[ BC \int_{S'} ||\phi|| \]
exists and
\[ BC \int_{S'} ||\phi|| \leq BC \int_S ||\phi|| < +\infty, \]
from which it follows that \( ||\phi|| \) is quasi additive on \( S' \). Given \( \varepsilon > 0 \), determine the parameters in the definition of quasi additive in such a way that the relevant relations are simultaneously satisfied per \( \phi \) on \( S \) and per \( ||\phi|| \) on \( S' \), hence
\[ \sum_{I_0} \sum_{S'} s(I_0, S') \left| \sum_I s(I, I_0) \phi(I) - \phi(I_0) \right| \]
\[ \leq \sum_{I_0} \sum_I s(I_0, S) \left| \sum_I s(I, I_0) \phi(I) - \phi(I_0) \right| < \varepsilon \]
and
\[ \sum_{I} s(I, I_0) \left[ 1 - \sum_{I_0} s(I, I_0) s(I_0, S') \right] ||\phi(I)|| < \varepsilon. \]
Therefore $\phi$ is quasi additive on $S'$.

24: APPLICATION If $\phi: I \to \mathbb{R}^M$ is quasi additive and if

$$BC \int_A ||\phi|| < +\infty,$$

then $\phi$ is quasi additive on every subset of $A$.

Here is a summary of certain fundamental points of this §. Work with $\phi$ and $||\phi||$.

- Suppose that $||\phi||$ is quasi subadditive on $S$ and

$$BC \int_S ||\phi|| < +\infty.$$

Then $||\phi||$ is quasi additive on $S$.

- Suppose that $\phi$ is quasi additive on $S$ -- then $||\phi||$ is quasi subadditive on $S$.

So: If $\phi$ is quasi additive on $S$ AND if

$$BC \int_S ||\phi|| < +\infty,$$

then $||\phi||$ is quasi additive on $S$.

[Note: It is not true in general that $||\phi||$ quasi additive implies $\phi$ quasi additive.]

25: EXAMPLE Take $A = [a,b]$ and let $I, D$, and $c$ be as at the beginning.

Given a continuous curve

$$C \leftrightarrow f: [a,b] \to \mathbb{R}^M,$$

define a quasi additive interval function $\phi: I \to \mathbb{R}^M$ by the rule

$$\phi(I) = (\phi_1(I), \ldots, \phi_M(I))$$

$$= (f_1(d) - f_1(c), \ldots, f_M(d) - f_M(c)).$$
where $I = [c,d] \subset [a,b]$, thus

$$||\phi(I)|| = ||f(d) - f(c)||,$$

so if $P \in P[a,b]$ corresponds to

$$D \leftrightarrow \{[x_{i-1}, x_i]: i = 1, \ldots, n\},$$

then

$$\sum_{I \in D} ||\phi(I)|| = \sum_{i=1}^n ||f(x_i) - f(x_{i-1})||$$

$$= \lim_{|P| \to 0} \sum_{i=1}^n ||f(x_i) - f(x_{i-1})||$$

$$= \ell(C).$$

Therefore $C$ is rectifiable iff

$$BC \int_A ||\phi|| < +\infty.$$

And when this is the case, $||\phi||$ is quasi additive on $A$.

[Note: A priori,

$$\ell(C) = \sup_{P \in P[a,b]} \sum_{i=1}^n ||f(x_i) - f(x_{i-1})||.$$]

But here, thanks to the continuity of $f$, the sup can be replaced by $\lim$.]

26: EXAMPLE Take $A = [a,b]$ and let $I$ and $D$ be as above. Suppose that

$$C \leftrightarrow f: [a,b] \to \mathbb{R}^M$$

is a rectifiable curve, potentially discontinuous.

- Given $a \leq x_0 < b$, put

$$s^+(x_0) = \lim \sup_{x \downarrow x_0} ||f(x) - f(x_0)||$$
and let \( s^+(b) = 0 \).

- Given \( a < x_0 \leq b \), put

\[
s^-(x_0) = \limsup_{x \rightarrow x_0^+} |f(x) - f(x_0)|
\]

and let \( s^-(a) = 0 \). Combine the data and set

\[
s(x) = s^+(x) + s^-(x) \quad (a \leq x \leq b).
\]

Then \( s(x) \) is zero everywhere save for at most countably many \( x \) and

\[
s(x) \leq l(C).
\]

Take \( \phi \) as above and define a mesh \( \delta \) by the rule

\[
\delta(D) = ||\phi|| + \sigma - \sum_{i=0}^{n} s(x_i).
\]

One can then show that \( \phi \) is quasi additive and

\[
BC \int_{A} ||\phi|| = l(C).
\]

27: **NOTATION** Given a quasi additive interval function \( \phi \), let

\[
V[\phi,S] = \sup_{D \in \mathcal{D}} \Sigma[||\phi||,S,D].
\]

28: **N.B.** By definition,

\[
BC \int_{S} ||\phi|| = \lim_{\delta(D) \rightarrow 0} \Sigma[||\phi||,S,D],
\]

so

\[
BC \int_{S} ||\phi|| \leq V[\phi,S]
\]

and strict inequality may hold.
29: **Lemma** Given a quasi additive $\phi$ and a subset $S \subset A$, suppose that for every $\varepsilon > 0$ and any $D_0 = [I_0]$ there exists $\lambda(\varepsilon,S,D_0) > 0$ such that for every system $D = [I]$ with $\delta(D) < \lambda(\varepsilon,S,D_0)$ the relation

$$\sum_{I_0} s(I_0,S) \left[ \sum_I s(I,I_0) |\phi(I)| |\phi(I_0)| \right] < \varepsilon$$

obtains. Then

$$BC \int_S |\phi| = V[\phi,S].$$
§10. LINE INTEGRALS (bis)

Throughout this §, the situation will be absolute, where \( A = [a, b] \) and \( I, v, \) and \( \delta \) have their usual connotations.

If

\[
C \leftrightarrow \xi : [a, b] \rightarrow \mathbb{R}^M
\]

is a curve, continuous and rectifiable, then

\[
\text{BC } \int_A ||\phi|| = \ell(C).
\]

And if \( F \) is a parametric integrand, then

\[
\int_C F = \lim_{||P|| \to 0} \sum_{i=1}^{n} F(f(\xi_i), f(x_i) - f(x_{i-1}))
\]

exists, the result being independent of the \( \xi_i \).

1. N.B. Recall the procedure: Introduce the integral

\[
I(C) = \int_a^b F(\xi(x), \phi(x)) \, dx
\]

and prove that

\[
\lim_{||P|| \to 0} \sum_{i=1}^{n} F(f(\xi_i), f(x_i) - f(x_{i-1}))
\]

exists and equals \( I(C) \), the result being denoted by the symbol.

\[
\int_C F
\]

and called the line integral of \( F \) along \( C \).

There is another approach to all this which does not use measure theory. Thus define an interval function \( \phi : I \rightarrow \mathbb{R} \) by the prescription

\[
\phi(I; \xi) = F(\xi(\xi), \phi(I)),
\]
where $\xi \in I$ is arbitrary.

[Note: By definition,
\[ \phi(I) = (\phi_1(I), \ldots, \phi_M(I)) \]
\[ = (f_1(d) - f_1(c), \ldots, f_M(d) - f_M(c)), \]
$I$ being $[c,d] \subset [a,b]$. Moreover, $\phi$ is quasi additive.]

2: THEOREM $\phi$ is quasi additive.

Admit the contention -- then

\[
\lim_{\delta(D) \to 0} \sum_{I \in D} \phi(I; \xi)
\]
\[= \lim_{||P|| \to 0} \sum_{i=1}^{n} F(f(\xi_i), f(x_i) - f(x_{i-1}))
\]
exists, call it

\[(\xi) \int_C F.\]

3: N.B. Needless to say, it turns out that

\[(\xi) \int_C F \]
is independent of the $\xi$ (this follows by a standard "$\varepsilon/3$" argument) (details at the end).

[Note: This is one advantage of the approach via $I(C)$ in that independence is manifest.]

To simplify matters, it will be best to generalize matters.

Assume from the outset that $\phi:I \to \mathbb{R}^M$ is now an arbitrary interval function which is quasi additive with

\[(BC) \int_A ||\phi|| < + \infty,\]
hence that \( \| \phi \| \) is also quasi additive as well.

Introduce another interval function \( \zeta : I \to R^N \) and expand the definition of parametric integrand so that

\[
F : X \times R^M \to R,
\]

where \( X \subseteq R^N \) is compact and \( \zeta(I) \subseteq X \).

4: EXAMPLE To recover the earlier setup, take \( N = M \), keep \( \phi : I \to R^M \), let \( \omega : I \to [a, b] \) be a choice function, i.e., suppose that \( \omega(I) \in I \subseteq [a, b] \), let \( \zeta(I) = \mathcal{F} (\omega(I)) \), and take \( X = [C] \subseteq R^M \).

5: CONDITION (\( \zeta \)) \( \forall \epsilon > 0, \exists \, t(\epsilon) > 0 \) such that if \( D_0 = [I_0] \) is any system subject to \( \delta(D_0) < t(\epsilon) \) there also exists \( T(\epsilon, D_0) \) such that for any system \( D = [I] \) with \( \delta(D) < T(\epsilon, D_0) \), the relation

\[
\max_{I_0} \max_{I \subseteq I_0} \| \zeta(I) - \zeta(I_0) \| < \epsilon
\]

obtains.

6: N.B. Owing to the uniform continuity of \( \mathcal{F} \), this condition is automatic in the special case supra.

7: THEOREM Let \( F \) be a parametric integrand, form the interval function \( \phi : I \to R \) defined by the prescription

\[
\phi(I) = F(\zeta(I), \phi(I)),
\]

and impose condition (\( \zeta \)) — then \( \phi \) is quasi additive.

The proof will emerge from the discussion below but there are some preliminaries that have to be dealt with first.
4.

Start by writing down simultaneously (qa$_1$-A) and (qa$_2$-A) for $\phi$ and $||\phi||$ (both are quasi additive), $\bar{\epsilon}$ to be determined.

\[
\Sigma \sum_{I \subset I_0} |\phi(I) - \phi(I_0)| < \bar{\epsilon}
\]

\[
\Sigma \sum_{I \not\subset I_0} |\phi(I)| < \bar{\epsilon}
\]

\[
\Sigma \left| \sum_{I \subset I_0} |\phi(I)| - \sum_{I \not\subset I_0} |\phi(I)| \right| < \bar{\epsilon}
\]

\[
\Sigma |\phi(I)| < \bar{\epsilon}
\]

for $\delta(D_0) < \eta(\bar{\epsilon})$ and $\delta(D) < \lambda(\bar{\epsilon},D_0)$ and in addition

\[
\Sigma |\phi(I)| - \text{BC} \int_A |\phi|| < \bar{\epsilon}
\]

for $\delta(D) < \sigma(\bar{\epsilon})$.

Fix $\epsilon > 0$. Put

\[V = \text{BC} \int_A |\phi|| (< + \infty).
\]

- (F) $X \times U(M)$ is a compact set on which $F$ is bounded:

\[|F(x,t)| \leq C \ (x \in X, \ t \in U(M))
\]

and uniformly continuous: $\exists \gamma$ such that

\[
\frac{|t - t'|}{|x - x'|} < \gamma \Rightarrow |F(x_0, t) - F(x_1, t_1)| < \frac{\epsilon}{3(\gamma + \epsilon)}.
\]
\[ \alpha(I_0) = \frac{\phi(I_0)}{||\phi(I_0)||} \quad \text{if } \phi(I_0) \neq 0 \]
but 0 otherwise and
\[ \alpha(I) = \frac{\phi(I)}{||\phi(I)||} \quad \text{if } \phi(I) \neq 0 \]
but 0 otherwise.

8: NOTATION Denote by
\[ \sum_{\gamma^+} (I_0) \]
the sum over the \( I \subset I_0 \) for which
\[ ||\alpha(I_0) - \alpha(I)|| \geq \gamma \]
and denote by
\[ \sum_{\gamma^-} (I_0) \]
the sum over the \( I \subset I_0 \) for which
\[ ||\alpha(I_0) - \alpha(I)|| < \gamma. \]

Therefore
\[ \sum_{I \subset I_0} = \sum_{\gamma^+} (I_0) + \sum_{\gamma^-} (I_0). \]

9: LEMMA
\[ \frac{\gamma^2}{2} \sum_{I_0} \sum_{\gamma^+} ||\phi(I)|| \]
PROOF The inequality

\[ ||\alpha(I_0) - \alpha(I) || \geq \gamma \]

implies that

\[ \gamma^2 \leq ||\alpha(I_0) - \alpha(I) ||^2 \]

\[ = (\alpha(I_0) - \alpha(I)) \cdot (\alpha(I_0) - \alpha(I)) \]

\[ = ||\alpha(I_0) ||^2 - 2\alpha(I_0) \cdot \alpha(I) + ||\alpha(I) ||^2 \]

\[ = 2 - 2\alpha(I_0) \cdot \alpha(I), \]

so

\[ \frac{\gamma^2}{2} \leq 1 - \alpha(I_0) \cdot \alpha(I) \]

\[ \Rightarrow \]

\[ \frac{\gamma^2}{2} ||\phi(I) || \leq ||\phi(I) || - \alpha(I_0) \cdot \phi(I). \]

But for any I,

\[ 0 \leq ||\phi(I) || - \alpha(I_0) \cdot \phi(I). \]

Proof: In fact,

\[ ||\phi(I) || - \frac{\phi(I_0) \cdot \phi(I)}{||\phi(I_0)||} \]

\[ = \frac{1}{||\phi(I_0)||} [ ||\phi(I) || ||\phi(I_0)|| - \phi(I_0) \cdot \phi(I)]. \]
Now quote Schwarz's inequality. Thus we may write

\[
\frac{\gamma^2}{2} \sum_{\gamma^+} ||\phi(I)||
\]

\[
\leq \sum_{\gamma^+} (||\phi(I) - \alpha(I_0) \cdot \phi(I)||)
\]

\[
\leq \sum_{I \subseteq I_0} (||\phi(I) - \alpha(I_0) \cdot \phi(I)||)
\]

\[
= \sum_{I \subseteq I_0} ||\phi(I)|| - ||\phi(I_0)|| + \alpha(I_0) \cdot (\phi(I_0) - \sum_{I \subseteq I_0} \phi(I))
\]

\[
\leq \sum_{I \subseteq I_0} ||\phi(I)|| - ||\phi(I_0)|| + ||\phi(I_0) - \sum_{I \subseteq I_0} \phi(I)|| (\text{Schwarz}).
\]

To finish, sum over $I_0$.

- (D) Assume

\[
\delta(D_0) < \min\{t(\gamma), \eta(\varepsilon), \eta(\varepsilon \gamma^2)\}.
\]

- (D) Assume

\[
\delta(D) < \min\{\sigma(\varepsilon), \lambda(\varepsilon, D_0), \lambda(\varepsilon \gamma^2, D_0), T(\gamma, D_0)\}.
\]

- (e) Assume

\[
\varepsilon < \min\{\gamma, \frac{\varepsilon}{3C}, \frac{\varepsilon \gamma^2}{24C}\}.
\]

Then

\[
\sum_{I_0} \sum_{I \subseteq I_0} ||\phi(I) - \phi(I_0)||
\]
\[
= \sum_{I_0} \sum_{I \in I_0} F(z(I), \phi(I)) \\
- \sum_{I \in I_0} F(z(I_0), \alpha(I_0)) | | \phi(I) | | \\
+ \sum_{I \in I_0} F(z(I_0), \alpha(I_0)) | | \phi(I) | | \\
- F(z(I_0), \alpha(I_0)) | | \phi(I_0) | | \\
= \sum_{I_0} \sum_{I \in I_0} (F(z(I), \alpha(I)) - F(z(I_0), \alpha(I_0))) | | \phi(I) | | \\
+ \sum_{I \in I_0} F(z(I_0), \alpha(I_0)) (| | \phi(I) | | - | | \phi(I_0) | |) \\
\leq \sum_{I_0} |F(z(I_0), \alpha(I_0))| | \sum_{I \in I_0} | \phi(I) | | - | | \phi(I_0) | | | | \\
+ \sum_{I_0} \sum_{I \in I_0} |F(z(I), \alpha(I)) - F(z(I_0), \alpha(I_0))| | \phi(I) | | \\
= \sum_{I_0} |F(z(I_0), \alpha(I_0))| | \sum_{I \in I_0} | \phi(I) | | - | | \phi(I_0) | | | | \\
+ \sum_{I_0} (\sum_{I \in I_0} \lambda(I_0) \lambda(I_0)) |F(z(I_0), \alpha(I_0)) - F(z(I), \alpha(I))| | \phi(I) | |. \\
\text{First:} \\
\sum_{I_0} |F(z(I_0), \alpha(I_0))| | \sum_{I \in I_0} | \phi(I) | | - | | \phi(I_0) | | | | \\
\leq C \sum_{I_0} \sum_{I \in I_0} | \phi(I) | | - | | \phi(I_0) | | | | 
\]
\[ \langle C \varepsilon, C \frac{\varepsilon}{3C} = \frac{\varepsilon}{3} \].

**Second:** Consider

\[ \sum_{I, I_0} \sum_{y} |F(y(I), a(I)) - F(y(I), a(I))| |\phi(I)|. \]

Here

\[ |a(I_0)| = 1, |a(I)| = 1, |a(I_0) - a(I)| < \gamma, \]

\[ |\zeta(I_0) - \zeta(I)| < \gamma \]

\[ \Rightarrow \]

\[ |F(y(I_0), a(I_0)) - F(y(I), a(I))| < \frac{\varepsilon}{3(V+\varepsilon)}. \]

The entity in question is thus majorized by

\[ \frac{\varepsilon}{3(V+\varepsilon)} \sum_{I, I_0} \sum_{y} |\phi(I)| \leq \frac{\varepsilon}{3(V+\varepsilon)} \sum_{I \in D} |\phi(I)| \]

\[ \leq \frac{\varepsilon}{3(V+\varepsilon)} (V + \varepsilon) = \frac{\varepsilon}{3}. \]

**Third:**

\[ \sum_{I, I_0} \sum_{y} |F(y(I_0), a(I_0)) - F(y(I), a(I))| |\phi(I)| \]

\[ \leq 2C \sum_{I, I_0} \sum_{y} |\phi(I)| \]

\[ \leq \frac{4C}{\gamma^2} \left[ \sum_{I, I_0} \sum_{I \in I_0} |\phi(I) - \phi(I_0)| \right] \]

\[ + \sum_{I, I_0} \sum_{I \in I_0} |\phi(I)| - |\phi(I_0)| \]

\[ + \sum_{I, I_0} |\phi(I)| - |\phi(I_0)| \]

\[ \leq \frac{4C}{\gamma^2} \left[ \sum_{I, I_0} \sum_{I \in I_0} |\phi(I) - \phi(I_0)| \right] \]
\[
\leq \frac{4C}{\gamma^2} (\varepsilon + \varepsilon) \\
= \frac{8C}{\gamma^2} \varepsilon \\
< \frac{8C}{\gamma^2} \cdot \frac{\varepsilon \gamma^2}{24C} = \frac{\varepsilon}{3}.
\]

In total then:
\[
\sum_{I_0} | \sum_{I \subset I_0} \phi(I) - \phi(I_0) | < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
= \varepsilon.
\]

And finally
\[
\sum_{I \not\subset I_0} | \phi(I) | \\
= \sum_{I \not\subset I_0} | F(\xi(I), \phi(I)) | \\
= \sum_{I \not\subset I_0} | F(\xi(I), \alpha(I)) | \cdot | \phi(I) | \\
\leq C \sum_{I \not\subset I_0} | \phi(I) | \\
< C \varepsilon < C \cdot \frac{\varepsilon}{3} = \frac{\varepsilon}{3} < \varepsilon.
\]

Therefore \( \phi \) is quasi additive. And since the conditions on \( F \) carry over to \( |F| \), it follows that \( ||\phi|| \) is also quasi additive, hence
\[
BC \int_A ||\phi||
\]
exists and is finite.

To tie up one loose end, return to the beginning and consider the line integrals
\[
(\xi) \int_C F, \ (\xi') \int_C F,
\]
the claim being that they are equal. That this is so can be seen by writing

\[ |(\xi) \int_C F - (\xi') \int_C F| \]

\[ = |(\xi) \int_C F - \sum_{i=1}^{n} F(f(\xi_i), f(x_i) - f(x_{i-1})) \]

\[ + \sum_{i=1}^{n} F(f(\xi_i), f(x_i) - f(x_{i-1})) \]

\[ - \sum_{i=1}^{n} F(f(\xi_i'), f(x_i) - f(x_{i-1})) \]

\[ + \sum_{i=1}^{n} F(f(\xi_i'), f(x_i) - f(x_{i-1})) - (\xi') \int_C F| \]

and proceed from here in the obvious way.

**10: EXAMPLE** Take \( N = 1, M = 1 \) and define an interval function \(| . | : I \to \mathbb{R}\) by sending \( I \) to its length \(| I |\). Fix a choice function \( \omega : I \to [a, b] \). Consider a curve

\[ C \leftrightarrow f : [a, b] \to \mathbb{R}. \]

Assume: \( f \) is continuous and of bounded variation, thus

\[ \ell(C) = T_f[a, b] < + \infty. \]

Work with the parametric integrand \( F(x, t) = xt \) -- then the data

\[ I \to F(\xi(I), |I|) \]

\[ = F(\xi(\omega(I)), |I|) \]

\[ = f(\omega(I)) |I| \]
leads to sums of the form

\[ \sum_{i=1}^{n} f(\xi_i) (x_i - x_{i-1}), \]

hence to

\[ \int_{C} F = \int_{a}^{b} f, \]

the Riemann integral of \( f \).