LOCAL AND GLOBAL ANALYSIS

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The objective of this book is to give an introduction to p-adic analysis along the lines of Tate's thesis, as well as incorporating material of a more recent vintage, for example Weil groups.
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§1. ABSOLUTE VALUES

1: DEFINITION Let \( F \) be a field -- then an absolute value (a.k.a. a valuation of order 1) is a function

\[ |\cdot|: F \to \mathbb{R}_{\geq 0} \]

satisfying the following conditions.

AV-1 \( |a| = 0 \iff a = 0. \)

AV-2 \( |ab| = |a||b|. \)

AV-3 \( \exists M > 0: \)

\[ |a + b| \leq M \sup(|a|, |b|). \]

2: EXAMPLE Let \( F = \mathbb{R} \) or \( \mathbb{C} \) with the usual absolute value \(|.|_\infty\) -- then one can take \( M = 2 \).

3: DEFINITION The trivial absolute value is defined by the rule

\[ |a| = 1 \forall a \neq 0. \]

4: LEMMA If \(|.|\) is an absolute value, then

\[ |1| = 1. \]

5: APPLICATION If \( a^n = 1 \), then

\[ |a^n| = |a|^n = |1| = 1 \]

\[ \implies |a| = 1. \]

6: RAPPEL Let \( G \) be a cyclic group of order \( r < \infty \) -- then the order of any subgroup of \( G \) is a divisor of \( r \) and if \( n|r \), then \( G \) possesses one and only one
subgroup of order n (and this subgroup is cyclic).

7: RAPPEL Let $G$ be a cyclic group of order $r < \infty$ — then the order of $x \in G$ is, by definition, $\#<x>$, the latter being the smallest positive integer $n$ such that $x^n = 1$.

8: SCHOLIUM Every absolute value on a finite field $F_q$ is trivial.

[In fact, $F_q^\times$ is cyclic of order $q - 1$.]

9: DEFINITION Two absolute values $|.|_1, |.|_2$ on a field $F$ are equivalent if $\exists r > 0$:

$$|.|_2 = |.|_1^r.$$

[Note: Equivalence is an equivalence relation.]

10: N.B. If $|.|$ is an absolute value, then so is $|.|^r (r > 0)$, the $M$ per $|.|$ being $M^r$ per $|.|^r$.

11: LEMMA Every absolute value is equivalent to one with $M \leq 2$.

PROOF Assume from the beginning that $M > 2$, hence

$$M^r \leq 2 \quad (r > 0)$$

if

$$r \log M \leq \log 2$$

or still, if

$$r \leq \frac{\log 2}{\log M} \quad (< 1).$$
12: DEFINITION An absolute value $|.|$ satisfies the triangle inequality if

$$|a+b| \leq |a| + |b|.$$ 

13: LEMMA Suppose given a function $|.|:F \to \mathbb{R}_{\geq 0}$ satisfying AV-1 and AV-2 -- then AV-3 holds with $M \leq 2$ iff the triangle inequality obtains.

PROOF Obviously, if 

$$|a+b| \leq |a| + |b|,$$

then 

$$|a+b| \leq 2 \sup(|a|,|b|).$$

In the other direction, by induction on $m$,

$$\sum_{k=1}^{2^m} |a_k| \leq 2^m \sup_k |a_k| \quad (1 \leq k \leq 2^m).$$

Next, given $n$ choose $m$: $2^m \geq n > 2^{m-1}$, so upon inserting $2^m-n$ zero summands,

$$\sum_{k=1}^{2^m} |a_k| \leq M \sup \left( \sum_{k=1}^{2^{m-1}} |a_k|, \sum_{k=2^{m-1}+1}^{2^m} |a_k| \right)$$

$$\leq 2 \sup \left( \sum_{k=1}^{2^{m-1}} |a_k|, \sum_{k=2^{m-1}+1}^{2^m} |a_k| \right)$$

$$\leq 2 \sup (2^{m-1} \sup_{k \leq 2^{m-1}} |a_k|, 2^{m-1} \sup_{k > 2^{m-1}} |a_k|)$$

$$\leq 2 \cdot 2^{m-1} \sup_{1 \leq k \leq n} |a_k| \leq 2 \cdot n \sup_{1 \leq k \leq n} |a_k|. $$
I.e.:

\[ \left| \sum_{k=1}^{n} a_k \right| \leq 2n \sup_{1 \leq k \leq n} |a_k| \leq 2n \sum_{k=1}^{n} |a_k|. \]

In particular:

\[ \left| \sum_{k=1}^{n} 1 \right| = |n| \leq 2n. \]

Finally,

\[ |a + b|^n = |(a + b)^n| \quad \text{(AV-2)} \]

\[ = \left| \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \right| \]

\[ \leq 2(n+1) \sum_{k=0}^{n} \binom{n}{k} |a_k b^{n-k}| \]

\[ = 2(n+1) \sum_{k=0}^{n} \binom{n}{k} |a_k b^{n-k}| \quad \text{(AV-2)} \]

\[ \leq 2(n+1)2 \sum_{k=0}^{n} \binom{n}{k} |a_k b^{n-k}| \]

\[ = 4(n+1)(|a| + |b|)^n \]

\[ \Rightarrow \]

\[ |a + b| \leq 4^{1/n}(n+1)^{1/n}(|a| + |b|) \]

\[ \rightarrow (|a| + |b|) (n \rightarrow \infty). \]
14: SCHOLIUM Every absolute value is equivalent to one that satisfies the triangle inequality.

15: DEFINITION A place of $F$ is an equivalence class of nontrivial absolute values.

Accordingly, every place admits a representative for which the triangle inequality is in force.

16: DEFINITION An absolute value $|\cdot|$ is non-archimedean if it satisfies the ultrametric inequality:

$$|a + b| \leq \sup(|a|, |b|) \ (\text{so } M = 1).$$

17: N.B. A non-archimedean absolute value satisfies the triangle inequality.

18: LEMMA Suppose that $|\cdot|$ is non-archimedean and let $|b| < |a|$ -- then

$$|a + b| = |a|.$$

PROOF

$$|a| = |(a + b) - b| \leq \sup(|a + b|, |b|)$$

$$= |a + b|$$

since $|a| \leq |b|$ is untenable. Meanwhile,

$$|a + b| \leq \sup(|a|, |b|) = |a|.$$

19: EXAMPLE Fix a prime $p$ and take $F = \mathbb{Q}$. Given a rational number $x \neq 0$, write

$$x = p^k \frac{m}{n} \ (k \in \mathbb{Z}),$$
where \( p \nmid m, p \nmid n \), and then define the \( p \)-adic absolute value \( | \cdot |_p \) by the prescription

\[
|x|_p = p^{-k} \quad (|0|_p = 0).
\]

[AV-1 is obvious. To check AV-2, write

\[
x = p^k \frac{m}{n}, \quad y = p^\ell \frac{u}{v},
\]

where \( m, n, u, v \) are coprime to \( p \) — then

\[
xy = p^{k+\ell} \frac{mu}{nv}
\]

\[
\Rightarrow |xy|_p = p^{-(k+\ell)} = p^{-k}p^{-\ell} = |x|_p |y|_p.
\]

As for AV-3, \( | \cdot |_p \) satisfies the ultrametric inequality. To establish this, assume without loss of generality that \( k \leq \ell \) and write

\[
x + y = p^k \frac{m}{n} + p^{\ell-k} \frac{u}{v}
\]

\[
= p^k \frac{mv + p^{\ell-k}nu}{nv}.
\]

\* \( |x|_p \neq |y|_p \), so \( \ell - k > 0 \), hence

\[
mv + p^{\ell-k}nu
\]

is coprime to \( p \) (otherwise

\[
mv = p^rN - p^{\ell-k}nu \quad (r \geq 1)
\]

\[
= p(p^{r-1}N - p^{\ell-k-1}nu) \Rightarrow p|mvy
\]

\[
\Rightarrow |x + y|_p = p^{-k}
\]
= |x|_p = \sup(|x|_p', |y|_p),

since

\ell - k > 0 \Rightarrow p^{-\ell} < p^{-k}

\Rightarrow |y|_p < |x|_p.

\bullet \ |x|_p = |y|_p', \text{ so } \ell = k, \text{ hence }

mv + nu = p^r N \ (r \geq 0) \ (p \nmid N)

\Rightarrow

x + y = p^{k+r} \frac{N}{nv}

\Rightarrow

|x + y|_p = p^{-k-r}.

And

\[
\begin{cases}
  p^{-k} = |x|_p \\
p^{-k-r} = |y|_p
\end{cases}
\]

\Rightarrow

|x + y|_p \leq \sup(|x|_p', |y|_p'). \}

20: REMARK It can be shown that every nontrivial absolute value on \( Q \) is equivalent to a \(|.|_p\) for some \( p \) or to \(|.|_\infty\).
21: **LEMMA** \( \forall x \in \mathbb{Q}^x \),
\[
\prod_{p \leq \infty} |x|_p = 1,
\]
all but finitely many of the factors being equal to 1.

**PROOF** Write
\[
x = \pm p_1^{k_1} \cdots p_n^{k_n} \quad (k_1, \ldots, k_n \in \mathbb{Z})
\]
for pairwise distinct primes \( p_j \) -- then \( |x|_p = 1 \) if \( p \) is not equal to any of the \( p_j \). In addition,
\[
|x|_{p_j} = p_j^{-k_j}, \quad |x|_{\infty} = p_1^{k_1} \cdots p_n^{k_n}
\]
\[
\Rightarrow 
\prod_{p \leq \infty} |x|_p = (\prod_{j=1}^{n} p_j^{-k_j}) \cdot p_1^{k_1} \cdots p_n^{k_n}
= 1.
\]

22: **REMARK** If \( p_1, p_2 \) are distinct primes, then \( |.|_{p_1} \) is not equivalent to \( |.|_{p_2} \).

[Consider the sequence \( \{p_1^n\} \):
\[
|p_1^1|_{p_1} = p_1^{-1} \Rightarrow |p_1^n|_{p_1} = p_1^{-n} + 0.
\]
Meanwhile,
\[
|p_1|_{p_2} = |p_2 p_1|_{p_2} = p_2^{-1} = 1
\]
\[
\Rightarrow |p_1^n|_{p_2} \equiv 1.\]

23: CRITERION Let $|.|$ be an absolute value on $F$ — then $|.|$ is non-archimedean iff \{ $|n|: n \in \mathbb{N}$ \} is bounded.

[Note: In either case, $|n|$ is bounded by 1:

$$|n| = |1 + 1 + \cdots + 1| \leq 1.$$]
Let $|\cdot|$ be an absolute value on a field $F$. Given $a \in F$, $r > 0$, put
\[ N_r(a) = \{b : |b - a| < r\}. \]

1: **Lemma** There is a topology on $F$ in which a basis for the neighborhoods of $a$ are the $N_r(a)$.

**Proof** The nontrivial point is to show that given $V \in B_a$, there is a $V_0 \in B_a$ such that if $a_0 \in V_0$, then there is a $W \in B_{a_0}$ such that $W \subset V$. So let $V = N_r(a)$, $V_0 = N_{r/2M}(a)$, $W = N_{r/2M}(a_0)$ $(a_0 \in V_0)$ -- then $W \subset V$:
\[ b \in W \Rightarrow |b - a| = |(b - a_0) + (a_0 - a)| \leq M \sup(|b - a_0|, |a_0 - a|) \leq M \sup(r/2M, r/2M) = M(r/2M) = r/2 < r. \]

2: **Example** The topology induced by $|\cdot|$ is the discrete topology iff $|\cdot|$ is the trivial absolute value.

3: **Fact** Absolute values $|\cdot|_1$, $|\cdot|_2$ are equivalent iff they give rise to the same topology.

4: **Lemma** The topology induced by $|\cdot|$ is metrizable.

**Proof** This is because $|\cdot|$ is equivalent to an absolute value satisfying the
2. triangle inequality (cf. §1, #14), the underlying metric being
\[ d(a,b) = |a - b|. \]

5: **THEOREM** A field with a topology defined by an absolute value is a topological field, i.e., the operations sum, product, and inversion are continuous.

Assume now that \(|\cdot|\) is non-archimedean, hence that the ultrametric inequality
\[ |a - b| \leq \sup(|a|, |b|) \]
is in force.

6: **LEMMA** \(N_r(a)\) is closed (open is automatic).

**PROOF** Let \(p\) be a limit point of \(N_r(a)\) -- then \(\forall t > 0, \)
\[ (N_r(p) - \{p\}) \cap N_r(a) \neq \emptyset. \]
Take \(t = \frac{r}{2}\) and choose \(b \in N_r(a)\):
\[ d(p,b) < \frac{r}{2} \quad (p \neq b). \]
Then
\[ d(a,p) \leq \sup(d(a,b), d(b,p)) \]
\[ < r \]
\[ \Rightarrow \]
\[ p \in N_r(a). \]
Therefore \(N_r(a)\) contains all its limit points, hence is closed.

7: **LEMMA** If \(a' \in N_r(a)\), then \(N_r(a') = N_r(a)\).

**PROOF** E.g.:
\[ b \in N_r(a) \Rightarrow |b - a| < r \]
3.

\[ = \frac{|b - a'|}{|b - a| + (a - a')} \]

\[ \leq \sup(|b - a|, |a - a'|) \]

\[ < r \Rightarrow N_r(a) \subset N_r(a'). \]

**8: REMARK** Put

\[ B_r(a) = \{b: |b - a| < r\}. \]

Then a priori, \( B_r(a) \) is closed. But \( B_r(a) \) is also open and if \( a' \in B_r(a) \), then \( B_r(a') = B_r(a) \).

**9: LEMMA** If

\[ a_1 + a_2 + \ldots + a_n = 0, \]

then \( \exists i \neq j \) such that

\[ |a_i| = |a_j| = \sup|a_k|. \]
§3. COMPLETIONS

Let $|.|$ be an absolute value on a field $F$ which satisfies the triangle inequality -- then per $|.|$, $F$ might or might not be complete.

1: EXAMPLE Take $F = \mathbb{R}$ or $\mathbb{Q}$ and let $|.| = |.|_\infty$ -- then $\mathbb{R}$ is complete but $\mathbb{Q}$ is not.

2: EXAMPLE Take $F = \mathbb{Q}$ and let $|.| = |.|_p$ -- then $\mathbb{Q}$ is not complete.

[To illustrate this, choose $p = 5$ and starting with $x_1 = 2$, define inductively a sequence $\{x_n\}$ of integers subject to

$$
\begin{align*}
  x_n^2 + 1 &\equiv 0 \mod 5^n \\
  x_{n+1} &= x_n \mod 5^n.
\end{align*}
$$

Then

$$
|x_m - x_n|_5 \leq 5^{-n} \quad (m > n),
$$

so $\{x_n\}$ is a Cauchy sequence and, to get a contradiction, assume that it has a limit $x$ in $\mathbb{Q}$, thus

$$
|x_n^2 + 1|_5 \leq 5^{-n} \implies |x^2 + 1|_5 = 0
$$

$$
\implies x^2 + 1 = 0 \ldots .]
$$

3: DEFINITION If an absolute value is not non-archimedean, then it is said to be archimedean.
4: FACT Suppose that $F$ is a field which is complete with respect to an archimedean absolute value $|.|$ -- then $F$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ and $|.|$ is equivalent to $|.|_\infty$.

5: RAPPEL Every metric space $X$ has a completion $\bar{X}$. Moreover, there is an isometry $\phi:X \to \bar{X}$ such that $\phi(X)$ is dense in $\bar{X}$ and $\bar{X}$ is unique up to isometric isomorphism.

6: CONSTRUCTION The standard model for $\bar{X}$ is the set of all Cauchy sequences in $X$ modulo the equivalence relation $\sim$, where

$\{x_n\} \sim \{y_n\} \iff d(x_n, y_n) \to 0$,

the map $\phi:X \to \bar{X}$ being the rule that sends $x \in X$ to the equivalence class of the constant sequence $x_n = x$.

[Note: The metric on $\bar{X}$ is specified by $\bar{d}([x_n],[y_n]) = \lim_{n \to \infty} d(x_n, y_n)$.]

Take $X = F$ and

$d(x, y) = |x - y|$.

Then the claim is that $\bar{F}$ is a field. E.g.: Let us deal with addition. Given $\bar{x}, \bar{y} \in \bar{F}$, how does one define $\bar{x} + \bar{y}$? To this end, choose sequences $\begin{bmatrix} x_n \\ y_n \end{bmatrix}$ in $F$ such that $\begin{bmatrix} x_n \to \bar{x} \\ y_n \to \bar{y} \end{bmatrix}$ -- then

$d(x_n + y_n, x_m + y_m)$
Therefore \( \{x_n + y_n\} \) is a Cauchy sequence in \( F \), hence converges in \( \bar{F} \) to an element \( \bar{z} \). If \( \begin{bmatrix} x_n' \\ y_n' \end{bmatrix} \) are sequences in \( F \) converging to \( \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \) as well, then \( \{x_n' + y_n'\} \) converges in \( \bar{F} \) to an element \( \bar{z}' \). And
\[
\bar{z} = \bar{z}'.
\]

Proof: Choose \( n \in \mathbb{N} \) such that
\[
\begin{align*}
|\bar{z} - (x_n + y_n)| &< \frac{\varepsilon}{3} \\
|\bar{z}' - (x_n' + y_n')| &< \frac{\varepsilon}{3}
\end{align*}
\]
and
\[
| (x_n + y_n) - (x_n' + y_n') | \leq |x_n - x_n'| + |y_n - y_n'| < \frac{\varepsilon}{3}.
\]
Then
\[
|\bar{z} - \bar{z}'| \leq |\bar{z} - (x_n + y_n)| + |\bar{z}' - (x_n + y_n)|
\]

\[
\leq |\bar{z} - (x_n + y_n)| + |\bar{z}' - (x_n' + y_n')| + |(x_n' + y_n') - (x_n + y_n)| < \varepsilon
\]

\[
=> \bar{z} = \bar{z}'.
\]

Therefore addition in \( F \) extends to \( \bar{F} \). The same holds for multiplication and
inversion. Bottom line: $\overline{F}$ is a field. Furthermore, the prescription
\[
|x| = \overline{d}(x,0) \quad (x \in F)
\]

is an absolute value on $\overline{F}$ whose underlying topology is the metric topology. It thus follows that $\overline{F}$ is a topological field (cf. §2, #5).

7: EXAMPLE Take $F = \mathbb{Q}$, $|.| = |.|_p$ -- then the completion $\overline{F} = \overline{\mathbb{Q}}$ is denoted by $\mathbb{Q}_p$, the field of $p$-adic numbers.

8: LEMMA If $|.|$ is non-archimedean per $F$, then $|.|$ is non-archimedean per $\overline{F}$.

PROOF Given
\[
\begin{align*}
\frac{x}{y} \in F, \text{ choose } x_n \in F \text{ such that } & x_n \to x \\
y_n \to y \\
|\frac{x}{y}| \leq |x - x_n + x_n - y_n + y_n - y|
\end{align*}
\]

\[\leq |x - x_n| + |x_n - y_n| + |y - y_n|.
\]

And
\[
|\frac{x}{y}| \leq \sup(|x_n|, |y_n|)
\]

\[= \frac{1}{2} (|x_n| + |y_n| + |x_n - y_n|)
\]

\[+ \frac{1}{2} (|\frac{x}{y}| + |\frac{y}{y}| + |\frac{x}{y} - \frac{y}{y}|)
\]

\[= \sup(|\frac{x}{y}|, |\frac{y}{y}|).
\]
9: **LEMMA** If $|\cdot|$ is non-archimedean per $|\cdot|$, then

$$\{|x|: x \in F\} = \{|x|: x \in F\}.$$  

**PROOF** Take $\bar{x} \in F: \bar{x} \neq 0$. Choose $x \in F: |\bar{x} - x| < |\bar{x}|$. Claim: $|\bar{x}| = |x|$. Thus consider the other possibilities.

- $|x| < |\bar{x}|$:
  
  $$|\bar{x} - x| = |\bar{x} + (-x)| = |\bar{x}|$$

  (cf. §1, #18) < $|\bar{x}|$...

- $|\bar{x}| < |x|$:
  
  $$|\bar{x} - x| = |-x + \bar{x}| = |-x|$$

  (cf. §1, #18) = $|x| < |\bar{x}|$...

10: **EXAMPLE** The image of $Q_p$ under $|\cdot|_p$ is the same as the image of $Q$ under $|\cdot|_p$, namely

$$\{p^k: k \in Z\} \cup \{0\}.$$  

Let $K$ be a field, $L \supset K$ a finite field extension.

11: **EXTENSION PRINCIPLE** Let $|\cdot|_K$ be a complete absolute value on $K$ --- then there is one and only one extension $|\cdot|_L$ of $|\cdot|_K$ to $L$ and it is given by

$$|x|_L = |N_{L/K}(x)|_K^{1/n},$$

where $n = [L:K]$. In addition, $L$ is complete with respect to $|\cdot|_L$.

[Note: $|\cdot|_L$ is non-archimedean if $|\cdot|_K$ is non-archimedean.]

12: **SCHOLIUM** There is a unique extension of $|\cdot|_K$ to the algebraic closure $K^{cl}$ of $K$.

[Note: It is not true in general that $K^{cl}$ is complete.]
6.

Suppose further that $L \supset K$ is a Galois extension. Given $\sigma \in \text{Gal}(L/K)$, define $|\cdot|_\sigma$ by $|x|_\sigma = |\sigma x|_L$ -- then

$$|\cdot|_\sigma|_K = |\cdot|_K.$$

so by uniqueness, $|\cdot|_\sigma = |\cdot|_L$. But

$$N_{L/K}(x) = \prod_{\sigma \in \text{Gal}(L/K)} \sigma x,$$

so

$$|N_{L/K}(x)|_K = |N_{L/K}(x)|_L = \left| \prod_{\sigma \in \text{Gal}(L/K)} \sigma x \right|_L$$

$$= \prod_{\sigma \in \text{Gal}(L/K)} |\sigma x|_L$$

$$= \prod_{\sigma \in \text{Gal}(L/K)} |x|_L$$

$$= |x|^{|\text{Gal}(L/K)|}$$

$$= |x|^{[L:K]}_L = |x|^{n}_L.$$

**APPENDIX**

**APPROXIMATION PRINCIPLE** Let $|\cdot|_1, \ldots, |\cdot|_N$ be pairwise inequivalent non-trivial absolute values on $F$. Fix elements $a_1, \ldots, a_N$ in $F$ -- then $\forall \varepsilon > 0, \exists a_\varepsilon \in F$:

$$|a_\varepsilon - a_k|_k < \varepsilon \quad (k = 1, \ldots, N).$$
Let $\bar{F}_1, \ldots, \bar{F}_N$ be the associated completions and let
\[
\Delta: F \to \prod_{k=1}^{N} \bar{F}_k
\]
be the diagonal map -- then the image $\Delta F$ is dense (i.e., its closure is the whole of $\prod_{k=1}^{N} \bar{F}_k$).

[Fix $\varepsilon > 0$ and elements $\bar{a}_1, \ldots, \bar{a}_N$ in $\bar{F}_1, \ldots, \bar{F}_N$ respectively -- then there exist elements $a_k \in F$:
\[
|a_k - \bar{a}_k|_k < \varepsilon \quad (k = 1, \ldots, N).
\]
Choose $a_\varepsilon \in F$:
\[
|a_\varepsilon - a_k| < \varepsilon \quad (k = 1, \ldots, N).
\]
Then
\[
|a_\varepsilon - \bar{a}_k|_k = |(a_\varepsilon - a_k) + (a_k - \bar{a}_k)|_k
\]
\[
\leq |a_\varepsilon - a_k|_k + |a_k - \bar{a}_k|_k
\]
\[
< 2\varepsilon.]
\]

N.B. The product $\prod_{k=1}^{N} \bar{F}_k$ carries the product topology and the prescription
\[
d((\bar{a}_1, \ldots, \bar{a}_N), (\bar{b}_1, \ldots, \bar{b}_N))
\]
\[
= \sup_{1 \leq k \leq N} d_k(\bar{a}_k, \bar{b}_k)
\]
\[
= \sup_{1 \leq k \leq N} |\bar{a}_k - \bar{b}_k|_k
\]
metrizes the product topology. Therefore

\[ d((a_1, \ldots, a_n), (a'_1, \ldots, a'_n)) \]

\[ = \sup_{1 \leq k \leq N} d_k(a_k, a'_k) \]

\[ = \sup_{1 \leq k \leq N} |a_k - a'_k|_k \]

\[ < 2\varepsilon. \]
§4. p-ADIC STRUCTURE THEORY

Fix a prime \( p \) and recall that \( Q_p \) is the completion of \( Q \) per the p-adic absolute value \( |\cdot|_p \).

1: NOTATION Let

\[ A = \{0,1,\ldots,p-1\}. \]

2: SCHOLIUM Structurally, \( Q_p \) is the set of all Laurent series in \( p \) with coefficients in \( A \) subject to the restriction that only finitely many negative powers of \( p \) occur, thus generically a typical element \( x \neq 0 \) of \( Q_p \) has the form

\[ x = \sum_{n=N}^{\infty} a_n p^n \quad (a_n \in A, N \in \mathbb{Z}). \]

3: N.B. It follows from this that \( Q_p \) is uncountable, so \( Q \) is not complete per \( |\cdot|_p \).

The exact formulation of the algebraic rules (i.e., addition, multiplication, inversion) is elementary (but technically a bit of a mess) and will play no role in the sequel, hence can be omitted.

4: LEMMA Every positive integer \( N \) admits a base \( p \) expansion:

\[ N = a_0 + a_1 p + \cdots + a_n p^n, \]

where the \( a_k \in A \).

5: EXAMPLE

\[ 1 = 1 + 0p + 0p^2 + \ldots . \]
6: EXAMPLE Take \( p = 3 \) -- then

\[
\begin{align*}
24 &= 0 + 2 \times 3 + 2 \times 3^2 = 2p + 2p^2 \\
17 &= 2 + 2 \times 3 + 1 \times 3^2 = 2 + 2p + p^2
\end{align*}
\]

\[\Rightarrow \]

\[
\frac{24}{17} = \frac{2p + 2p^2}{2 + 2p + p^2} = p + p^3 + 2p^5 + p^7 + p^8 + 2p^9 + \cdots.
\]

7: LEMMA

\[-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots.\]

PROOF Add 1:

\[1 + (p-1) + (p-1)p + (p-1)p^2 + (p-1)p^3 + \cdots = p + (p-1)p + (p-1)p^2 + (p-1)p^3 + \cdots = p^2 + (p-1)p^2 + (p-1)p^3 + \cdots = p^3 + (p-1)p^3 + \cdots = 0.\]

8: APPLICATION

\[-N = (-1) \cdot N\]

\[= (\sum_{i=0}^{\infty} (p-1)p^i)(a_0 + a_1p + \cdots + a_np^n) = \cdots\]

9: LEMMA A p-adic series

\[
\sum_{n=1}^{\infty} x_n \quad (x_n \in Q_p)
\]
is convergent iff \( |x_n|_p \to 0 \ (n \to \infty) \).

**PROOF** The usual argument establishes necessity. So suppose that \( |x_n|_p \to 0 \) \((n \to \infty)\). Given \( K > 0 \), \( \exists N: \)

\[
n > N \Rightarrow |x_n|_p < p^{-K}.
\]

Let

\[
s_n = \sum_{k=1}^{n} x_k.
\]

Then

\[
m > n > N \Rightarrow \|s_m - s_n\|_p = |x_{n+1} + \cdots + x_m|_p
\]

\[
\leq \sup(|x_{n+1}|_p, \ldots, |x_m|_p)
\]

\[
< p^{-K}.
\]

Therefore the sequence \( \{s_n\} \) of partial sums is Cauchy, thus is convergent \((Q_p\)

10: **EXAMPLE** The \( p \)-adic series

\[
\sum_{i=0}^{\infty} \frac{p^i}{p-1}
\]

is convergent (to \( \frac{1}{1-p} \)).

11: **EXAMPLE** The \( p \)-adic series

\[
\sum_{n=0}^{\infty} n!
\]

is convergent.

[Note that

\[
|x|_p = p^{-N},
\]
where

\[ N = [n/p] + [n/p^2] + \cdots \]

12: **EXAMPLE** The p-adic series

\[ \sum_{n=0}^{\infty} n \cdot n! \]

is convergent (to -1).

13: **LEMMA** $\mathbb{Q}_p$ is a topological field (cf. §2, #5).

14: **LEMMA** $\mathbb{Q}_p$ is 0-dimensional, hence is totally disconnected.

**PROOF** A basic neighborhood $N_r(x)$ is open (by definition) and closed (cf. §2, #6).

15: **NOTATION**

- $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$
- $p\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p < 1\}$
- $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p : |x|_p = 1\}$

16: **LEMMA** $\mathbb{Z}_p$ is a commutative ring with unit (the ring of p-adic integers), in fact $\mathbb{Z}_p$ is an integral domain.

17: **LEMMA** $p\mathbb{Z}_p$ is an ideal in $\mathbb{Z}_p$, in fact $p\mathbb{Z}_p$ is a maximal ideal in $\mathbb{Z}_p$, in fact $p\mathbb{Z}_p$ is the unique maximal ideal in $\mathbb{Z}_p$, hence $\mathbb{Z}_p$ is a local ring.

18: **LEMMA** $\mathbb{Z}_p^\times$ is a group under multiplication, in fact $\mathbb{Z}_p^\times$ is the set of
p-adic units in $\mathbb{Z}_p$, i.e., the set of elements in $\mathbb{Z}_p$ that have a multiplicative inverse in $\mathbb{Z}_p$.

Obviously,

$$\mathbb{Z}_p = \mathbb{Z}_p^\times \sqcup (\mathbb{Z}_p - \mathbb{Z}_p^\times)$$

or still,

$$\mathbb{Z}_p = \mathbb{Z}_p^\times \sqcup p\mathbb{Z}_p.$$
Then

$$\left| \sum_{n=0}^{\infty} a_n p^n \right|_p \leq \sup_n \left| a_n p^n \right|_p \leq \sup_n \left| p^n \right|_p = 1,$$

so it converges to an element $x$ of $\mathbb{Z}_p$. Conversely:

20: **THEOREM** Every $x \in \mathbb{Z}_p$ admits a unique representation

$$x = \sum_{n=0}^{\infty} a_n p^n \quad (a_n \in A).$$

**PROOF** Let $x \in \mathbb{Z}_p$ be given. Choose uniquely $a_0 \in A$ such that $|x - a_0|_p < 1$,

hence $x = a_0 + px_1$ for some $x_1 \in \mathbb{Z}_p$. Choose uniquely $a_1 \in A$ such that $|x_1 - a_1|_p < 1$,

hence $x_1 = a_1 + px_2$ for some $x_2 \in \mathbb{Z}_p$. Continuing:

$$x = a_0 + a_1 p + \cdots + a_N p^N + x_{N+1} p^{N+1},$$

where $a_n \in A$ and $x_{N+1} \in \mathbb{Z}_p$. But

$$x_{N+1} p^{N+1} \to 0.$$

21: **APPLICATION** $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.

22: **EXAMPLE** Let $x \in \mathbb{Z}_p$ -- then $\forall n \in \mathbb{N}$,

$$x \left( \begin{array}{c} x \frac{1}{n!} \cdots \frac{x-n+1}{n!} \end{array} \right) = \frac{x(x-1) \cdots (x-n+1)}{n!} \in \mathbb{Z}_p.$$

23: **LEMMA**

$$\mathbb{Z}_p^x = \bigcup_{1 \leq k \leq p-1} (k + p\mathbb{Z}_p).$$
Consequently, if

\[ x = \sum_{n=0}^{\infty} a_n p^n \quad (a_n \in \Lambda) \]

and if \( x \in \mathbb{Z}_p^x \), then \( a_0 \neq 0 \).

[In fact, there is a unique \( k \) (1 \( \leq k \leq p-1 \)) such that \( x \in k + p\mathbb{Z}_p \) and this "k" is \( a_0 \).]

24: THEOREM An element

\[ x = \sum_{n=0}^{\infty} a_n p^n \quad (a_n \in \Lambda) \]

in \( \mathbb{Z}_p \) is a unit iff \( a_0 \neq 0 \).

PROOF To establish the characterization, construct a multiplicative inverse \( y \) for \( x \) as follows. First choose uniquely \( b_0 \) (1 \( \leq b_0 \leq p-1 \)) such that \( a_0 b_0 \equiv 1 \mod p \). Proceed from here by recursion and assume that \( b_1, \ldots, b_M \) between 0 and \( p-1 \) have already been found subject to

\[ x(\sum_{0 \leq m \leq M} b_m p^m) \equiv 1 \mod p^{M+1}. \]

Then there is exactly one \( 0 \leq b_{M+1} \leq p - 1 \) such that

\[ x(\sum_{0 \leq m \leq M+1} b_m p^m) \equiv 1 \mod p^{M+2}. \]

Now put \( y = \sum_{m=0}^{\infty} b_m p^m \), thus \( xy = 1 \).

25: EXAMPLE \( 1-p \) is invertible in \( \mathbb{Z}_p \) but \( p \) is not invertible in \( \mathbb{Z}_p \).
26: REMARK The arrow

$$\varepsilon: \mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z}$$

that sends

$$x = \sum_{n=0}^{\infty} a_n p^n \quad (a_n \in A)$$

to $$a_0 \mod p$$ is a homomorphism of rings called \textit{reduction mod p}. It is surjective with kernel $$p\mathbb{Z}_p$$, hence $$[\mathbb{Z}_p:p\mathbb{Z}_p] = p$$.

Consider now the topological aspects of $$\mathbb{Z}_p$$:

- $$\mathbb{Z}_p$$ is totally disconnected.
- $$\mathbb{Z}_p$$ is closed, hence complete.
- $$\mathbb{Z}_p$$ is open.

[As regards the last point, observe that

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p < r\}$$

$$\equiv N_r(0) \quad (1 < r < p).$$]

27: THEOREM $$\mathbb{Z}_p$$ is compact.

PROOF Since $$\mathbb{Z}_p$$ is a metric space, it suffices to show that $$\mathbb{Z}_p$$ is sequentially compact. So let $$x_1, x_2, \ldots$$ be an infinite sequence in $$\mathbb{Z}_p$$. Choose $$a_0 \in A$$ such that $$a_0 + p\mathbb{Z}_p$$ contains infinitely many of the $$x_n$$. Write

$$a_0 + p\mathbb{Z}_p$$

$$= a_0 + p \left( \bigcup_{a \in A} (a + p\mathbb{Z}_p) \right)$$
Choose $a_1 \in A$ such that $a_0 + a_1 p + p^2 \mathbb{Z}_p$ contains infinitely many of the $x_n$. ETC.

The construction thus produces a descending sequence of cosets of the form

$$A_j + p^j \mathbb{Z}_p,$$

each of which contains infinitely many of the $x_n$. But

$$A_j + p^j \mathbb{Z}_p = \{ x \in \mathbb{Z}_p : |x - A_j|_p \leq p^{-j} \}$$

$$= B_{p^{-j}}(A_j),$$

a closed ball in the $p$-adic metric of radius $p^{-j} \to 0$ ($j \to \infty$), hence by the completeness of $\mathbb{Z}_p$,

$$\bigcap_{j=1}^{\infty} B_{p^{-j}}(A_j) = \{ A \}.$$  

Finally, choose

$$x_{n_1} \in B_{p^{-1}}(A_1), x_{n_2} \in B_{p^{-2}}(A_2), \ldots.$$  

Then

$$\lim_{j \to \infty} x_{n_j} = A.$$  

28: APPLICATION $\mathbb{Q}_p$ is locally compact.
[Since \(Q_p\) is Hausdorff, it is enough to prove that each \(x \in Q_p\) has a compact neighborhood. But \(Z_p\) is a compact neighborhood of 0, so \(x + Z_p\) is a compact neighborhood of \(x\).]

The set \(p^{-n}Z_p\) \((n \geq 0)\) is the set of all \(x \in Q_p\) such that \(|x|_p \leq p^n\). Therefore

\[
Q_p = \bigcup_{n=0}^{\infty} p^{-n}Z_p.
\]

Accordingly, \(Q_p\) is \(\sigma\)-compact (the \(p^{-n}Z_p\) being compact).

29: SCHOLIUM A subset of \(Q_p\) is compact iff it is closed and bounded.

30: LEMMA Given \(n,m \in \mathbb{Z}\),

\[
p^nZ_p \subseteq p^mZ_p \iff m \leq n.
\]

31: REMARK Take \(n \geq 1\) -- then the \(p^nZ_p\) are principal ideals in \(Z_p\) and, apart from \(\{0\}\), these are the only ideals in \(Z_p\), thus \(Z_p\) is a principal ideal domain.

32: LEMMA For every \(x_0 \in Q_p\) and \(r > 0\), there is an integer \(n\) such that

\[
N_r(x_0) = \{x \in Q_p : |x - x_0|_p < r\}
= N_{p^{-n}}(x_0) = \{x \in Q_p : |x - x_0|_p < p^{-n}\}
= x_0 + p^{n+1}Z_p.
\]

33: SCHOLIUM The basic open sets in \(Q_p\) are the cosets of some power of \(pZ_p\).
[Note: It is a corollary that every nonempty open subset of $\mathbb{Q}_p$ can be written as a disjoint union of cosets of the $p^n\mathbb{Z}_p$ ($n \in \mathbb{Z}$).]

34: **Lemma**

$$p^n\mathbb{Z}_p = p^n\mathbb{Z}_p - p^{n+1}\mathbb{Z}_p.$$ 

35: **Definition** The $p^n\mathbb{Z}_p$ are called shells.

36: **N.B.** There is a disjoint decomposition

$$\mathbb{Q}_p^x = \bigcup_{n \in \mathbb{Z}} p^n\mathbb{Z}_p,$$

where

$$p^n\mathbb{Z}_p = \bigcup_{1 \leq k \leq p-1} (p^n \mathbb{Z}_p + p^{n+1}\mathbb{Z}_p).$$

[Note: For the record, $\mathbb{Q}_p^x$ is totally disconnected and, being open in $\mathbb{Q}_p$, is Hausdorff and locally compact. Moreover, $\mathbb{Z}_p^x$ is open-closed (indeed, open-compact).]

Let $x \in \mathbb{Q}_p^x$ — then there is a unique $v(x) \in \mathbb{Z}$ and a unique $u(x) \in \mathbb{Z}_p^x$ such that $x = p^v(x) u(x)$. Consequently,

$$\mathbb{Q}_p^x \simeq \langle p \rangle \times \mathbb{Z}_p^x$$

or still,

$$\mathbb{Q}_p^x \simeq \mathbb{Z} \times \mathbb{Z}_p^x.$$ 

37: **Notation** For $n = 1, 2, \ldots$, put

$$U_{p,n} = 1 + p^n\mathbb{Z}_p.$$
12.

[Note:]

\[1 + p^n Z_p = \{x \in Z_p^\times : |1 - x|_p < p^{-n}\}.\]

The \(U_{p,n}\) are open-compact subgroups of \(Z_p^\times\) and

\[Z_p^\times \supset U_{p,1} \supset U_{p,2} \supset \cdots .\]

38: **Lemma** The collection \(\{U_{p,n} : n \in \mathbb{N}\}\) is a neighborhood basis at \(1\).

39: **Definition** \(U_{p,1} = 1 + pZ_p\) is called the group of **principal units** of \(Z_p\).

40: **Lemma** The quotient \(Z_p^\times / U_{p,1}\) is isomorphic to \(F_p^\times\) and the index of \(U_{p,1}\) in \(Z_p^\times\) is \(p - 1\).

A generator of \(F_p^\times\) can be "lifted" to \(Z_p^\times\).

41: **Theorem** There exists a \(\zeta \in Z_p^\times\) such that \(\zeta^{p-1} = 1\) and \(\zeta^k \neq 1\) \((0 < k < p-1)\).

[This is a straightforward application of Hensel's lemma.]

42: **N.B.** \(\zeta \notin U_{p,1}\) \((p\ \text{odd})\).

[If \(x \in Z_p\) and if for some \(n \geq 1\),

\[(1 + px)^n = 1,\]

then using the binomial theorem one finds that \(x = 0\). This said, suppose that]
$\zeta \in U_{p,1}$:

$$\zeta = 1 + pu(u \in \mathbb{Z}_p) \Rightarrow (1 + pu)^{p-1} = 1 \Rightarrow u = 0,$$

a contradiction.

43: SCHOLIUM $\mathbb{Z}_p$ can be written as a disjoint union

$$\mathbb{Z}_p^x = U_{p,1} \cup \zeta U_{p,1} \cup \zeta^2 U_{p,1} \cup \cdots \cup \zeta^{p-2} U_{p,1}. $$

Therefore

$$\mathbb{Q}_p^x \approx \mathbb{Z} \times \mathbb{Z}_p^x \approx \mathbb{Z} \times \mathbb{Z}/(p-1) \times U_{p,1}. $$

44: LEMMA Any root of unity in $\mathbb{Q}_p$ lies in $\mathbb{Z}_p^x$.

PROOF If $x = p^y(x)u(x)$ and if $x^n = 1$, then $nv(x) = 0$, so $v(x) = 0$, thus

$x \in \mathbb{Z}_p^x$.

The roots of unity in $\mathbb{Z}_p^x$ are a subgroup (as in any abelian group), call it $T_p$. If, on the other hand, $G_{p-1}$ is the cyclic subgroup of $\mathbb{Z}_p^x$ generated by $\zeta$,

then $G_{p-1}$ consists of $(p-1)^{st}$ roots of unity, hence $G_{p-1} \subset T_p$.

45: LEMMA If $p \neq 2$, then $G_{p-1} = T_p$ but if $p = 2$, then $T_p = \{ \pm 1 \}$.

46: APPLICATION If $p_1, p_2$ are distinct primes, then $\mathbb{Q}_{p_1}$ is not field isomorphic to $\mathbb{Q}_{p_2}$.
47: **REMARK** \( \mathbb{Q}_p \) is not field isomorphic to \( \mathbb{R} \).

[\( \mathbb{Q}_p \) has algebraic extensions of arbitrarily large linear degree which is not the case of \( \mathbb{R} \) (cf. §5, #26).]

48: **LEMMA** Let \( x \in \mathbb{Q}_p^\times \) -- then \( x \in \mathbb{Z}_p^\times \) iff \( x^{p-1} \) possesses \( n \)th roots for infinitely many \( n \).

**PROOF** If \( x \in \mathbb{Z}_p^\times \) and if \( n \) is not a multiple of \( p \), then one can use Hensel's lemma to infer the existence of a \( y_n \in \mathbb{Z}_p \) such that \( y_n^n = x^{p-1} \). Conversely, if \( y_n^n = x^{p-1} \), then

\[
v(y_n) = (p-1)v(x),
\]

thus \( n \) divides \( (p-1)v(x) \). But this can happen for infinitely many \( n \) only if \( v(x) = 0 \), implying thereby that \( x \) is a unit.

49: **APPLICATION** Let \( \phi: \mathbb{Q}_p \to \mathbb{Q}_p \) be a field automorphism -- then \( \phi \) preserves units.

[In fact, if \( x \in \mathbb{Z}_p^\times \), then

\[
y_n^n = x^{p-1} \Rightarrow \phi(y_n)^n = (\phi(x))^{p-1}.
\]

50: **THEOREM** The only field automorphism \( \phi \) of \( \mathbb{Q}_p \) is the identity.

**PROOF** Given \( x \in \mathbb{Q}_p^\times \), write \( x = p^{v(x)}u(x) \), hence

\[
\phi(x) = \phi(p^{v(x)}u(x))
\]

\[
= \phi(p^{v(x)}\phi(u(x))) = p^{v(x)}\phi(u(x)),
\]

hence

\[
v(\phi(x)) = v(x) \quad (\phi(u(x)) \in \mathbb{Z}_p^\times).
\]
Therefore $\phi$ is continuous. Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, it then follows that $\phi = \text{id}_{\mathbb{Q}_p}$.

[Note:

$x_k \to 0 \Rightarrow |x_k|_p \to 0 \Rightarrow p^{-v(x_k)} \to 0$

$\Rightarrow p^{-v(\phi(x_k))} \to 0 \Rightarrow |\phi(x_k)|_p \to 0 \Rightarrow \phi(x_k) \to 0$.]

The final structural item to be considered is that of quadratic extensions and to this end it is necessary to explicate $(\mathbb{Q}_p^x)^2$, bearing in mind that

$$\mathbb{Q}_p^x \approx \mathbb{Z} \times \mathbb{Z}_p^x \approx \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \times U_{p,1}.$$  

51: **Lemma** If $p \neq 2$, then $U_{2,1}^2 = U_{p,1}$ but if $p = 2$, then $U_{2,1}^2 = U_{2,3}$.

52: **Application** If $p \neq 2$, then

$$(\mathbb{Q}_p^x)^2 \approx 2\mathbb{Z} \times 2(\mathbb{Z}/(p-1)\mathbb{Z}) \times U_{p,1}$$

but if $p = 2$, then

$$(\mathbb{Q}_2^x)^2 \approx 2\mathbb{Z} \times U_{2,3}.$$  

53: **Theorem** If $p \neq 2$, then

$$[\mathbb{Q}_p^x : (\mathbb{Q}_p^x)^2] = 4$$

but if $p = 2$, then

$$[\mathbb{Q}_2^x : (\mathbb{Q}_2^x)^2] = 8.$$  

54: **Remark** If $p \neq 2$, then

$$(\mathbb{Q}_p^x / (\mathbb{Q}_p^x)^2) \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

if $p = 2$. If $p = 2$, then

$$(\mathbb{Q}_2^x / (\mathbb{Q}_2^x)^2) \approx \mathbb{Z}/2\mathbb{Z}.$$
but if $p = 2$, then

$$\frac{Q^x_p}{(Q^x_p)^2} \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$ 

55: CRITERION Suppose that $p \neq 2$.

- $p$ is not a square.

[If $p = x^2$, write $x = p^{v(x)} u(x)$ to get

$$1 = v(p) = v(x^2) = 2v(x),$$

an untenable relation.]

- $\zeta$ is not a square.

[Assume that $\zeta = x^2$ — then

$$\zeta^{p-1} = 1 \Rightarrow x^{2(p-1)} = 1,$$

thus $x$ is a root of unity, thus $x \in T_p$, thus $x \in G_{p-1}$ (cf. #45), thus $x = \zeta^k$ ($0 < k < p-1$), thus $\zeta = (\zeta^k)^2 = \zeta^{2k}$, thus $1 = \zeta^{2k-1}$. But

$$2k < 2p-2 \Rightarrow 2k-1 < 2p-1.$$ 

And

$$2k - 1 = p - 1 \Rightarrow 2k = p \Rightarrow p \text{ even}...$$

$$2k - 1 = 2p - 2 \Rightarrow 2k - 1 = 2(p-1) \Rightarrow 2k - 1 \text{ even}... .$$]

- $p\zeta$ is not a square.

[For if $p\zeta = p^{2n+2} u^2$ ($n \in \mathbb{Z}$), then

$$\zeta = p^{2n-1} u^2 \Rightarrow 1 = |\zeta|_p = |p^{2n-1}|_p = p^{1-2n}$$

$$\Rightarrow 1 - 2n = 0,$$

an untenable relation.]
56: THEOREM If \( p \neq 2 \), then up to isomorphism, \( \mathbb{Q}_p \) has three quadratic extensions, viz.

\[ \mathbb{Q}_p(\sqrt{p}), \mathbb{Q}_p(\sqrt{\zeta}), \mathbb{Q}_p(\sqrt{\zeta^2}). \]

[Note: If \( \tau_1 = p \), \( \tau_2 = \zeta \), \( \tau_3 = p\zeta \), then these extensions of \( \mathbb{Q}_p \) are inequivalent since \( \tau_i \tau_j^{-1} \) (\( i \neq j \)) is not a square in \( \mathbb{Q}_p \).]

57: REMARK Another choice for the three quadratic extensions of \( \mathbb{Q}_p \) when \( p \neq 2 \) is

\[ \mathbb{Q}_p(\sqrt{p}), \mathbb{Q}_p(\sqrt{a}), \mathbb{Q}_p(\sqrt{pa}), \]

where \( 1 < a < p \) is an integer that is not a square mod \( p \).

58: REMARK It can be shown that up to isomorphism, \( \mathbb{Q}_2 \) has seven quadratic extensions, viz

\[ \mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{5}), \mathbb{Q}_2(\sqrt{10}). \]

59: EXAMPLE Take \( p = 5 \) -- then \( 2 \notin (\mathbb{Q}_5^x)^2 \), \( 3 \notin (\mathbb{Q}_5^x)^2 \) but \( 6 \in (\mathbb{Q}_5^x)^2 \). And

\[ \mathbb{Q}_5(\sqrt{2}) = \mathbb{Q}_5(\sqrt{3}). \]

[Working within \( \mathbb{Z}_5^x \), consider the equation \( x^2 = 2 \) and expand \( x \) as usual:

\[ x = \sum_{n=0}^{\infty} a_n 5^n \quad (a_n \in A). \]

Then

\[ a_0^2 \equiv 2 \text{ mod } 5. \]

But the possible values of \( a_0 \) are 0, 1, 2, 3, 4, thus the congruence is impossible,
so \( 2 \not\in (Q_3^x)^2 \). Analogously, \( 3 \not\in (Q_5^x)^2 \). On the other hand, \( 6 \in (Q_5^x)^2 \) (by direct verification or Hensel's lemma), hence \( 6 = \gamma^2 \) (\( \gamma \in Q_5 \)). Finally, to see that 

\[ Q_5(\sqrt{2}) = Q_5(\sqrt{3}), \]

it need only be shown that \( \sqrt{2} = a + b \sqrt{3} \) for certain \( a, b \in Q_5 \). To this end, note that \( \sqrt{2} / \sqrt{3} = \pm \gamma \), from which

\[ \sqrt{2} = \pm \frac{\gamma}{\sqrt{3}} = \pm \frac{\gamma}{3} \sqrt{3}. \]

60: EXAMPLE If \( p \) is odd, then \( p - 1 \) is even and \(-1 \in G_{p-1}\). In addition, \( -1 \in (Q_p^x)^2 \) iff \((p-1)/2\) is even, i.e., iff \( p \equiv 1 \mod 4 \). Accordingly, to start \( \sqrt{-1} \) exists in \( Q_5, Q_{13}, \ldots \).

[Note: \( \sqrt{-1} \) does not exist in \( Q_2 \).]

APPENDIX

Let \( Q_p^{cl} \) be the algebraic closure of \( Q_p \) — then \( |.|_p \) extends uniquely to \( Q_p^{cl} \) (cf. §3, #12) (and satisfies the ultrametric inequality). Furthermore, the range of \( |.|_p \) per \( Q_p^{cl} \) is the set of all rational powers of \( p \) (plus 0).

1: THEOREM \( Q_p^{cl} \) is not second category.

2: APPLICATION The metric space \( Q_p^{cl} \) is not complete.

3: APPLICATION The Hausdorff space \( Q_p^{cl} \) is not locally compact (cf. §5, #5).
4: NOTATION Put

\[ C_p = \overline{\mathbb{Q}_p^{\text{cl}}} \]

the completion of \( \mathbb{Q}_p^{\text{cl}} \) per \( | \cdot |_p \).

5: THEOREM \( C_p \) is algebraically closed.

6: N.B. The metric space \( C_p \) is separable but the Hausdorff space \( C_p \) is not locally compact (cf. §5, #5).
§5. LOCAL FIELDS

Let $K$ be a field of characteristic 0 equipped with a non-archimedean absolute value $|\cdot|$.

1: NOTATION Let

$$
R = \{a \in K : |a| \leq 1\}
$$

$$
R^\times = \{a \in K : |a| = 1\}.
$$

2: LEMMA $R$ is a commutative ring with unit and $R^\times$ is its multiplicative group of invertible elements.

3: NOTATION Let

$$
P = \{a \in K : |a| < 1\}.
$$

4: LEMMA $P$ is a maximal ideal.

Therefore the quotient $R/P$ is a field, the **residue field** of $K$.

5: THEOREM $K$ is locally compact iff the following conditions are satisfied.

1. $K$ is a complete metric space.
2. $R/P$ is a finite field.
3. $|K^\times|$ is a nontrivial discrete subgroup of $R_{>0}$.

6: DEFINITION A **local field** is a locally compact field of characteristic 0.

7. EXAMPLE $\mathbb{R}$ and $\mathbb{C}$ are local fields.

8. EXAMPLE $\mathbb{Q}_p$ is a local field.
2.

Assume that $K$ is a non-archimedean local field.

9: **Lemma** $R$ is compact.

10: **Lemma** $P$ is principal, say $P = \pi R$, and

$$|K^x| = |\pi|^Z,$$

where $0 < |\pi| < 1$.

[Note: Such a $\pi$ is said to be a prime element.]

11: **Remark** A nontrivial discrete subgroup $\Gamma$ of $R_{>0}$ is free on one generator $0 < \gamma < 1$:

$$\Gamma = \{\gamma^n : n \in \mathbb{Z}\}.$$

This said, choose $\pi$ with the largest absolute value $< 1$, thus $\pi \in P \subset R \Rightarrow \pi R \subset P$.

In the other direction,

$$a \in P \Rightarrow |a| \leq |\pi| \Rightarrow \frac{a}{\pi} \in R.$$  

And

$$a = \pi \cdot \frac{a}{\pi} \Rightarrow a \in \pi R.$$

12: **Fact** A locally compact topological vector space over a local field is necessarily finite dimensional.

13: **Theorem** $K$ is a finite extension of $Q_p$ for some $p$.

**Proof** First, $K \supset Q$ (since char $K = 0$). Second, the restriction of $|.|$ to $Q$ is equivalent to $|.|_p$ ($\exists p$) (cf. §1, #20), hence the closure of $Q$ in $K$ "is" $Q_p$ (since $K$ is complete). Third, $K$ is finite dimensional over $Q_p$ (since $K$ is locally compact).
There is also a converse.

14: THEOREM Let $K$ be a finite extension of $\mathbb{Q}_p$ -- then $K$ is a local field.

PROOF In view of 5, it suffices to equip $K$ with a non-archimedean absolute value subject to conditions 1,2,3. But, by the extension principle (cf. §3, #11), $|\cdot|_p$ extends uniquely to $K$. This extension is non-archimedean and points 1,3 are manifest. As for point 2, it suffices to observe that the canonical arrow $\mathbb{Z}_p/p\mathbb{Z}_p \to R/P$ is injective and

$$[R/P:F_p] \leq [K:Q_p] < \infty.$$ 

[Details: To begin with, $\mathbb{Q}_p \cap P = p\mathbb{Z}_p$, thus the inclusion $\mathbb{Z}_p \to R$ induces an injection

$$\mathbb{Z}_p/p\mathbb{Z}_p \to R/P.$$ 

Put now $n = [K:Q_p]$ and let $A_1, \ldots, A_{n+1} \in R$ -- then the claim is that the residue classes $\bar{A}_1, \ldots, \bar{A}_{n+1} \in R/P$ are linearly dependent over $\mathbb{Z}_p/p\mathbb{Z}_p$. In any event, there are elements $x_1, \ldots, x_{n+1} \in \mathbb{Q}_p$ such that

$$\sum_{i=1}^{n+1} x_i A_i = 0,$$

matters being arranged in such a way that

$$\max |x_i|_p = 1.$$ 

Therefore the $x_i \in \mathbb{Z}_p$ and not every residue class $\bar{x}_i \in \mathbb{Z}_p/p\mathbb{Z}_p$ is zero. But then
4.

\[
\sum_{i=1}^{n+1} x_i \bar{A}_i = 0
\]

is a nontrivial dependence relation.]

15: SCHOLIUM A non-archimedean field of characteristic zero is a local field iff it is a finite extension of \( \mathbb{Q}_p \) (\( \exists p \)).

Let \( K \supset \mathbb{Q}_p \) be a finite extension of linear degree \( n \) -- then the canonical absolute value on \( K \) is given by

\[
|a|_p = \left| N_{K/\mathbb{Q}_p}(a) \right|_p^{1/n}.
\]

[Note: The normalized absolute value on \( K \) is given by

\[
|a|_K = |a|_p^n.
\]

Its intrinsic significance will emerge in due course but for now observe that \( |.|_K \) is equivalent to \( |.|_p \) and is non-archimedean (cf. §1, #23).]

16: LEMMA The range of \( |.|_p|K^\times \) is \( |\pi|^\mathbb{Z}_p \).

17: DEFINITION The ramification index of \( K \) over \( \mathbb{Q}_p \) is the positive integer

\[
e = [|K^\times|_p : |\mathbb{Q}_p^\times|_p].
\]

I.e.

\[
e = [|\pi|^\mathbb{Z}_p : |p|^\mathbb{Z}_p].
\]

Therefore

\[
|\pi|_p^e = |p|_p = \left( \frac{1}{p} \right).
\]
5.

[Consider $\mathbb{Z}$ and $\mathbb{Z}^{e}$ -- then the generator $1$ of $\mathbb{Z}$ is related to the generator $e$ of $\mathbb{Z}^{e}$ by the triviality $1 + \cdots + 1 = e \cdot 1 = e$.]

18: N.B. If $\pi' \equiv \pi^{v(\pi)} u$, then $\pi'$ is a prime element.

[Using obvious notation, write $\pi' = \pi^{v(\pi)} u$, thus

\[|\pi'|_{p} = |\pi'|_{p}^{e} = (|\pi'|_{p}^{v(\pi)})^{e} = (|\pi'|_{p}^{v(\pi)}) = |p|^{v(\pi)},\]

thus $v(\pi) = 1$.]

19: NOTATION

\[q \equiv \text{card } R/P = (\text{card } F_{p})^{f} = p^{f},\]

so

\[f = [R/P:F_{p}],\]

the residual index of $K$ over $Q_{p}$.

20: THEOREM Let $K \supset Q_{p}$ be a finite extension of linear degree $n$ -- then

\[n = [K:Q_{p}] = ef.\]

21: APPLICATION

\[|\pi|_{K} = |\pi|^{n} = |p|^{n/e} = \left(\frac{1}{p}\right)^{n/e} = \left(\frac{1}{p}\right)^{f} = \frac{1}{q}.\]
View \( p \) as an element of \( K \):

- \( |p|_p = |N_{K/Q}(p)|_p^{1/n} = |p^n|_p^{1/n} = |p|_p \).
- \( |p|_K = |N_{K/Q}(p)|_p = |p^n|_p = \frac{1}{p^n \text{ef}} = (\frac{1}{p^e})^e = q^{-e} \).

22: DEFINITION A finite extension \( K \) of \( \mathbb{Q}_p \) is

- \textbf{unramified} if \( e = 1 \)
- \textbf{ramified} if \( f = 1 \).

Take the case \( K = \mathbb{Q}_p \) -- then \( e = 1 \), hence \( K \) is unramified, and \( f = 1 \), hence \( K \) is ramified.

23: LEMMA If \( K \supset \mathbb{Q}_p \) is unramified, then \( p \) is a prime element.

24: THEOREM \( \forall n = 1, 2, \ldots \), there is up to isomorphism one unramified extension \( K \) of \( \mathbb{Q}_p \) of linear degree \( n \).

Let \( K \) be a finite extension of \( \mathbb{Q}_p \).

25: LEMMA The group \( M^x \) of roots of unity of order prime to \( p \) in \( K \) is cyclic of order \( p^f - 1 \) (\( = q-1 \)).

26: LEMMA The set \( M = M^x \cup \{0\} \) is a set of coset representatives for \( R/P \).

Therefore (cf. §4, #43)

\[
K^x \cong \mathbb{Z} \times R^x \cong \mathbb{Z} \times \mathbb{Z}/(q-1) \mathbb{Z} \times 1 + P.
\]
27: NOTATION Let
\[ K_{ur} = Q_p(M^\times). \]

28: LEMMA \( K_{ur} \) is the maximal unramified extension of \( Q_p \) in \( K \) and
\[ [K_{ur}:Q_p] = f. \]

29: REMARK The maximal unramified extension \( (Q_p^c)^{ur} \subset Q_p^c \) is the field extension generated by all roots of unity of order prime to \( p \).

30: QUADRATIC EXTENSIONS (cf. §4, #56) Suppose that \( p \neq 2 \), let
\[ \tau \in Q_p^\times - (Q_p^\times)^2, \]
and form the quadratic extension
\[ Q_p(\tau) = \{ x + y\sqrt{\tau} : x, y \in Q_p \}. \]
Then the canonical absolute value on \( Q_p(\sqrt{\tau}) \) is given by
\[ |x + y\sqrt{\tau}|_p = \left| N_{Q_p(\sqrt{\tau})/Q_p}(x + y\sqrt{\tau}) \right|_p^{1/2} \]
\[ = |x^2 - \tau y^2|_p^{1/2}. \]

31: CLASSIFICATION Consider the three possibilities
\[ Q_p(\sqrt{p}), Q_p(\sqrt{\tau}), Q_p(\sqrt[p]{\tau}), \]
thus here \( 2 = ef \).
- \( Q_p(\sqrt{p}) \) is ramified or still, \( e = 2 \).

[Note that
\[ |\sqrt{p}|_p^2 = |0^2 - (p)1^2|_p = |p|_p = \frac{1}{p}. \]
• \( \mathbb{Q}_p(\sqrt{\zeta}) \) is ramified or still, \( e = 2 \).

[Note that
\[
|\sqrt{\zeta}|^2 = |0^2 - (\zeta)\overline{\zeta}|^2 = |\zeta|_p^2 = |p|_p \cdot |\zeta|_p = |p|_p = \frac{1}{p^2}.
\]

If \( e = 1 \), then in either case, the value group would be \( p^Z \), an impossibility since \( \frac{1}{\sqrt{p}} \not\in p^Z \), so \( e = 2 \).

• \( \mathbb{Q}_p(\sqrt{\zeta}) \) is unramified or still, \( e = 1 \).

[There is up to isomorphism one unramified extension \( K \) of \( \mathbb{Q}_p \) of linear degree 2 (cf. #24).]

[Instead of quoting theory, one can also proceed directly, it being simplest to work instead with \( \mathbb{Q}_p(\sqrt{a}) \), where \( 1 < a < p \) is an integer that is not a square mod \( p \) (cf. §4, #57) -- then the residue field of \( \mathbb{Q}_p(\sqrt{a}) \) is \( F_p(\sqrt{a}) \), hence \( f = 2 \), hence \( e = 1 \) (since \( n = 2 \)).]

The preceding developments are absolute, i.e., based at \( \mathbb{Q}_p \). It is also possible to relativize the theory. Thus let \( L \supset K \supset \mathbb{Q}_p \) be finite extensions of \( \mathbb{Q}_p \). Append subscripts to the various quantities involved:

\[
\begin{align*}
& R_K \supset P_K, \ P_K^e, \ M_K^e \\
& R_L \supset P_L, \ P_L^e, \ M_L^e.
\end{align*}
\]

Introduce

\[
\begin{align*}
e(L/K) &= [|L^e| : |K^e|] \\
f(L/K) &= [R_L/P_L : R_K/P_K].
\end{align*}
\]
32: LEMMA

\[ [L:K] = e(L/K)f(L/K). \]

PROOF We have

\[ [L:Q_p] = e_{L/L}, \]

(cf. #20)

\[ [K:Q_p] = e_{K/K}. \]

Therefore

\[ [L:K] = \frac{[L:Q_p]}{[K:Q_p]} = \frac{e_{L/L}}{e_{K/K}} = e(L/K)f(L/K). \]

33: THEOREM Let \( L \supset K \supset Q_p \) be finite extensions of \( Q_p \) — then there exists a unique maximal intermediate extension \( K \subset K_{ur} \subset L \) that is unramified over \( K \).

[In fact,

\[ K_{ur} = K(M_{L}^{\times}) \subset L. \]

[Note: The extension \( L \supset K_{ur} \) is ramified.]
§6. HAAR MEASURE

Let X be a locally compact Hausdorff space.

1. DEFINITION A Radon measure is a measure $\mu$ defined on the Borel $\sigma$-algebra of X subject to the following conditions.

1. $\mu$ is finite on compacta, i.e., for every compact set $K \subset X$, $\mu(K) < \infty$.
2. $\mu$ is outer regular, i.e., for every Borel set $A \subset X$,
   $$\mu(A) = \inf_{U \supseteq A} \mu(U),$$
   where $U \subset X$ is open.
3. $\mu$ is inner regular, i.e., for every open set $A \subset X$,
   $$\mu(A) = \sup_{K \subseteq A} \mu(K),$$
   where $K \subset X$ is compact.

Let $G$ be a locally compact abelian group.

2. DEFINITION A Haar measure on $G$ is a Radon measure $\mu_G$ which is translation invariant: $\forall$ Borel set $A$, $\forall x \in G$,
   $$\mu_G(x+A) = \mu_G(A) = \mu_G(A+x),$$
   or still, $\forall f \in C_c(G)$, $\forall y \in G$,
   $$\int_G f(x+y)d\mu_G(x) = \int_G f(x)d\mu_G(x).$$

3. THEOREM $G$ admits a Haar measure and any two Haar measures $\mu_G$, $\nu_G$ differ by a positive constant: $\mu_G = c\nu_G$ ($c > 0$).
4: **Lemma** Every open subset of $G$ has positive Haar measure.

5: **Lemma** $G$ is compact iff $G$ has finite Haar measure.

6: **Lemma** $G$ is discrete iff every point of $G$ has positive Haar measure.

7: **Example** Take $G = \mathbb{R}$ -- then $\mu_{\mathbb{R}} = dx$ (Lebesgue measure) is a Haar measure ($\mu_{\mathbb{R}}([0,1]) = \int_0^1 dx = 1$).

8: **Example** Take $G = \mathbb{R}^\times$ -- then $\mu_{\mathbb{R}^\times} = \frac{dx}{|x|}$ (Lebesgue measure) is a Haar measure ($\mu_{\mathbb{R}^\times}([1,e]) = \int_1^e \frac{dx}{|x|} = 1$).

9: **Example** Take $G = \mathbb{Z}$ -- then $\mu_{\mathbb{Z}}$ = counting measure is a Haar measure.

10: **Lemma** Let $G'$ be a closed subgroup of $G$ and put $G'' = G/G'$. Fix Haar measures $\mu_G$, $\mu_{G'}$ on $G$, $G'$ respectively -- then there is a unique determination of the Haar measure $\mu_{G''}$ on $G''$ such that $\forall f \in C_c(G)$,

\[ \int_{G''} f(x) \mu_{G''}(x) = \int_{G'} (\int_{G'} f(x+x') \mu_{G'}(x')) \mu_{G''}(x'). \]

[Note: The function

\[ x \mapsto \int_{G'} f(x+x') \mu_{G'}(x') \]

is $G'$-invariant, hence is a function on $G''$.]

11: **Example** Take $G = \mathbb{R}$, $G' = \mathbb{Z}$ with the usual choice of Haar measures. Determine $\mu_{\mathbb{R}/\mathbb{Z}}$ per #10 -- then $\mu_{\mathbb{R}/\mathbb{Z}}(\mathbb{R}/\mathbb{Z}) = 1$. 


Let \( \chi \) be the characteristic function of \([0,1]\) — then

\[
\sum_{n \in \mathbb{Z}} \chi(x+n)
\]

is \( \equiv 1 \), hence when integrated over \( \mathbb{R}/\mathbb{Z} \) gives the volume of \( \mathbb{R}/\mathbb{Z} \). On the other hand, \( \int_{\mathbb{R}} \chi = 1 \).

Let \( K \) be a local field (cf. §5, #6). Given \( a \in K \), let \( M_a : K \to K \) be the automorphism that sends \( x \) to \( ax = xa \) — then for any Haar measure \( \mu_K \) on \( K \), the composite \( \mu_K \circ M_a \) is again a Haar measure on \( K \), hence there exists a positive constant \( \text{mod}_K(a) \) such that for every Borel set \( A \),

\[
\mu_K(M_a(A)) = \text{mod}_K(a) \mu_K(A)
\]

or still, \( \forall f \in C_c(K) \),

\[
\int_K f(a^{-1}x) d\mu_K(x) = \text{mod}_K(a) \int_K f(x) d\mu_K(x).
\]

[Note: \( \text{mod}_K(a) \) is independent of the choice of \( \mu_K \).]

Extend \( \text{mod}_K \) to all of \( K \) by setting \( \text{mod}_K(0) \) equal to 0.

12: LEMMA Let \( K, L \) be local fields, where \( L \supset K \) is a finite field extension — then \( \forall x \in L \),

\[
\text{mod}_L(x) = \text{mod}_K(N_{L/K}(x))
\]

\( \equiv \text{mod}_K(\det(M_x)) \).

[Let \( n = [L:K] \), view \( L \) as a vector space of dimension \( n \), and identify \( L \) with \( K^n \) by choosing a basis. Proceed from here by breaking \( M_x \) into a product of \( n \).]
"elementary" transformations.]

13: EXAMPLE Take \( K = \mathbb{R} \), \( L = \mathbb{R} \) — then \( \forall \ a \in \mathbb{R} \),

\[
\text{mod}_R(a) = |a|.
\]

[\( \forall f \in C_c(\mathbb{R}) \),

\[
f(R f(a^{-1}x)dx = |a| \int_R f(x)dx.]
\]

14: EXAMPLE Take \( K = \mathbb{R} \), \( L = \mathbb{C} \) — then \( \forall \ z \in \mathbb{C} \),

\[
\text{mod}_C(z) = \text{mod}_R(N_c/R(z))
\]

\[
= |\bar{z}z| = |z|^2.
\]

15: LEMMA

\[
\text{mod}_p = |.|_p.
\]

To prove this, we need a preliminary.

16: LEMMA The arrow

\[
\varepsilon_k: \mathbb{Z}_p \rightarrow \mathbb{Z}/p^k\mathbb{Z}
\]

that sends

\[
x = \sum_{n=0}^{\infty} a_n p^n \ (a_n \in \mathbb{A})
\]

to

\[
\sum_{n=0}^{k-1} a_n p^n \mod p^k
\]

is a homomorphism of rings. It is surjective with kernel \( p^k\mathbb{Z}_p \), so \([\mathbb{Z}_p:p^k\mathbb{Z}_p] = p^k\)
(cf. §4, #26), thus there is a disjoint decomposition of $Z_p$:

$$Z_p = \bigcup_{j=1}^{p^k} (x_j + p^kZ_p).$$

Normalize the Haar measure on $Q_p$ by stipulating that

$$\mu_{Q_p}(Z_p) = 1.$$

[Note: In this connection, recall that $Z_p$ is an open-compact set.]

The claim now is that for every Borel set $A$,

$$\mu_{Q_p}(\mathcal{M}_x(A)) = |x|_p \mu_{Q_p}(A).$$

Since the Borel $\sigma$-algebra is generated by the open sets, it is enough to take $A$ open. But any open set can be written as a disjoint union of cosets of the subgroups $p^kZ_p$ (cf. §4, #33), hence, thanks to translation invariance, it suffices to deal with these alone:

$$\mu_{Q_p}(p^kZ_p) = \text{mod}_{Q_p}(p^k) \mu_{Q_p}(Z_p)$$

$$= \text{mod}_{Q_p}(p^k) = |p^k|_p.$$

1. $k \geq 0$:

$$1 = \mu_{Q_p}(Z_p) = \mu_{Q_p}\left( \bigcup_{j=1}^{p^k} (x_j + p^kZ_p) \right)$$

$$= p^k \mu_{Q_p}(p^kZ_p)$$

$$\Rightarrow$$

$$\mu_{Q_p}(p^kZ_p) = p^{-k} = |p^k|_p.$$
2. $k < 0$:

$$1 = \mu_{Q_p}(Z_p) = \mu_{Q_p}(p^{-k} p^k Z_p)$$

$$= \text{mod}_{Q_p}(p^{-k}) \mu_{Q_p}(p^k Z_p)$$

$$= |p^{-k}|_p \mu_{Q_p}(p^k Z_p)$$

$$\Rightarrow$$

$$\mu_{Q_p}(p^k Z_p) = |p^{-k} - 1|_p = |p^k|_p.$$  

17: SCHOLIUM If $K$ is a finite extension of $Q_p$, then $\forall \ a \in K$,

$$\text{mod}_K(a) = |N_{K/Q_p}(a)|_p,$$

the normalized absolute value on $K$ mentioned in §5:

$$\text{mod}_K(a) = |a|_K (= |a|^n|_p, \ n = [K:Q_p]).$$

18: CONVENTION Integration w.r.t. $\mu_{Q_p}$ will be denoted by $dx$:

$$\int_{Q_p} f(x) d\mu_{Q_p}(x) = \int_{Q_p} f(x) dx.$$  

[Note: Points are of Haar measure zero:

$$\{0\} = \cap_{k=1}^{\infty} p^k Z_p$$

$$\Rightarrow$$

$$\mu_{Q_p}(\{0\}) = \lim_{k \to \infty} \mu_{Q_p}(p^k Z_p)$$

$$= \lim_{k \to \infty} p^{-k} = 0.]$$
19: EXAMPLE

\[ Z_p^x = \bigcup_{1 \leq k \leq p-1} (k + pZ_p) \] (cf. §4, #23).

Therefore

\[ \text{vol}_{dx}(Z_p^x) = (p-1)\text{vol}_{dx}(pZ_p) = \frac{p-1}{p}. \]

20: EXAMPLE

\[ \text{vol}_{dx}(p^nZ_p^x) = \text{vol}_{dx}(p^nZ_p - p^{n+1}Z_p) \] (cf. §4, #34)

\[ = \text{vol}_{dx}(p^nZ_p) - \text{vol}_{dx}(p^{n+1}Z_p) \]

\[ = |p^n|_p \text{vol}_{dx}(Z_p) - |p^{n+1}|_p \text{vol}_{dx}(Z_p) \]

\[ = p^{-n} - p^{-n-1}. \]

21: EXAMPLE Write

\[ Z_p - \{0\} = \bigcup_{n \geq 0} p^nZ_p^x. \]

Then

\[ \int_{Z_p - \{0\}} \log |x|_p \, dx = \sum_{n=0}^{\infty} \int_{p^nZ_p} \log |x|_p \, dx \]

\[ = \sum_{n=0}^{\infty} \log p^{-n} \text{vol}_{dx}(p^nZ_p^x) \]

\[ = - \log p \sum_{n=0}^{\infty} n(p^{-n} - p^{-n-1}) \]
8.

\[ = - \log p \left( \sum_{n=0}^{\infty} \frac{n}{p^n} - \frac{1}{p} \sum_{n=0}^{\infty} \frac{n}{p^n} \right) \]

\[ = -(1 - \frac{1}{p}) \log p \sum_{n=0}^{\infty} \frac{n}{p^n} \]

\[ = -(1 - \frac{1}{p}) \log p \frac{p}{(p-1)^2} \]

\[ = - \frac{\log p}{p-1} . \]

22: EXAMPLE

\[ \int_{Z_p^x} \log |1 - x|_p \, dx = - \frac{\log p}{p-1} . \]

[Break \( Z_p^x \) up via the scheme

\( (Z_p^x:a_0 \neq 1)u (Z_p^x:a_0 = 1, a_1 \neq 0)u (Z_p^x:a_0 = 1, a_1 = 0, a_2 \neq 0)u \ldots . \)]

23: LEMMA The measure \( \frac{dx}{|x|_p} \) is a Haar measure on the multiplicative group \( Q_p^x \).

PROOF \( \forall y \in Q_p^x \),

\[ \int_{Q_p^x} f(y^{-1}x) \left( \frac{dx}{|x|_p} \right) \]

\[ = |y|^{-1}_p \int_{Q_p^x} f(y^{-1}x) \frac{1}{|y^{-1}x|_p} \, dx \]

\[ = |y|^{-1}_p \text{mod}_{Q_p^x}(y) \int_{Q_p^x} f(x) \left( \frac{dx}{|x|_p} \right) \]
9.

\[ = |y|^{-1} \frac{dx}{|x|} \int_{Q_p} f(x) \frac{dx}{|x|} \]

\[ = \int_{Q_p} f(x) \frac{dx}{|x|}. \]

24: **EXAMPLE**

\[ \text{vol} \int_{Q_p} (p^n z^x_p) = \text{vol} \int_{Q_p} (z^x_p) \]

\[ = \int_{z^x_p} \frac{dx}{|x|} = \int_{z^x_p} dx \]

\[ = \text{vol} \int_{z^x_p} = \frac{p^{-1}}{p}. \]

25: **DEFINITION** The normalized Haar measure on the multiplicative group \( Q^x_p \) is given by

\[ d^x x = \frac{p}{p-1} \frac{dx}{|x|}. \]

Accordingly,

\[ \text{vol} \int_{d^x x} (z^x_p) = 1, \]

this condition characterizing \( d^x x \).

26: **EXAMPLE** Let \( s \) be a complex variable with \( \text{Re}(s) > 1 \). Write

\[ Z_p - \{0\} = \bigcup_{n \geq 0} p^n z^x_p. \]
Then

$$\int_{\mathbb{Z}_p - \{0\}} |x|_p^s \, dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{1-p^{-s}}$$

the $p^{th}$ factor in the Euler product for the Riemann zeta function.

Let $K$ be a finite extension of $\mathbb{Q}_p$. Given a Haar measure $da$ on $K$, put

$$d^x a = \frac{q}{q-1} \frac{da}{|a|_K}.$$ 

Then $\frac{da}{|a|_K}$ is a Haar measure on $K^\times$ and we have

$$\text{vol}_{d^x a} (R^\times) = \int_{R^\times} \frac{q}{q-1} \frac{da}{|a|_K}$$

$$= \frac{q}{q-1} \int_{R^\times} da$$

$$= \sum_{n=0}^{\infty} q^{-n} \int_{R^\times} da$$

$$= \sum_{n=0}^{\infty} \int_{R^\times} q^{-n} da$$

$$= \sum_{n=0}^{\infty} \int_{\pi R^\times} da$$
\[ = \int_{\bigcup_{n \geq 0} \pi_R^n}^\times \, da \]
\[ = \int_R^\times \, da = \text{vol}_\times^\times (R). \]
§7. HARMONIC ANALYSIS

Let $G$ be a locally compact abelian group.

1. DEFINITION A character of $G$ is a continuous homomorphism $\chi: G \to \mathbb{C}^\times$.

2. NOTATION Write $\tilde{G}$ for the group whose elements are the characters of $G$.

3. DEFINITION A unitary character of $G$ is a continuous homomorphism $\chi: G \to T$.

4. NOTATION Write $\hat{G}$ for the group whose elements are the unitary characters of $G$.

5. LEMMA There is a decomposition

$$\tilde{G} \cong \tilde{G}_+ \times \hat{G},$$

where $\tilde{G}_+$ is the group of positive characters of $G$.

PROOF The only positive unitary character is trivial, so $\tilde{G}_+ \cap \hat{G} = \{1\}$. On the other hand, if $\chi$ is a character, then $|\chi|$ is a positive character, $\chi/|\chi|$ is a unitary character, and $\chi = |\chi| \left(\frac{\chi}{|\chi|}\right)$.

6. LEMMA Every bounded character of $G$ is a unitary character.

PROOF The only compact subgroup of $\mathbb{R}_{>0}$ is the trivial subgroup $\{1\}$.

7. APPLICATION If $G$ is compact, then every character of $G$ is unitary.

8. EXAMPLE Take $G = \mathbb{Z}$ — then $\tilde{G} \cong \mathbb{C}^\times$, the isomorphism being given by the map $\chi \mapsto \chi(1)$. 
2.

9: EXAMPLE Take $G = \mathbb{R}$ — then $\tilde{G} \cong \mathbb{R} \times \mathbb{R}$ and every character has the form 
$\chi(x) = e^{zx} \ (z \in \mathbb{C}).$

10: EXAMPLE Take $G = \mathbb{C}$ — then $\tilde{G} \cong \mathbb{C} \times \mathbb{C}$ and every character has the form 
$\chi(x) = \exp(z_1 \Re(x) + z_2 \Im(x)) \ (z_1, z_2 \in \mathbb{C}).$

11: EXAMPLE Take $G = \mathbb{R}^\times$ — then $\tilde{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{C}$ and every character has the form 
$\chi(x) = (\text{sgn } x)^\sigma |x|^s \ (\sigma \in \{0, 1\}, \ s \in \mathbb{C}).$

12: EXAMPLE Take $G = \mathbb{C}^\times$ — then $\tilde{G} \cong \mathbb{Z} \times \mathbb{C}$ and every character has the form 
$\chi(x) = \exp(n \Im x) |x|^s (n \in \mathbb{Z}, \ s \in \mathbb{C}).$

13: DEFINITION The dual group of $G$ is $\hat{G}.$

14: RAPPEL Let $X, Y$ be topological spaces and let $F$ be a subspace of $C(X, Y).$
Given a compact set $K \subset X$ and an open subset $V \subset Y,$ let $W(K, V)$ be the set of all 
f $\in F$ such that $f(K) \subset V$ — then the collection $\{W(K, V)\}$ is a subbasis for the 
compact open topology on $F.$

[Note: The family of finite intersections of sets of the form $W(K, V)$ is then 
a basis for the compact open topology: Each member has the form $\bigcap_{i=1}^{n} W(K_i, V_i),$ 
where the $K_i \subset X$ are compact and the $V_i \subset Y$ are open.]

Equip $\hat{G}$ with the compact open topology.

15: FACT The compact open topology on $\hat{G}$ coincides with the topology of 
uniform convergence on compact subsets of $G.$
16: LEMMA \( \hat{G} \) is a locally compact abelian group.

17: REMARK \( \hat{G} \) is also a locally compact abelian group and the decomposition
\[ \tilde{G} = \tilde{G}_+ \times \hat{G} \]
is topological.

18: EXAMPLE Take \( G = \mathbb{R} \) and given a real number \( t \), let \( \chi_t(x) = e^{\sqrt{-1} tx} \) — then \( \chi_t \) is a unitary character of \( G \) and for any \( \chi \in \hat{G} \), there is a unique \( t \in \mathbb{R} \) such that \( \chi = \chi_t \), hence \( G \) can be identified with \( \hat{G} \).

19: EXAMPLE Take \( G = \mathbb{R}^2 \) and given a point \( (t_1, t_2) \), let \( \chi_{(t_1, t_2)}(x_1, x_2) = e^{\sqrt{-1} (t_1 x_1 + t_2 x_2)} \) — then \( \chi_{(t_1, t_2)} \) is a unitary character of \( G \) and for any \( \chi \in \hat{G} \), there is a unique \( (t_1, t_2) \in \mathbb{R}^2 \) such that \( \chi = \chi_{(t_1, t_2)} \), hence \( G \) can be identified with \( \hat{G} \).

20: EXAMPLE Take \( G = \mathbb{Z}/n\mathbb{Z} \) and given an integer \( m = 0, 1, \ldots, n-1 \), let \( \chi_m(k) = \exp(2\pi \sqrt{-1} \frac{km}{n}) \) — then \( \chi_0, \chi_1, \ldots, \chi_{n-1} \) are the characters of \( G \), hence \( G \) can be identified with \( \hat{G} \).

21: LEMMA If \( G \) is compact, then \( \hat{G} \) is discrete.

22: EXAMPLE Take \( G = \mathbb{T} \) and given \( n \in \mathbb{Z} \), let \( \chi_n(e^{\sqrt{-1} \theta}) = e^{\sqrt{-1} n\theta} \) — then \( \chi_n \) is a unitary character of \( G \) and all such have this form, so \( \mathbb{T} \cong \mathbb{Z} \).
23: **LEMMA** If $G$ is discrete, then $\hat{G}$ is compact.

24: **EXAMPLE** Take $G = \mathbb{Z}$ and given $e^{i\theta} \in T$, let $\chi_\theta(n) = e^{i\theta n}$ -- then $\chi_\theta$ is a unitary character of $G$ and all such have this form, so $\hat{\mathbb{Z}} \cong T$.

25: **LEMMA** If $G_1, G_2$ are locally compact abelian groups, then $G_1 \times G_2$ is topologically isomorphic to $\hat{G}_1 \times \hat{G}_2$.

26: **EXAMPLE** Take $G = \mathbb{R}$ -- then $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}_{>0} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$, thus $\hat{G}$ is topologically isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$:

$$(u, t) \mapsto \chi_{u, t} \quad (u \in \mathbb{Z}/2\mathbb{Z}, \ t \in \mathbb{R}),$$

where

$$\chi_{u, t}(x) = \left(\frac{x}{|x|}\right)^u |x|^{i\theta} t.$$  

27: **EXAMPLE** Take $G = \mathbb{C}$ -- then $G \cong T \times \mathbb{R}_{>0} \cong T \times \mathbb{R}$, thus $\hat{G}$ is topologically isomorphic to $T \times \mathbb{R}$:

$$(n, t) \mapsto \chi_{n, t} \quad (n \in \mathbb{Z}, \ t \in \mathbb{R}),$$

where

$$\chi_{n, t}(z) = \left(\frac{z}{|z|}\right)^n |z|^{i\theta} t.$$  

Denote by $\text{ev}_G$ the canonical arrow $G \to \hat{G}$:

$$\text{ev}_G(x)(\chi) = \chi(x).$$
28: REMARK If $G,H$ are locally compact abelian groups and if $\phi: G \to H$ is a continuous homomorphism, then there is a commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{ev_G} & \hat{G} \\
\downarrow \phi & & \downarrow \hat{\phi} \\
H & \xrightarrow{ev_H} & \hat{H}
\end{array}
\]

29: PONTRYAGIN DUALITY $ev_G$ is an isomorphism of groups and a homeomorphism of topological spaces.

30: SCHOLIUM Every compact abelian group is the dual of a discrete abelian group and every discrete abelian group is the dual of a compact abelian group.

31: REMARK Every finite abelian group is isomorphic to its dual $\hat{G}: G \cong \hat{G}$ (but the isomorphism is not "functorial").

Let $H$ be a closed subgroup of $G$.

32: NOTATION Put

$$H^\perp = \{ \chi \in \hat{G} : \chi|_{H} = 1 \}.$$ 

33: LEMMA $H^\perp$ is a closed subgroup of $\hat{G}$ and $H = H^{\perp\perp}$.

Let $\pi_H : G \to G/H$ be the projection and define

\[
\begin{array}{c}
\phi : G/H \to H^\perp \\
\psi : \hat{G}/H^\perp \to \hat{H}
\end{array}
\]
by

\[ \phi(\chi) = \chi \circ \pi_H \]
\[ \psi(\chi H^\perp) = \chi|_H. \]

34: **Lemma** \( \phi \) and \( \psi \) are isomorphisms of topological groups.

35: **Application** Every unitary character of \( H \) extends to a unitary character of \( G \).

36: **Example** Let \( G \) be a finite abelian group and let \( H \) be a subgroup of \( G \) — then \( G \) contains a subgroup isomorphic to \( G/H \).

[In fact,]

\[ G/H \cong \widehat{G/H} \cong H^\perp \subset \hat{G} \cong G. \]

37: **Remark** Denote by \( \text{LCA} \) the category whose objects are the locally compact abelian groups and whose morphisms are the continuous homomorphisms — then

\[ ^\sim : \text{LCA} \to \text{LCA} \]

is a contravariant functor. This said, consider the short exact sequence

\[ 1 \to H \to G \xrightarrow{\pi_H} G/H \to 1 \]

and apply \( ^\sim \):

\[ 1 \to \widehat{G/H} \cong H^\perp \to \hat{G} \to \hat{H} \cong \widehat{G/H^\perp} \to 1, \]

which is also a short exact sequence.
Given \( f \in L^1(G) \), its \textbf{Fourier transform} is the function

\[
\hat{f}: G \to \mathbb{C}
\]
defined by the rule

\[
\hat{f}(\chi) = \int_G f(x)\chi(x) \, d\mu_G(x).
\]

\textbf{38: EXAMPLE} Take \( G = \mathbb{R} \) -- then \( \hat{\mathbb{R}} \cong \mathbb{R} \) and

\[
\hat{f}(\chi_t) = \hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-itx} \, dx.
\]

\textbf{39: EXAMPLE} Take \( G = \mathbb{R}^2 \) -- then \( \hat{\mathbb{R}^2} \cong \mathbb{R}^2 \) and

\[
\hat{f}(\chi_{(t_1,t_2)}) = \hat{f}(t_1,t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1,x_2) e^{-it_1x_1 - it_2x_2} \, dx_1 \, dx_2.
\]

\textbf{40: EXAMPLE} Take \( G = \mathbb{T} \) -- then \( \hat{\mathbb{T}} \cong \mathbb{Z} \) and

\[
\hat{f}(\chi_n) = \hat{f}(n) = \int_{0}^{2\pi} f(\theta) e^{-in\theta} \, d\theta.
\]

\textbf{41: EXAMPLE} Take \( G = \mathbb{Z} \) -- then \( \hat{\mathbb{Z}} \cong \mathbb{T} \) and

\[
\hat{f}(\chi_\theta) = \hat{f}(\theta) = \sum_{n=-\infty}^{\infty} f(n) e^{-in\theta}.
\]

\textbf{42: EXAMPLE} Take \( G = \mathbb{Z}/n\mathbb{Z} \) -- then \( \hat{\mathbb{Z}/n\mathbb{Z}} \cong 
\hat{\mathbb{Z}/n\mathbb{Z}} \) and

\[
\hat{f}(\chi_m) = \hat{f}(m) = \sum_{k=0}^{n-1} f(k) \exp(2\pi i km/n).
\]

\textbf{43: LEMMA} \( \hat{f}: \hat{G} \to \mathbb{C} \) is a continuous function on \( \hat{G} \) that vanishes at infinity.
and

$$||\hat{f}||_\infty \leq ||f||_1.$$  

44: NOTATION INV(G) is the set of continuous functions \(f \in L^1(G)\) with the property that \(\hat{f} \in L^1(\hat{G})\).

45: FOURIER INVERSION Given a Haar measure \(\mu_G\) on \(G\), there exists a unique Haar measure \(\mu_\wedge\) on \(\hat{G}\) such that \(\forall f \in INV(G),\)

\[f(x) = \int_{\hat{G}} \hat{f}(\chi) \overline{\chi(x)} d\mu_\wedge(\chi).\]

If \(G\) is compact, then it is customary to normalize \(\mu_G\) by the requirement \(\int_G 1 d\mu_G = 1\).

46: LEMMA

\[
\int_G \chi(x) d\mu_G(x) = \begin{cases} 1 & \text{if } \chi = 0 \\ 0 & \text{if } \chi \neq 0. \end{cases}
\]

PROOF The case \(\chi = 0\) is clear. On the other hand, if \(\chi \neq 0\), then there exists \(x_0; \chi(x_0) \neq 1\), hence

\[
\int_G \chi(x) d\mu_G(x) = \int_G \chi(x-x_0 + x_0) d\mu_G(x)
\]

\[
= \chi(x_0) \int_G \chi(x-x_0) d\mu_G(x)
\]

\[
= \chi(x_0) \int_G \chi(x) d\mu_G(x)
\]

\[
\Rightarrow \int_G \chi(x) d\mu_G(x) = 0.
\]
Assuming still that $G$ is compact ($\Rightarrow \hat{G}$ is discrete), take $f \equiv 1$:

$$\hat{f}(0) = 1, \hat{f}(\chi) = 0 \ (\chi \neq 0).$$

I.e.: $\hat{f}$ is the characteristic function of $\{0\}$, hence is integrable, thus $f \in \text{INV}(G)$. Accordingly, if $\mu_G$ is the Haar measure on $\hat{G}$ per Fourier inversion, then

$$\int_G f(\chi) d\mu_G(\chi) = \mu_G(\{0\}),$$

so $\forall \chi \in \hat{G}$,

$$\mu_G(\{\chi\}) = 1.$$

47: EXAMPLE Take $G = \mathbb{T}$ -- then $d\mu_G = \frac{d\theta}{2\pi}$, so for $f \in \text{INV}(G)$,

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{-in\theta},$$

where

$$\hat{f}(n) = \int_0^{2\pi} f(\theta) e^{-in\theta} \frac{d\theta}{2\pi}.$$

If $G$ is discrete, then it is customary to normalize $\mu_G$ by stipulating that singletons are assigned measure 1.

48: REMARK There is a conflict if $G$ is both compact and discrete, i.e., if $G$ is finite.

Assuming still that $G$ is discrete ($\Rightarrow \hat{G}$ is compact), take $f(0) = 1$, 

f(x) = 0 (x \neq 0):

\[ \hat{f}(\chi) = \int_G f(x) \chi(x) \, d\mu_G(x) \]

\[ = f(0) \chi(0) \mu_G(\{0\}) \]

\[ = 1. \]

I.e.: \( \hat{f} \) is the constant function 1, hence is integrable, thus \( f \in \text{INV}(G) \).

Accordingly, if \( \mu_G \) is the Haar measure on \( \hat{G} \) per Fourier inversion, then

\[ \mu_G(\hat{G}) = \int_G 1 \, d\mu_G(\chi) \]

\[ = \int_G \hat{f}(\chi) \, d\mu_G(\chi) \]

\[ = \int_G \hat{f}(\chi) \chi(0) \, d\mu_G(\chi) \]

\[ = f(0) = 1. \]

49: EXAMPLE Take \( G = \mathbb{Z}/n\mathbb{Z} \) and let \( \mu_G \) be the counting measure (thus here \( \mu_G(G) = n \)) -- then \( \hat{\mu}_G \) is the counting measure divided by \( n \) and for \( f \in \text{INV}(G) \),

\[ f(k) = \frac{1}{n} \sum_{m=0}^{n-1} \hat{f}(m) \exp\left(-2\pi \sqrt{-1} \frac{km}{n}\right), \]

where

\[ \hat{f}(m) = \sum_{k=0}^{n-1} f(k) \exp\left(2\pi \sqrt{-1} \frac{km}{n}\right). \]

50: EXAMPLE Take \( G = \mathbb{R} \) and let \( \mu_G = \alpha \, dx \) (\( \alpha > 0 \)), hence \( \hat{\mu}_G = \beta \, dt \) (\( \beta > 0 \)).
and we claim that

\[ l = 2\alpha\beta\pi. \]

To establish this, recall first that the formalism is

\[
\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ıtx} \, dx
\]

\[
f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{ıtx} \, dt.
\]

Let \( f(x) = e^{-|x|} \) — then

\[
\frac{2\alpha}{1+t^2} = \int_{-\infty}^{\infty} e^{-|x|} e^{-ıtx} \, dx,
\]

so \( f \in \text{INV}(G) \), thus

\[
e^{-|x|} = \int_{-\infty}^{\infty} \frac{2\alpha}{1+t^2} e^{-ıtx} \, dt
\]

\[
= 2\alpha\beta \int_{-\infty}^{\infty} \frac{e^{-ıtx}}{1+t^2} \, dt.
\]

Now put \( x = 0 \):

\[
l = 2\alpha\beta \int_{-\infty}^{\infty} \frac{dt}{1+t^2} = 2\alpha\beta\pi,
\]

as claimed. One choice is to take

\[
\alpha = \beta = \frac{1}{\sqrt{2\pi}},
\]

the upshot then being that the Haar measure of \([0,1]\) is not \(1\) but rather \(\frac{1}{\sqrt{2\pi}}\).

51: NOTATION Given \( f \in L^1(\mathbb{R}) \), let

\[
F_R f(t) = \int_{-\infty}^{\infty} f(x) e^{2\piıtx} \, dx.
\]
Therefore
\[ F_R \vec{f}(t) = \sqrt{2\pi} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{2\pi \sqrt{-1} tx} dx \]
\[ = \sqrt{2\pi} \hat{f}(2\pi t). \]

52: STANDARDIZATION \((G = \mathbb{R})\) Let \( f \in \text{INV}(\mathbb{R}) \) -- then
\[ F_R \hat{F}_R f(x) = f(-x). \]

[In fact,
\[ F_R \hat{F}_R f(x) = \int_{-\infty}^{\infty} F_R \vec{f}(t) e^{2\pi \sqrt{-1} tx} dt \]
\[ = \int_{-\infty}^{\infty} \sqrt{2\pi} \hat{f}(2\pi t) e^{2\pi \sqrt{-1} tx} dt \]
\[ = \sqrt{2\pi} \int_{-\infty}^{\infty} \hat{f}(u) e^{\sqrt{-1} ux} \frac{du}{2\pi} \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(t) e^{\sqrt{-1} tx} dt \]
\[ = f(-x). \]

Fourier inversion in the plane takes the form
\[
\begin{bmatrix}
\hat{f}(t_1, t_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2) e^{\sqrt{-1}(t_1 x_1 + t_2 x_2)} dx_1 dx_2 \\
\hat{f}(t_1, t_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(t_1, t_2) e^{-\sqrt{-1}(t_1 x_1 + t_2 x_2)} dt_1 dt_2.
\end{bmatrix}
\]

One may then introduce
\[ F_R^2 \vec{f}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2) e^{2\pi \sqrt{-1}(t_1 x_1 + t_2 x_2)} dx_1 dx_2. \]
13.

\[ = 2\pi \hat{f}(2\pi t_1, 2\pi t_2) \]

and, proceeding as above, find that

\[ \mathcal{F}_R^2 \mathcal{F}_R^2 f(x_1, x_2) = f(-x_1, -x_2). \]

Now identify \( \mathbb{R}^2 \) with \( \mathbb{C} \) and recall that \( \text{tr}_{\mathbb{C}/\mathbb{R}}(z) = z + \overline{z} \). Write

\[
\begin{cases}
  w = a + \sqrt{-1} \ b \\
  z = x + \sqrt{-1} \ y.
\end{cases}
\]

Then

\[ wz + \overline{wz} = 2\text{Re}(wz) = 2(ax - by). \]

Therefore

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{2\sqrt{-1}(ax-by)} \, dx \, dy = \hat{f}(2a,-2b).
\]

[Note: Let \( \chi_w(z) = \exp(\sqrt{-1} (wz + \overline{wz})) \) -- then \( \chi_w \) is a unitary character of \( \mathbb{C} \) and for any \( \chi \in \hat{\mathbb{C}} \), there is a unique \( w \in \mathbb{C} \) such that \( \chi = \chi_w \), hence \( \hat{\mathbb{C}} \cong \mathbb{C} \).]

53: NOTATION Given \( f \in L^1(\mathbb{R}^2) \), let

\[ F_\mathbb{C} f(w) = F_\mathbb{C} f(a,b) \]

\[ = 2\mathcal{F}_R^2 f(2a,-2b) \]

\[ = 4\pi \hat{f}(4\pi a,-4\pi b) \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{4\pi \sqrt{-1}(ax-by)} \, dx \, dy. \]
14.

54: STANDARDIZATION \((G = C)\) Let \(f \in \text{INV}(C)\) — then

\[
F_C F_C f(x,y) = f(-x,-y).
\]

[In fact,

\[
F_C F_C f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_C f(a,b) e^{4\pi i \sqrt{-1}(ax-by)} 2dadb
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 4\pi^2 (4\pi a, -4\pi b) e^{4\pi i \sqrt{-1}(ax-by)} 2dadb
\]

\[
= \frac{4\pi^2}{(4\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(u,-v) e^{\sqrt{-1}(ux-uy)} 2dudv
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(u,-v) e^{-\sqrt{-1}(ux-uy)} dudv
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(u,v) e^{-\sqrt{-1}(ux-uy)} dudv
\]

\[
= f(-x,-y).\]

55: PLANCHEREL THEOREM The Fourier transform restricted to \(L^1(G) \cap L^2(G)\)
is an isometry (with respect to \(L^2\) norms) onto a dense linear subspace of \(L^2(\hat{G})\), hence can be extended uniquely to an isometric isomorphism \(L^2(G) \rightarrow L^2(\hat{G})\).

56: PARSEVAL FORMULA \(\forall f, g \in L^2(G),\)

\[
\int_G f(x) g(x) d_G(x) = \int_{\hat{G}} \hat{f}(\chi) \overline{g(\chi)} d_\chi(\chi).
\]

57: N.B. In both of these results, the Haar measure on \(\hat{G}\) is per Fourier inversion.
§8. ADDITIVE p-ADIC CHARACTER THEORY

1: FACT Every proper closed subgroup of $T$ is finite.

Suppose that $G$ is compact abelian and totally disconnected.

2: LEMMA If $\chi \in \hat{G}$, then the image $\chi(G)$ is a finite subgroup of $T$.

PROOF Ker $\chi$ is closed and

$$\chi(G) \cong G/\ker \chi.$$ 

But the quotient $G/\ker \chi$ is 0-dimensional, hence totally disconnected. Therefore $\chi(G)$ is totally disconnected. Since $T$ is connected, it follows that $T \cong \chi(G)$, thus $\chi(G)$ is finite.

3: N.B. The torsion of $R/Z$ is $Q/Z$, so $\chi$ factors through the inclusion $Q/Z \to R/Z$, i.e., $\chi(G) \subset Q/Z$.

The foregoing applies in particular to $G = \mathbb{Z}_p$.

4: LEMMA Every character of $Q_p$ is unitary.

PROOF This is because

$$Q_p = \bigcup_{n \in \mathbb{Z}} p^nZ_p,$$

where the $p^nZ_p$ are compact, thus §7, §7 is applicable.

5: If $\chi \in \hat{Q}_p$ is nontrivial, then there exists an $n \in \mathbb{Z}$ such that $\chi \equiv 1$ on $p^nZ_p$ but $\chi \not\equiv 1$ on $p^{n-1}Z_p$.

PROOF Consider a ball $B$ of radius $\frac{1}{2}$ about 1 in $\mathbb{C}^\times$ — then the only subgroup of $\mathbb{C}^\times$ contained in $B$ is the trivial subgroup and, by continuity, $\chi(p^nZ_p)$ must be
inside B for all sufficiently large \( n \), thus must be identically 1 there.

6: Definition The conductor \( \text{con} \) \( \chi \) of a nontrivial \( \chi \in \hat{Q}_p \) is the largest subgroup \( p^n\mathbb{Z}_p \) on which \( \chi \) is trivial (and \( n \) is the minimal integer with this property).

A typical \( x \neq 0 \) of \( \hat{Q}_p \) has the form

\[
x = \sum_{n=\nu(x)}^{\infty} a_n p^n (a_n \in A, \nu(x) \in \mathbb{Z})
\]

\[
= f(x) + [x].
\]

Here the fractional part \( f(x) \) of \( x \) is defined by the prescription

\[
f(x) = \begin{cases} 
-\frac{1}{p} \sum_{n=\nu(x)}^{\infty} a_n p^n & \text{if } \nu(x) < 0 \\
0 & \text{if } \nu(x) \geq 0 
\end{cases}
\]

and the integral part \( [x] \) of \( x \) is defined by the prescription

\[
[x] = \sum_{n=0}^{\infty} a_n p^n,
\]

with \( f(0) = 0 \), \( [0] = 0 \) by convention.

7: N.B.

\[
f(x) \in \mathbb{Z}[\frac{1}{p}] \subset Q,
\]

where

\[
\mathbb{Z}[\frac{1}{p}] = \{ \frac{n}{p^k} : n \in \mathbb{Z}, k \in \mathbb{Z} \},
\]
3.

while \([x] \in \mathbb{Z}_p\).

8: OBSERVATION

\[
0 \leq f(x) = \sum_{1 \leq j \leq v(x)} \frac{a_j}{p^j}\n\]

\[
< (p-1) \sum_{j=1}^{\infty} \frac{1}{p^j} = 1
\]

\[
\Rightarrow
\]

\[
f(x) \in [0,1[ \cap \mathbb{Z}_{\frac{1}{p}}.
\]

Let \(\mu_\infty\) stand for the group of roots of unity in \(\mathbb{C}^x\) having order a power of \(p\). There, \(\mu_\infty\) is a p-group and \(\mu_\infty\) is an increasing sequence of cyclic groups

\[
\mu_p \subset \mu_{p^2} \subset \cdots \subset \mu_{p^k} \subset \cdots
\]

\[
\mu_\infty = \bigcup_{p^{k+1}} \mu_{p^k},
\]

where

\[
\mu_{p^k} = \{z \in \mathbb{C}^x : z^{p^k} = 1\}.
\]

9: REMARK Denote by \(\mu\) the group of all roots of unity in \(\mathbb{C}^x\), hence

\[
\mu = \bigcup_{m \geq 1} \mu_{m'}, \mu_m = \{z \in \mathbb{C}^x : z^m = 1\}.
\]

Then \(\mu\) is an abelian torsion group and \(\mu_\infty\) is the p-Sylow subgroup of \(\mu\), i.e., the maximal p-subgroup of \(\mu\).
Put
\[ \chi_p(x) = \exp(2\pi \sqrt{-1} f(x)) \quad (x \in \mathbb{Q}_p). \]

Then
\[ \chi_p: \mathbb{Q}_p \to \mathbb{C} \]
and \( \mathbb{Z}_p \subset \text{Ker} \chi_p \).

\textbf{10: EXAMPLE} Suppose that \( v(x) = -1 \), so \( x = \frac{k}{p} + y \) with \( 0 < k \leq p-1 \) and \( y \in \mathbb{Z}_p \):
\[ \chi_p(x) = \exp(2\pi \sqrt{-1} \frac{k}{p}) = \zeta^k, \]
where \( \zeta = \exp(2\pi \sqrt{-1}/p) \) is a primitive \( p \)th root of unity.

\textbf{11: LEMMA} \( \chi_p \) is a unitary character.

\textbf{PROOF} Given \( x, y \in \mathbb{Q}_p \), write
\[ f(x+y) - f(x) - f(y) = x + y - [x+y] - (x - [x]) - (y - [y]) = [x] + [y] - [x+y] \in \mathbb{Z}_p. \]
But at the same time
\[ f(x+y) - f(x) - f(y) \in \mathbb{Z}[\frac{1}{p}]. \]
Thus
\[ f(x+y) - f(x) - f(y) \in \mathbb{Z}[\frac{1}{p}] \cap \mathbb{Z}_p = \mathbb{Z} \]
and so
\[ \exp(2\pi \sqrt{-1}(f(x+y) - f(x) - f(y))) = 1 \]
or still,

\[ \chi_p(x+y) = \chi_p(x)\chi_p(y). \]

Therefore, \( \chi_p : \mathbb{Q}_p \to T \) is a homomorphism. As for continuity, it suffices to check this at 0, matters then being clear (since \( \chi_p \) is trivial in a neighborhood of 0) (\( \mathbb{Z}_p \) is open and \( 0 \in \mathbb{Z}_p \)).

12: LEMMA The kernel of \( \chi_p \) is \( \mathbb{Z}_p \).

[A priori, the kernel of \( \chi_p \) consists of those \( x \in \mathbb{Q}_p \) such that \( f(x) \in \mathbb{Z} \).

Therefore

\[ \ker \chi_p = \mathbb{Z}_p. \]

13: LEMMA The image of \( \chi_p \) is \( \mu_p \).

[A priori, the image of \( \chi_p \) consists of the complex numbers of the form

\[ \exp(2\pi i \sqrt{\frac{1}{p^m}}) = \exp(2\pi i \sqrt{1/p^m})^k. \]

Since \( \exp(2\pi i \sqrt{1/p^m}) \) is a root of unity of order \( p^m \), these roots generate \( \mu_p \) as \( m \) ranges over the positive integers.]

14: SCHOLIUM \( \chi_p \) implements an isomorphism

\[ \mathbb{Q}_p / \mathbb{Z}_p \cong \mu_p \]

15: REMARK

\[ x \in p^{-k} \mathbb{Z}_p \iff p^k x \in \mathbb{Z}_p \]
6.

\[ \Leftrightarrow \chi_p(x^k) = 1 \]
\[ \Leftrightarrow \chi_p(x)^p = 1 \]
\[ \Leftrightarrow \chi_p(x) \in \mu_p^k. \]

16: RAPPEL Let \( p \) be a prime -- then a group is \( p \)-primary if every element has order a power of \( p \).

17: RAPPEL Every abelian torsion group \( G \) is a direct sum of its \( p \)-primary subgroups \( G_p \).

[Note: The \( p \)-primary component \( G_p \) is the \( p \)-Sylow subgroup of \( G \).]

18: NOTATION \( Z(p^\infty) \) is the \( p \)-primary component of \( \mathbb{Q}/\mathbb{Z} \).

Therefore

\[ \mathbb{Q}/\mathbb{Z} = \bigoplus_p Z(p^\infty). \]

19: LEMMA \( Z(p^\infty) \) is isomorphic to \( \bigoplus_p \mu_p^\infty \).

[\( Z(p^\infty) \) is generated by the \( 1/p^n \) in \( \mathbb{Q}/\mathbb{Z} \).]

Therefore

\[ \mathbb{Q}/\mathbb{Z} \simeq \bigoplus_p \mu_p^\infty \simeq \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p. \]

[Note: Consequently,

\[ \text{End}(\mathbb{Q}/\mathbb{Z}) \simeq \text{End}(\bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p). \]
20: REMARK \( \hat{\mathbb{Z}}_p \) is isomorphic to \( \mu_\infty \) (cf. #26 infra).

Given \( t \in \mathbb{Q}_p \), let \( L_t \) be left multiplication by \( t \) and put \( \chi_{p,t} = \chi_p \circ L_t \).

Then \( \chi_{p,t} \) is continuous and \( \forall x \in \mathbb{Q}_p \),

\[ \chi_{p,t}(x) = \chi_p(tx). \]

[Note: Trivially, \( \chi_{p,0} \equiv 1 \). And \( \forall t \neq 0 \),

\[ \text{con } \chi_{p,t} = p^{-\nu(t)}\mathbb{Z}_p. \]

Proof:

\[ x \in \text{con } \chi_{p,t} \iff tx \in \mathbb{Z}_p \]

\[ \iff |tx|_p \leq 1 \]

\[ \iff |x|_p \leq \frac{1}{|t|_p} = p^{\nu(t)} \]

\[ \iff x \in p^{-\nu(t)}\mathbb{Z}_p. \]

Next

\[ \chi_{p,t}(x+y) = \chi_p(t(x+y)) \]

\[ = \chi_p(tx+ty) \]

\[ = \chi_p(tx)\chi_p(ty) \]
Therefore \( \chi_{p,t} \in \hat{Q}_p \).

Next

\[
\chi_{p,t+s}(x) = \chi_p((t+s)x)
\]

\[
= \chi_p(tx+sx)
\]

\[
= \chi_p(tx)\chi_p(sx)
\]

\[
= \chi_{p,t}(x)\chi_{p,s}(x).
\]

Therefore the arrow \( \Xi_p:Q_p \to \hat{Q}_p \) that sends \( t \) to \( \chi_{p,t} \) is a homomorphism.

21: LEMMA If \( t \neq s \), then \( \chi_{p,t} \neq \chi_{p,s} \).

PROOF If to the contrary, \( \chi_{p,t} = \chi_{p,s} \), then \( \forall x \in Q_p, \chi_p(tx) = \chi_p(sx) \) or still, \( \forall x \in Q_p, \chi_p((t-s)x) = 1 \). But \( L_{t-s}:Q_p \to Q_p \) is an automorphism, hence \( \chi_p \) is trivial, which it isn't.

22: LEMMA The set

\[
\Xi_p(Q_p) = \{ \chi_{p,t}: t \in Q_p \}
\]

is dense in \( \hat{Q}_p \).

PROOF Let \( H \) be the closure in \( \hat{Q}_p \) of the \( \chi_{p,t} \). Consider the quotient \( \hat{Q}_p/H \) and to get a contradiction, assume that \( H \neq \hat{Q}_p \), thus that there is a nontrivial \( \xi \in \hat{Q}_p \). By definition, \( H^\perp \) is computed in \( \hat{Q}_p \), which by Pontryagin duality, is which is trivial on \( H \).
identified with \( \hat{Q}_p \), so spelled out

\[ H^1 = \{ x \in \hat{Q}_p : ev_{\hat{Q}_p}(x) | H = 1 \}. \]

Accordingly, for some \( x, \xi = ev_{\hat{Q}_p}(x) \), hence \( \forall t, \xi(x_p,t) = ev_{\hat{Q}_p}(x)(x_p,t) \)

\[ = x_p(t) = x_p(tx) = 1, \]

which is possible only if \( x = 0 \) and this implies that \( \xi \) is trivial.

23: **LEMMA** The arrows

\[
\begin{array}{c}
\hat{Q}_p \\
\uparrow \xi_p(Q_p) \quad Q_p \\
\xi_p(Q_p) \rightarrow \hat{Q}_p \\
\end{array}
\]

are continuous.

Therefore \( \xi_p(Q_P) \) is a locally compact subgroup of \( \hat{Q}_p \). But a locally compact subgroup of a locally compact group is closed. Therefore \( \xi_p(Q_p) = \hat{Q}_p \).

In summary:

24: **THEOREM** \( \hat{Q}_p \) is topologically isomorphic to \( Q_p \) (via the arrow \( \xi_p : Q_p \rightarrow \hat{Q}_p \)).

25: **LEMMA** Fix \( t \) -- then \( x_p,t|_Z_p = 1 \) iff \( t \in Z_p \).

PROOF Recall that the kernel of \( x_p \) is \( Z_p \).

- \( t \in Z_p, x \in Z_p \Rightarrow tx \in Z_p \Rightarrow x_p(tx) = 1 \Rightarrow x_p,t|_Z_p = 1. \)
10.

- \( \chi_{p,t}|z_p = 1 \Rightarrow \chi_{p,t}(1) = 1 \Rightarrow \chi_p(t) = 1 \Rightarrow t \in z_p. \)

26: APPLICATION \( \hat{z}_p \) is isomorphic to \( \mu. \)

\( \hat{z}_p \) can be computed as \( \hat{0}_p/z_p^l. \) But \( z_p^l, \) when viewed as a subset of \( q_p, \) consists of those \( t \) such that \( \chi_{p,t}|z_p = 1. \) Therefore

\[ \hat{z}_p \cong \hat{0}_p/z_p \cong q_p/z_p \cong \mu. \]

27: NOTATION Let

\( \chi_\infty(x) = \exp(-2\pi \sqrt{-1} x) \quad (x \in R). \)

28: PRODUCT PRINCIPLE \( \forall x \in Q, \)

\[ \prod_{p \leq \infty} \chi_p(x) = 1. \]

PROOF Take \( x \) positive — then there exist primes \( p_1, \ldots, p_n \) such that \( x \) admits a representation

\[ x = \frac{N_1}{\alpha_1} + \frac{N_2}{\alpha_2} + \ldots + \frac{N_n}{\alpha_n} + M, \]

where the \( \alpha_k \) are positive integers, the \( N_k \) are positive integers \( (1 \leq N_k < p_k^\alpha - 1), \) and \( M \in Z. \) Appending a subscript to \( f, \) we have

\[ f_{p_k}(x) = \frac{N_k}{\alpha_k}, \quad f_{p_k}(x) = 0 \quad (p \neq p_k, \quad k = 1, 2, \ldots, n). \]

Therefore

\[ \prod_{p \leq \infty} \chi_p(x) = \prod_{1 \leq k \leq n} \chi_{p_k}(x). \]
\[
11.
\]
\[
\prod_{1 \leq k \leq n} \exp(2\pi\sqrt{-1} f_{p_k}(x))
\]
\[
= \exp(2\pi\sqrt{-1} \sum_{k=1}^{n} f_{p_k}(x))
\]
\[
= \exp(2\pi\sqrt{-1} f(x) - M)
\]
\[
= \exp(2\pi\sqrt{-1} x)
\]
\[
\Rightarrow
\]
\[
\prod_{p<\infty} x_p(x) = \prod_{p<\infty} x_p(x) x_\infty(x)
\]
\[
= \exp(2\pi\sqrt{-1} x) \exp(-2\pi\sqrt{-1} x)
\]
\[
= 1.
\]

APPENDIX

Let $K$ be a finite extension of $\mathbb{Q}_p$.

1. THEOREM The topological groups $K$ and $\hat{K}$ are topologically isomorphic.

[Put
\[
\chi_{K,p}(a) = \exp(2\pi\sqrt{-1} f(\text{tr}_{K/Q_p}(a))
\]
\[
= \chi_p(\text{tr}_{K/Q_p}(a))
\]
and given $b \in K$, put
\[
\chi_{K,p,b}(a) = \chi_{K,p}(ab).
\]
Proceed from here as above.]
2: REMARK Every character of \( K \) is unitary.

3: LEMMA

\[
\begin{align*}
\text{if } a \in R &\Rightarrow \text{tr}_{K/Q_p}(a) \in \mathbb{Z}_p \\
\text{if } a \in P &\Rightarrow \text{tr}_{K/Q_p}(a) \in p\mathbb{Z}_p.
\end{align*}
\]

4: DEFINITION The differential of \( K \) is the set

\[ \Delta_K = \{ b \in K : \text{tr}_{K/Q_p}(Rb) \in \mathbb{Z}_p \}. \]

5: LEMMA \( \Delta_K \) is a proper \( R \)-submodule of \( K \) containing \( R \).

6: LEMMA There exists a unique nonnegative integer \( d \) -- the differential exponent of \( K \) -- characterized by the condition that

\[ \pi^{-d}R = \Delta_K. \]

[This follows from the theory of "fractional ideals" (details omitted).]

[Note: \( \chi_{K,p} \) is trivial on \( \pi^{-d}R \) but is nontrivial on \( \pi^{-d-1}R \).]

7: LEMMA Let \( e \) be the ramification index of \( K \) over \( \mathbb{Q}_p \) (cf. §5, #17) -- then

\[ a \in \mathbb{P}^{-e+1} \Rightarrow \text{tr}_{K/Q_p}(a) \in \mathbb{Z}_p. \]

PROOF Let

\[ a \in \mathbb{P}^{-e+1} = \pi^{-e+1}R = \pi^{-e}(\pi R) = \pi^{-e}P, \]
so \( a = \pi^{-e}b \) (\( b \in \mathbb{P} \)). Write \( p = \pi^e u \) and consider \( pa: \)

\[
pa = \pi^e u \pi^{-e}b = ub.
\]

But

\[
|u| = 1, \quad |b| < 1 \Rightarrow |ub| < 1
\]

\[
\Rightarrow ub \in \mathbb{P}
\]

\[
\Rightarrow \text{tr}_{K/\mathbb{Q}_p}(ub) \in p\mathbb{Z}_p
\]

\[
\Rightarrow \text{tr}_{K/\mathbb{Q}_p}(pa) \in p\mathbb{Z}_p
\]

\[
\Rightarrow p \text{tr}_{K/\mathbb{Q}_p}(a) \in p\mathbb{Z}_p \Rightarrow \text{tr}_{K/\mathbb{Q}_p} \in \mathbb{Z}_p.
\]

**8: APPLICATION**

\( d \geq e-1. \)

[It suffices to show that

\[
p^{-e+1} \subset \Delta_K (\equiv \pi^{-d} R).
\]

Thus let \( a \in p^{-e+1} \), say \( a = \pi^e b \) (\( b \in \mathbb{P} \)), and let \( r \in \mathbb{R} \) — then the claim is that

\[
\text{tr}_{K/\mathbb{Q}_p}(ar) \in \mathbb{Z}_p.
\]

But

\[
ar = \pi^{-e}br \in \pi^e \mathbb{P} \quad (|br| < 1)
\]

or still,

\[
ar \in p^{-e+1} \Rightarrow \text{tr}_{K/\mathbb{Q}_p}(ar) \in \mathbb{Z}_p.
\]
9: REMARK Therefore \( d = 0 \Rightarrow e = 1 \), hence in this situation, \( K \) is unramified.

[Note: There is also a converse, viz. if \( K \) is unramified, then \( d = 0 \).]

10: N.B. It can be shown that

\[ \text{tr}_{K/Q_p}(R) = \mathbb{Z}_p \]

iff \( d = e - 1 \).

11: CRITERION Fix \( b \in K \) -- then

\[ b \in \Delta_K \iff \forall a \in R, \chi_{K,p}(ab) = 1. \]

PROOF

• \( a \in R, b \in \Delta_K \Rightarrow ab \in \Delta_K \)

\[ \Rightarrow \text{tr}_{K/Q_p}(ab) \in \mathbb{Z}_p \]

\[ \Rightarrow \chi_{K,p}(ab) = \chi_p(\text{tr}_{K/Q_p}(ab)) = 1. \]

• \( \forall a \in R, \chi_{K,p}(ab) = 1 \)

\[ \Rightarrow \forall a \in R, \text{tr}_{K/Q_p}(ab) \in \mathbb{Z}_p \Rightarrow b \in \Delta_K. \]

Normalize the Haar measure on \( K \) by the condition

\[ \mu_K(R) = \int_R \text{da} = q^{-d/2}. \]

Let \( \chi_R \) be the characteristic function of \( R \) -- then
\[ f_K \chi_R(a) \chi_{K,p}(ab) da = f_R \chi_{K,p}(ab) da. \]

- \( b \in \Delta_K \Rightarrow \chi_{K,p}(ab) = 1 \quad (\forall a \in R) \)

\[ \Rightarrow f_R \chi_{K,p}(ab) da = \mu_K(R) = q^{-d/2}. \]

- \( b \not\in \Delta_K \Rightarrow \chi_{K,p}(ab) \neq 1 \quad (\exists a \in R) \)

\[ \Rightarrow f_R \chi_{K,p}(ab) da = 0. \]

Consequently, as a function of \( b \),

\[ f_R \chi_{K,p}(ab) da = q^{-d/2} \chi_{\Delta_K}(b), \]

\( \chi_{\Delta_K} \) the characteristic function of \( \Delta_K \).

**12: Lemma**

\[ [\pi^{-d} R: R] = q^d. \]

Therefore

\[ \mu_K(\Delta_K) = \mu_K(\pi^{-d} R) = q^d \mu_K(R) = q^d q^{-d/2} = q^{d/2}. \]

**13: Lemma** \( \forall a \in K, \)

\[ f_K q^{-d/2} \chi_{\Delta_K}(b) \chi_{K,p}(ab) db = \chi_R(a). \]
16.

PROOF The left hand side reduces to
\[ q^{-d/2} \int_{\Delta_K} \chi_{K,p}(ab) \, db \]
and there are two possibilities.

- \( a \in \mathbb{R} \Rightarrow ab \in \Delta_K \quad (\forall b \in \Delta_K) \)

\[ \Rightarrow \text{tr}_{K/Q_p}(ab) \in \mathbb{Z}_p \Rightarrow \chi_{K,p}(ab) = 1 \]
\[ \Rightarrow q^{-d/2} \int_{\Delta_K} \chi_{K,p}(ab) \, db = q^{-d/2} \cdot 1 = 1. \]

- \( a \not\in \mathbb{R} \quad \chi_{K,p}(ab) \neq 1 \quad (\exists b \in \Delta_K) \)

\[ \Rightarrow q^{-d/2} \int_{\Delta_K} \chi_{K,p}(ab) \, db = 0. \]

To detail the second point of this proof, work with the normalized absolute value (cf. §6, #18) and recall that \( |\pi|_K = \frac{1}{q} \) (cf. §5, #21). Accordingly,

\[ x \in \pi^n R \iff |x|_K \leq q^{-n}. \]

Fix \( a \not\in \mathbb{R} \) -- then the claim is that \( b \mapsto \chi_{K,p}(ab) \quad (b \in \Delta_K) \) is nontrivial. For
\[ \chi_{K,p}(ab) = 1 \iff ab \in \pi^{-d} R \]
\[ \iff |a+b|_K \leq q^d \]
\[ \iff |a|_K |b|_K \leq q^d \]
\[ \iff |b|_K \leq \frac{q^d}{|a|_K} = q^{d+v(a)}. \]

But

\[ a \not\in R \Rightarrow v(a) < 0 \]
\[ \Rightarrow -v(a) > 0 \Rightarrow -d-v(a) > -d \]
\[ \Rightarrow \pi^{-d-v(a)}_R \subset \pi^{-d}_R, \]

a proper containment.
§9. MULTIPlicative p-ADIC CHARACTER THEORY

Recall that

\[ \mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^\times \]

the abstract reflection of the fact that for every \( x \in \mathbb{Q}_p^\times \), there is a unique \( v(x) \in \mathbb{Z} \) and a unique \( u(x) \in \mathbb{Z}_p^\times \) such that \( x = p^{v(x)}u(x) \). Therefore

\[ \hat{\mathbb{Q}_p^\times} \cong \hat{\mathbb{Z}} \times \hat{\mathbb{Z}_p^\times} \cong \mathbb{T} \times \mathbb{T}_p. \]

1: N.B. A character of \( \mathbb{Q}_p \) is necessarily unitary (cf. §8, #4) but this is definitely not the case for \( \mathbb{Q}_p^\times \) (cf. infra).

2: DEFINITION A character \( \chi: \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times \) is unramified if it is trivial on \( \mathbb{Z}_p^\times \).

3: EXAMPLE Given any complex number \( s \), the arrow \( x \rightarrow |x|_p^s \) is an unramified character of \( \mathbb{Q}_p^\times \).

4: LEMMA If \( \chi: \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times \) is an unramified character, then there exists a complex number \( s \) such that \( \chi = |.|_p^s \).

PROOF Such a \( \chi \) factors through the projection \( \mathbb{Q}_p^\times \rightarrow \mathbb{P}^\times \) defined by \( x \rightarrow |x|_p \), hence gives rise to a character \( \tilde{\chi}: \mathbb{P}^\times \rightarrow \mathbb{C}^\times \) which is completely determined by its value on \( p \), say \( \tilde{\chi}(p) = p^s \) for the complex number

\[ s = \frac{\log \tilde{\chi}(p)}{\log p}, \]
itself determined up to an integral multiple of
\[ \frac{2\pi\sqrt{-1}}{\log p}. \]

Therefore
\[
\chi(x) = \tilde{\chi}(|x|_p) = \tilde{\chi}(p^{-v(x)}) = (\tilde{\chi}(p))^{-v(x)} = (p^s)^{-v(x)} = (p^{-v(x)})^s = |x|^s.
\]

[Note: For the record,
\[
|x|^\frac{2\pi\sqrt{-1}}{\log p} = (p^{-v(x)})^\frac{2\pi\sqrt{-1}}{\log p} = (e^{-v(x)\log p})^\frac{2\pi\sqrt{-1}}{\log p} = e^{-v(x)2\pi\sqrt{-1}} = 1.
\]

Suppose that \( \chi: \mathbb{Q}^\times_p \rightarrow \mathbb{C}^\times \) is a character -- then \( \chi \) can be written as
\[
\chi(x) = |x|^s \chi(u(x)),
\]
where \( s \in \mathbb{C} \) and \( \chi \equiv \chi|\mathbb{Z}_p^\times \in \hat{(\mathbb{Z}_p^\times)} \), thus \( \chi \) is unitary iff \( s \) is pure imaginary.

5: LEMMA If \( \chi \in \hat{(\mathbb{Z}_p^\times)} \) is nontrivial, then there is an \( n \in \mathbb{N} \) such that
\( \chi \equiv 1 \) on \( U_{p,n} \) but \( \chi \not\equiv 1 \) on \( U_{p,n-1} \) (cf. §8, #5).
Assume again that $\chi: \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is a character.

6: DEFINITION $\chi$ is ramified of degree $n \geq 1$ if $\chi|_{U_{p,n}} \equiv 1$ and $\chi|_{U_{p,n-1}} \neq 1$.

7: DEFINITION The conductor $\text{cond } \chi$ of $\chi$ is $\mathbb{Z}_p^\times$ if $\chi$ is unramified and $U_{p,n}$ if $\chi$ is ramified of degree $n$.

8: RAPPEL If $G$ is a finite abelian group, then the number of unitary characters of $G$ is $\text{card } G$.

9: LEMMA

$$[\mathbb{Z}_p^\times : U_{p,1}] = p-1 \quad (\text{cf. } \S 4, \#40)$$

and

$$[U_{p,1} : U_{p,n}] = p^{n-1}.$$ 

If $\chi$ is ramified of degree $n$, then $\chi$ can be viewed as a unitary character of $\mathbb{Z}_p^\times /U_{p,n}$. But the quotient $\mathbb{Z}_p^\times /U_{p,n}$ is a finite abelian group, thus has

$$\text{card } \mathbb{Z}_p^\times /U_{p,n} = [\mathbb{Z}_p^\times : U_{p,n}]$$

unitary characters. And

$$[\mathbb{Z}_p^\times : U_{p,n}] = [\mathbb{Z}_p^\times : U_{p,1}] \cdot [U_{p,1} : U_{p,n}]$$

$$= (p-1)p^{n-1},$$

this being the number of unitary characters of $\mathbb{Z}_p^\times$ of degree $\leq n$. Therefore the
4.

The group $Z_p^\ast$ has $p-2$ unitary characters of degree 1 and for $n \geq 2$, the group $Z_p^\ast$ has

$$(p-1)p^{n-1} - (p-1)p^{n-2} = p^{n-2}(p-1)^2$$

unitary characters of degree $n$.

10: LEMMA Let $\chi \in \mathbb{Q}_p^\ast$ then

$$\chi(x) = |x|_p^{\sqrt{-1}t} \chi(u(x)),$$

where $t$ is real and

$$- (\pi/\log p) < t \leq \pi/\log p.$$

APPENDIX

Suppose that $p \neq 2$, let $\tau \in \mathbb{Q}_p^\ast - (\mathbb{Q}_p^\ast)^2$, and form the quadratic extension

$$\mathbb{Q}_p(\tau) = \{x + y\sqrt{\tau} : x, y \in \mathbb{Q}_p\}.$$

1: NOTATION Let $\mathbb{Q}_{p,\tau}$ be the set of points of the form $x^2 - \tau y^2$ ($x \neq 0$, $y \neq 0$).

2: LEMMA $\mathbb{Q}_{p,\tau}$ is a subgroup of $\mathbb{Q}_p^\ast$ containing $(\mathbb{Q}_p^\ast)^2$.

3: LEMMA

$$[\mathbb{Q}_p^\ast : \mathbb{Q}_{p,\tau}] = 2 \text{ and } [\mathbb{Q}_{p,\tau} : (\mathbb{Q}_p^\ast)^2] = 2.$$  

[Note:

$$[\mathbb{Q}_p^\ast : (\mathbb{Q}_p^\ast)^2] = 4 \quad (\text{cf. } \S4, \#53).]$$
4: DEFINITION Given $x \in Q^\times_p$, let

$$sgn_\tau(x) = \begin{cases} 1 & \text{if } x \in Q_{p,\tau} \\ -1 & \text{if } x \not\in Q_{p,\tau} \end{cases}$$

5: LEMMA $sgn_\tau$ is a unitary character of $\hat{Q}_p$. 
§10. TEST FUNCTIONS

The Schwartz space $S(R^n)$ consists of those complex valued $C^\infty$ functions which, together with all their derivatives, vanish at infinity faster than any power of $||\cdot||$.

1: DEFINITION. The elements $f$ of $S(R^n)$ are the test functions on $R^n$.

2: EXAMPLE Take $n = 1$ --- then

$$f(x) = Cx^A \exp(-\pi x^2),$$

where $A = 0$ or $1$, is a test function, said to be standard. Here

$$\int_R x^A \exp(-\pi x^2)e^{2\pi i t x} dx = (\sqrt{-1})^A t^A \exp(-\pi t^2),$$

thus $F_R$ of a standard function is again standard (cf. §7, §51).

[Note: Henceforth, by definition, the Fourier transform of an $f \in L^1(R)$ will be the function

$$\hat{f} : R \rightarrow C$$

defined by the rule

$$\hat{f}(t) = F_R f(t)$$

$$= \int_R f(x)e^{2\pi i t x} dx.$$]

3: EXAMPLE Take $n = 2$ and identify $R^2$ with $C$ --- then

$$f(z) = Cz^A \bar{z}^B \exp(-2\pi |z|^2),$$

where $A, B \in \mathbb{Z}_{\geq 0}$ & $AB = 0$, is a test function, said to be standard. Here
\[
\int_{C} z^{A} \overline{z}^{B} \exp(-2\pi |z|^2) e^{2\pi \sqrt{-1} (wz + \overline{wz})} |dz \wedge d\overline{z}|
\]

\[
= (\sqrt{-1})^{A+B} w^{B} \overline{w}^{A} \exp(-2\pi |w|^2),
\]

thus \( F_c \) of a standard function is again standard (cf. §7, #53).

[Note: Henceforth, by definition, the Fourier transform of an \( f \in L^1(C) \) will be the function

\[
\hat{f} : C \to C
\]

defined by the rule

\[
\hat{f}(w) = F_c f(w)
\]

\[
= \int_{C} f(z) e^{2\pi \sqrt{-1} (wz + \overline{wz})} |dz \wedge d\overline{z}|.
\]

4: **Definition** Let \( G \) be a totally disconnected locally compact group -- then a function \( f : G \to C \) is said to be **locally constant** if for any \( x \in G \), there is an open subset \( U_x \) of \( G \) containing \( x \) such that \( f \) is constant on \( U_x \).

5: **Lemma** A locally constant function \( f \) is continuous.

**Proof** Fix \( x \in G \) and suppose that \( \{x_i\} \) is a net converging to \( x \) -- then \( x_i \) is eventually in \( U_x \), hence there \( f(x_i) = f(x) \).

6: **Definition** The **Bruhat space** \( \mathcal{B}(G) \) consists of those complex valued locally constant functions whose support is compact.

[Note: \( \mathcal{B}(G) \) carries a "canonical topology" but I shall pass in silence as regards to its precise formulation.]
7: DEFINITION The elements \( f \) of \( \mathcal{B}(G) \) are the test functions on \( G \).

8: LEMMA Given a test function \( f \), there exists an open-compact subgroup \( K \) of \( G \), an integer \( n \geq 0 \), elements \( x_1, \ldots, x_n \) in \( G \) and elements \( c_1, \ldots, c_n \) in \( \mathbb{C} \) such that the union \( \bigcup_{k=1}^{n} K x_k K \) is disjoint and

\[
f = \sum_{k=1}^{n} c_k \chi_{K x_k K} \chi_{K x_k K}
\]

\( \chi_{K x_k K} \) is the characteristic function of \( K x_k K \).

PROOF Since \( f \) is locally constant, for every \( z \in \mathbb{C} \) the preimage \( f^{-1}(z) \) is an open subset of \( G \). Therefore \( X = \{ x : f(x) \neq 0 \} \) is the support of \( f \). This said, given \( x \in X \), define a map \( \phi_x : G \times G \rightarrow \mathbb{C} \) by \( \phi_x(x_1, x_2) = f(x_1 xx_2) \), thus \( \phi_x(e, e) = f(x) \) and \( \phi_x \) is continuous if \( \mathbb{C} \) has the discrete topology. Consequently, one can find an open-compact subgroup \( K_x \) of \( G \) such that \( \phi_x \) is constant on \( K_x \times K_x \). Put

\( U_x = K_x x K_x \) — then \( U_x \) is open-compact and \( f \) is constant on \( U_x \). But \( X \) is covered by the \( U_x \), hence, being compact, is covered by finitely many of them. Bearing in mind that distinct double cosets are disjoint, consider now the intersection \( K \) of the finitely many \( K_x \) that occur.

Specialize and let \( G = \mathbb{Q}_p \).

9: EXAMPLE If \( K \subset \mathbb{Q}_p \) is open-compact, then its characteristic function \( \chi_K \) is a test function on \( \mathbb{Q}_p \).
10: **LEMMA** Every \( f \in B(Q_p) \) is a finite linear combination of functions of the form
\[
\chi_{x+pZ_p^n}(x \in Q_p, n \in Z).
\]
[This is an instance of \#8 or argue directly (cf. §4, \#33).]

11: **DEFINITION** Given \( f \in L^1(Q_p) \), its **Fourier transform** is the function
\[
\hat{f}: Q_p \rightarrow C
\]
defined by the rule
\[
\hat{f}(t) = \int_{Q_p} f(x) \chi_{p,t}(x) \, dx
\]
\[
= \int_{Q_p} f(x) \chi_{p}(tx) \, dx.
\]

12: **LEMMA** \( \forall f \in L^1(Q_p) \),
\[
\hat{f}(t) = \hat{f}(-t).
\]

**PROOF**
\[
\hat{f}(t) = \int_{Q_p} \overline{f(x)} \chi_{p}(tx) \, dx
\]
\[
= \int_{Q_p} \overline{f(x)} \chi_{p}(-tx) \, dx
\]
\[
= \int_{Q_p} \overline{f(x)} \chi_{p}((-t)x) \, dx
\]
\[
= \int_{Q_p} f(x) \chi_{p}((-t)x) \, dx
\]
\[
= \hat{f}(-t).
\]
5.

13: SUBLEMMA

\[ \int_{p^{-n}Z_p} \chi_p(x) dx = \begin{cases} p^{-n} & (n \geq 0) \\ 0 & (n < 0). \end{cases} \]

[Recall that

\[ \mu_0(p^{n}Z_p) = p^{-n} \]

and apply §7, #46 and §8, #12.]

14: LEMMA Take \( f = \chi_{p^{n}Z_p} \) -- then

\[ \hat{\chi}_{p^{n}Z_p} = p^{-n} \chi_{p^{n}Z_p}. \]

PROOF

\[ \hat{\chi}_{p^{n}Z_p}(t) = \int_{p^{n}Z_p} \chi_p(x) \chi_{p^{n}Z_p}(x) dx \]

\[ = \int_{p^{n}Z_p} \chi_p(tx) dx \]

\[ = |t|^{-1} \int_{p^{n+\nu(t)}Z_p} \chi_p(x) dx. \]

The last integral equals

\[ p^{-n-\nu(t)} \]

if \( n+\nu(t) \geq 0 \) and equals 0 if \( n+\nu(t) < 0 \) (cf. #13). But
6.

\[ t \in p^{-n} \mathbb{Z}_p \iff v(t) \geq -n \iff n + v(t) \geq 0. \]

Since

\[ |t|_p^{-1} \cdot v(t) = 1, \]

it therefore follows that

\[ \hat{\chi}_{p^{-n} \mathbb{Z}_p} = p^{-n} \cdot \hat{\chi}_{\mathbb{Z}_p}. \]

In particular:

\[ \hat{\chi}_{\mathbb{Z}_p} = \chi_{\mathbb{Z}_p}. \]

**THEOREM** Take \( f = \chi_{x + p^n \mathbb{Z}_p} \) — then

\[ \hat{\chi}_{x + p^n \mathbb{Z}_p}(t) = \begin{cases} \chi_p(tx) \cdot p^{-n} & (|t|_p \leq p^n) \\ 0 & (|t|_p > p^n). \end{cases} \]

**PROOF**

\[ \hat{\chi}_{x + p^n \mathbb{Z}_p}(t) = \int_{x + p^n \mathbb{Z}_p} \chi_p(ty) \chi_p(ty) \, dy \]

\[ = \int_{x + p^n \mathbb{Z}_p} \chi_p(ty) \, dy \]

\[ = \int_{x + p^n \mathbb{Z}_p} \chi_p(ty) \, dy \]

\[ = \int_{p^n \mathbb{Z}_p} \chi_p(t(x+y)) \, dy \]
7.

\[
= \int_{\mathbb{P}^n \mathbb{Z}_p} \chi_p(\tau x + \tau y) dy \\
= \int_{\mathbb{P}^n \mathbb{Z}_p} \chi_p(\tau x) \chi_p(\tau y) dy \\
= \chi_p(\tau x) \int_{\mathbb{P}^n \mathbb{Z}_p} \chi_p(\tau y) dy \\
= \chi_p(\tau x) \int_0^1 \chi_p(\tau y) \chi_p(\tau y) dy \\
= \chi_p(\tau x) \hat{\chi}^{\mathbb{P}^n \mathbb{Z}_p}(t) \\
= \chi_p(\tau x) \chi_p^{-\mathbb{Z}_p}(t).
\]

16: APPLICATION Taking into account #10,

\[f \in \mathcal{B}(\mathbb{Q}_p) \Rightarrow \hat{f} \in \mathcal{B}(\mathbb{Q}_p).\]

17: THEOREM \forall f \in \text{INV}(\mathbb{Q}_p),

\[\hat{f}(x) = f(-x) \quad (x \in \mathbb{Q}_p).\]

PROOF It suffices to check this for a single function, so take \( f = \chi_{\mathbb{Z}_p} \) — then, as noted above,

\[\hat{\chi}_{\mathbb{Z}_p} = \chi_{\mathbb{Z}_p}.\]
thus $\forall x$, 

$$
\hat{\chi}_P(x) = \chi_P(x) = \chi_P(-x).
$$

18: N.B. It is clear that 

$$B(Q_p) \subset \text{INV}(Q_p).$$

19: SCHOLIUM The arrow $f \rightarrow \hat{f}$ is a linear bijection of $B(Q_p)$ onto itself. 
[Injectivity is manifest. As for surjectivity, the arrow $f \rightarrow \check{f}$, where 

$$\check{f}(x) = f(-x),$$

maps $B(Q_p)$ into itself. And 

$$\check{f} = \check{f} = (f) = (f) = ((f) \check{f}) = (f \check{f}) \check{f}. ]$$

20: REMARK As is well-known, the same conclusion obtains if $Q_p$ is replaced by $\mathbb{R}$ or $\mathbb{C}$.

Pass now from $Q_p$ to $Q_p^\times$.

21: LEMMA Let $f \in B(Q_p^\times)$ -- then $\exists n \in \mathbb{N}$: 

$$
\begin{align*}
|x|_p < p^{-n} & \Rightarrow f(x) = 0 \\
|x|_p > p^n & \Rightarrow f(x) = 0.
\end{align*}
$$

Therefore an element $f$ of $B(Q_p^\times)$ can be viewed as an element of $B(Q_p)$ with the property that $f(0) = 0$. 
22: DEFINITION Given \( f \in L^1(Q_\mathbb{P}^\times, d^\times x) \), its **Mellin transform** \( \tilde{f} \) is the Fourier transform of \( f \) per \( Q_\mathbb{P}^\times \):

\[
\tilde{f}(\chi) = \int_{Q_\mathbb{P}^\times} f(x) \chi(x) d^\times x.
\]

[Note: By definition,]

\[
d^\times x = \frac{p}{p-1} \frac{dx}{|x|_p} \quad \text{(cf. §6, #26),}
\]

so

\[
\text{vol} \int_{d^\times x} (Z_\mathbb{P}^\times) = \text{vol} \int_{d^\times x} (Z_\mathbb{P}) = 1.
\]

23: EXAMPLE Take \( f = \chi_{Z_\mathbb{P}^\times} \) \( \quad \text{then} \)

\[
\tilde{\chi}_{Z_\mathbb{P}^\times} (\chi) = \int_{Z_\mathbb{P}^\times} \chi_{Z_\mathbb{P}^\times}(x) \chi(x) d^\times x
\]

\[
= \int_{Z_\mathbb{P}^\times} \chi(x) d^\times x.
\]

Decompose \( \chi \) as in §9, #10, hence

\[
\int_{Z_\mathbb{P}^\times} \chi(x) d^\times x = \int_{Z_\mathbb{P}^\times} |x|_p^{\sqrt{\frac{p}{p-1}}} \chi(p^{-\nu(x)} x) d^\times x
\]

\[
= \int_{Z_\mathbb{P}^\times} \chi(x) d^\times x
\]

\[
= \begin{cases} 
0 & (\chi \not\equiv 1) \\
1 & (\chi \equiv 1).
\end{cases}
\]
According to §9, #2, a unitary character \( \chi \in (\hat{Q}_p^x) \) is unramified if its restriction \( \chi \) to \( Z_p^x \) is trivial. Therefore the upshot is that the Mellin transform of \( Z_p^x \) is the characteristic function of the set of unramified elements of \( (Q_p^x) \).

**APPENDIX**

Let \( K \) be a finite extension of \( Q_p \) -- then

\[ K^x \cong Z \times R^x \]

and the generalities developed in §9 go through with but minor changes when \( Q_p \) is replaced by \( K \).

In particular: \( \forall \chi \in \hat{K}^x \), there is a splitting

\[ \chi(a) = |a|_K^{\sqrt{-1} t} \chi(\pi^{-\nu(a)} a), \]

where \( t \) is real and

\[-(\pi/\log q) < t \leq \pi/\log q.\]

[Note: \( \chi \) is **unramified** if it is trivial on \( R^x \).]

1. **N.B.** The "\( \pi \)" in the first instance is a prime element (cf. §5, #10) and \( |\pi|_K = \frac{1}{q} \). On the other hand, the "\( \pi \)" in the second instance is 3.14....

The extension of the theory from \( B(Q_p) \) to \( B(K) \) is straightforward, the point of departure being the observation that

\[
\int_{\pi_R^x} \chi_{K, R}(a) da = \nu_K(R)
\]

\[
\begin{align*}
0 & \quad (n = -d-1, -d-2, 
- \quad q^{-n} & \quad (n = -d, -d+1, 
\end{align*}
\]
2: CONVENTION Normalize the Haar measure on $K$ by stipulating that
\[ f_R da = q^{-d/2}. \]

3: DEFINITION Given $f \in L^1(K)$, its Fourier transform is the function \( \hat{f} : K \to \mathbb{C} \) defined by the rule
\[ \hat{f}(b) = \int_K f(a)\chi_{K,p}(ab)da \]
\[ = \int_K f(a)\chi_{K,p}(ab)da. \]

4: THEOREM \( \forall f \in \text{INV}(K), \)
\[ \hat{f}(a) = f(-a) \quad (a \in K). \]

PROOF It suffices to check this for a single function, so take $f = \chi_{R'}$ in which case the work has already been done in the Appendix to §8. To review:

- \[ \hat{\chi_R}(b) = \int_K \chi_R(a)\chi_{K,p}(ab)da \]
  \[ = \int_R \chi_{K,p}(ab)da \]
  \[ = q^{-d/2}\chi_{\Lambda_K}(b). \]

- \[ \int_K q^{-d/2}\chi_{\Lambda_K}(b)\chi_{K,p}(ab)db \]
  \[ = q^{-d/2}\int_{\Lambda_K} \chi_{K,p}(ab)db \]
  \[ = \chi_R(a) \quad (\text{loc. cit., #13}) \]
  \[ = \chi_R(-a). \]
5: N.B. It is clear that
\[ B(K) \subseteq \text{INV}(K). \]

6: SCHOLIUM The arrow \( f \to \hat{f} \) is a linear bijection of \( B(K) \) onto itself.

7: CONVENTION Put
\[
d^\chi a = \frac{q}{q-1} \frac{da}{|a|^K}.
\]
Then \( d^\chi a \) is a Haar measure on \( K^\times \) and
\[
\text{vol}_{d^\chi a}(R^\times) = \text{vol}_{da}(R) = q^{-d/2}.
\]

8: DEFINITION Given \( f \in L^1(K^\times, d^\chi a) \), its Mellin transform \( \hat{f} \) is the Fourier transform of \( f \) per \( K^\times \):
\[
\hat{f}(\chi) = \int_{K^\times} f(a) \chi(a) d^\chi a.
\]

9: EXAMPLE Take \( f = \chi_{R^\times} \) -- then
\[
\hat{\chi}_{R^\times}(\chi) = \begin{cases} 1 & (\chi \equiv 1) \\ 0 & (\chi \not\equiv 1) \end{cases}
\]

\[
q^{-d/2} \quad (\chi \equiv 1).
\]
§11. LOCAL ZETA FUNCTIONS: \( \mathbb{R}^x \) or \( \mathbb{C}^x \)

We shall first consider \( \mathbb{R}^x \), hence \( \tilde{\mathbb{R}}^x \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{C} \) and every character has the form

\[
\chi(x) = \chi_{\sigma,s}(x) = (\text{sgn } x)^\sigma |x|^s \quad (\sigma \in \{0,1\}, \ s \in \mathbb{C}) \quad (\text{cf. } \S 7, \#11).
\]

1. DEFINITION Given \( f \in S(\mathbb{R}) \) and a character \( \chi: \mathbb{R}^x \to \mathbb{C}^x \), the local zeta function attached to the pair \( (f, \chi) \) is

\[
Z(f, \chi) = \int_{\mathbb{R}^x} f(x) \chi(x) d^x x,
\]

where \( d^x x = \frac{dx}{|x|} \).

[Note: The parameters \( \sigma \) and \( s \) are implicit:]

\[
Z(f, \chi) = Z(f, \chi_{\sigma,s}).
\]

2: LEMMA The integral defining \( Z(f, \chi) \) is absolutely convergent for \( \text{Re}(s) > 0 \).

PROOF Since \( f \) is Schwartz, there are no issues at infinity. As for what happens at the origin, let \( I = [-1,1]\setminus \{0\} \) and fix \( C > 0 \) such that \( |f(x)| \leq C \) (\( x \in I \)) -- then

\[
|Z(f, \chi)| \leq \int_{\mathbb{R}\setminus \{0\}} |f(x)||x|^\text{Re}(s)-1 dx
\]

\[
\leq (\int_{\mathbb{R}\setminus I} + \int_{I}) |f(x)||x|^\text{Re}(s)-1 dx
\]

\[
\leq M + C \int_{I} |x|^\text{Re}(s)-1 dx,
\]

a finite quantity.
2.

3: **Lemma** \( Z(f, \chi) \) is a holomorphic function of \( s \) in the strip \( \text{Re}(s) > 0 \).

FORMALLY,

\[
\frac{d}{ds} Z(f, \chi) = \int_{\mathbb{R}} f(x) (\text{sgn } x)^{\sigma} (\log |x|) |x|^s d^x x,
\]

and while correct, "differentiation under the integral sign" does require a formal proof... .]

4: **Notation** Put

\[
\chi = \chi^{-1} |\cdot|.
\]

The integral defining \( Z(f, \chi) \) is absolutely convergent if \( \text{Re}(1-s) > 0 \), i.e., if \( 1 - \text{Re}(s) > 0 \) or still, if \( \text{Re}(s) < 1 \).

5: **Lemma** Let \( f, g \in S(\mathbb{R}) \) and suppose that \( 0 < \text{Re}(s) < 1 \) -- then

\[
Z(f, \chi) Z(\hat{g}, \chi) = Z(\hat{f}, \chi^n) Z(g, \chi).
\]

**Proof** Write

\[
Z(f, \chi) Z(\hat{g}, \chi)
\]

\[
= \int_{\mathbb{R}^x} \int_{\mathbb{R}^y} f(x) \hat{g}(y) \chi(xy^{-1}) |y| d^x x d^y
\]

and make the substitution \( t = yx^{-1} \) to get

\[
Z(f, \chi) Z(\hat{g}, \chi)
\]

\[
= \int_{\mathbb{R}^x} (\int_{\mathbb{R}^x} f(x) \hat{g}(tx) |x| d^x x) \chi(t^{-1}) |t| d^x t.
\]

The claim now is that the inner integral is symmetric in \( f \) and \( g \) (which then implies
3.

that

$$Z(f,\chi)Z(\hat{g},\chi) = Z(g,\chi)Z(\hat{f},\chi),$$

the desired equality). To see that this is so, observe first that

$$|x|du\cdot d^x x = |u|dx\cdot d^x u.$$ 

Since $R^x$ and $R$ differ by a single element, it therefore follows that

$$\int_{R^x} f(x)\hat{g}(tx)|x|d^x x$$

$$= \int_{R^x} f(x)|x| (\int_{R^x} g(u)e^{2\pi i txu}du)d^x x$$

$$= \int \int_{R\times R^x} f(x)g(u)|x|e^{2\pi i txu}du d^x x$$

$$= \int_{R^x} g(u)|u| (\int_{R^x} f(x)e^{2\pi i txu}dx)d^x u$$

$$= \int_{R^x} g(u)\hat{f}(tu)|u|d^x u.$$ 

Fix $\phi \in S(R)$ and put

$$\rho(\chi) = \frac{Z(\phi,\chi)}{Z(\hat{\phi},\chi)}.$$ 

Then $\rho(\chi)$ is independent of the choice of $\phi$ and $\forall f \in S(R)$, the functional equation

$$Z(f,\chi) = \rho(\chi)Z(\hat{f},\chi)$$

obtains.

6: **Lemma** $\rho(\chi)$ is a meromorphic function of $s$ (cf. infra).
7: APPLICATION ∀ f ∈ S(R), Z(f, χ) admits a meromorphic continuation to the whole s-plane.

8: NOTATION Set

\[ \Gamma_R(s) = \pi^{-s/2} \Gamma(s/2). \]

9: DEFINITION Write

\[ L(\chi) = \begin{cases} 
\Gamma_R(s) & (\sigma = 0) \\
\Gamma_R(s+1) & (\sigma = 1). 
\end{cases} \]

Proceeding to the computation of \( \rho(\chi) \), distinguish two cases.

- \( \sigma = 0 \) Take \( \phi_0(x) \) to be \( e^{-\pi x^2} \) — then

\[ Z(\phi_0, \chi) = \int_{R^x} e^{-\pi x^2} |x|^s \, dx \]

\[ = 2 \int_0^\infty e^{-\pi x^2} x^{s-1} \, dx \]

\[ = \pi^{-s/2} \Gamma(s/2) = \Gamma_R(s) = L(\chi). \]

Next \( \hat{\phi}_0 = \phi_0 \) (cf. §10, #2) so by the above argument,

\[ Z(\hat{\phi}_0, \chi) = L(\hat{\chi}), \]

from which

\[ \rho(\chi) = \frac{L(\chi)}{L(\hat{\chi})}. \]
\[
\frac{-s/2 \Gamma(s/2)}{-1-(1-s)/2 \Gamma(1-s/2)}
\]

\[
= 2^{1-s} \pi^{-s} \cos \left(\frac{\pi s}{2}\right) \Gamma(s).
\]

- \( s = 1 \) Take \( \phi_1(x) \) to be \( xe^{-\pi x^2} \) -- then

\[
z(\phi_1, \chi) = \int_{\mathbb{R}^+} xe^{-\pi x^2} \frac{x}{|x|} |x|^{s} \, dx
\]

\[
= \int_{\mathbb{R}^+} e^{-\pi x^2} |x|^{s+1} \, dx
\]

\[
= 2 \int_0^\infty e^{-\pi x^2} x^s \, dx
\]

\[
= \pi^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right)
\]

\[
= \Gamma_R(s+1) = L(\chi).
\]

Next

\[
\hat{\phi}_1(t) = \sqrt{-1} \, t \exp(-\pi t^2) \quad (\text{cf. \S10, \#2}).
\]

Therefore

\[
z(\hat{\phi}_1, \chi) = \sqrt{-1} \int_{\mathbb{R}^+} xe^{-\pi x^2} \cdot \frac{x}{|x|} \cdot |x|^{1-s} \, dx
\]

\[
= \sqrt{-1} \int_{\mathbb{R}^+} e^{-\pi x^2} |x|^{2-s} \, dx
\]

\[
= \sqrt{-1} \, 2 \int_0^\infty e^{-\pi x^2} x^{1-s} \, dx
\]
Accordingly

\[\rho(\chi) = -\frac{\sqrt{-1}}{\frac{\pi}{2} (s-2) \Gamma \left( \frac{2-s}{2} \right)} \frac{L(\chi)}{L(\chi)}\]

\[= -\frac{\sqrt{-1}}{\frac{\pi}{2} (s-2) \Gamma \left( \frac{2-s}{2} \right)} \frac{(s+1)/2 \Gamma(s+1)}{\Gamma \left( \frac{s+1}{2} \right)}\]

\[= -\sqrt{-1} 2^{1-s} \pi^{-s} \sin \left( \frac{\pi s}{2} \right) \Gamma(s).\]

10: FACT

\[\frac{\zeta(1-s)}{\zeta(s)} = 2^{1-s} \pi^{-s} \cos \left( \frac{\pi s}{2} \right) \Gamma(s)\]

\[\frac{\zeta(s)}{\zeta(1-s)} = 2^{s} s^{-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s).\]

To recapitulate: \(\rho(\chi)\) is a meromorphic function of \(s\) and

\[\rho(\chi) = \epsilon(\chi) \frac{L(\chi)}{L(\chi)} ,\]

where

\[\epsilon(\chi) = 1 \quad (\sigma = 0)\]

\[\epsilon(\chi) = -\sqrt{-1} \quad (\sigma = 1).\]
Having dealt with $R^\times$, let us now turn to $C^\times$, hence $C^\times \cong \mathbb{Z} \times C$ and every character has the form

$$\chi(x) \equiv \chi_{n,s}(x) = \exp(\sqrt{-1} n \arg x) |x|^s \quad (n \in \mathbb{Z}, s \in C) \quad (\text{cf. } \S 7, \#12).$$

Here, however, it will be best to make a couple of adjustments.

1. Replace $x$ by $z$.
2. Replace $|.|$ by $|.|_C$, the normalized absolute value, so

$$|z|_C = |z\bar{z}| = |z|^2 \quad (\text{cf. } \S 6, \#15).$$

11. **DEFINITION** Given $f \in S(C) (= S(R^2))$ and a character $\chi: C^\times \to C^\times$, the **local zeta function** attached to the pair $(f, \chi)$ is

$$Z(f, \chi) = \int_{C^\times} f(z) \chi(z) d^\times z,$$

where $d^\times z = \frac{|dz \wedge d\bar{z}|}{|z|_C}$.

[Note: The parameters $n$ and $s$ are implicit:

$$Z(f, \chi) \equiv Z(f, \chi_{n,s}).$$]

12: **NOTATION** Put

$$\check{\chi} = \chi^{-1}|.|_C.$$

The analogs of #2 and #3 are immediate, as is the analog of #5 (just replace $R^\times$ by $C^\times$ and $|.|$ by $|.|_C$), the crux then being the analog of #6.

13: **NOTATION** Set

$$\Gamma_C(s) = (2\pi)^{1-s} \Gamma(s).$$
14: DEFINITION Write

\[ L(\chi) = \Gamma_c(s + |n|/2). \]

To determine \( \rho(\chi) \) via a judicious choice of \( \phi \) per the relation

\[ \rho(\chi) = \frac{Z(\phi, \chi)}{Z(\hat{\phi}, \chi)}, \]

let

\[
\begin{align*}
\phi_n(z) &= z^n e^{-2\pi|z|^2} \quad (n \geq 0) \\
\hat{\phi}_n(z) &= z^{-n} e^{-2\pi|z|^2} \quad (n < 0).
\end{align*}
\]

Then

\[ \hat{\phi}_n = (\sqrt{-1})^{|n|} \phi_{-n} \quad (\text{cf. } \S 10, \#3). \]

15: N.B. In terms of polar coordinates \( z = re^{\sqrt{-1}\theta} \),

- \( \phi_n(z) = r^n e^{-2\pi r^2 - \sqrt{-1} n\theta} \)

- \( d^n z = \frac{2rdrd\theta}{r^2} = \frac{2}{r} drd\theta \)

- \( \chi(z) = e^{\sqrt{-1} n\theta} |z|^{s_c} = e^{\sqrt{-1} n\theta} r^{s_c} \)

Therefore

\[ Z(\phi_n, \chi) \]
Consequently

\[ \rho(\chi) = \frac{z(\phi_n, \chi)}{z(\hat{\phi}_n, \check{\gamma})} \]

\[ = (\sqrt{-1})^{-|\eta|} \frac{L(\chi)}{L(\check{\gamma})} \]

\[ = \epsilon(\chi) \frac{L(\chi)}{L(\check{\gamma})}, \]
where

\[ \varepsilon(\chi) = (\sqrt{-1})^{-|n|}. \]

And

\[ \frac{L(\chi)}{L(\chi)} = (2\pi)^{1-2s} \frac{\Gamma(s + \frac{|n|}{2})}{\Gamma(1-s + \frac{|n|}{2})}. \]
§12. LOCAL ZETA FUNCTIONS: $\mathbb{Q}_p^\times$

The theory set forth below is in the same spirit as that of §11 but matters are technically more complicated due to the presence of ramification.

1. DEFINITION Given $f \in \mathcal{B}(\mathbb{Q}_p)$ and a character $\chi: \mathbb{Q}_p^\times \to \mathbb{C}^\times$, the local zeta function attached to the pair $(f, \chi)$ is

$$Z(f, \chi) = \int_{\mathbb{Q}_p^\times} f(x) \chi(x) d^\times x,$$

where $d^\times x = \frac{1}{p-1} \frac{dx}{|x|_p} \log p$ (cf. §6, #26).

[Note: There are two parameters associated with $\chi$, viz. $s$ and $\chi$ (cf. §9).]

2. LEMMA The integral defining $Z(f, \chi)$ is absolutely convergent for $\text{Re}(s) > 0$.

PROOF It suffices to check absolute convergence for $f = \chi$ (cf. §10, #10) and then we might just as well take $n = 0$:

$$|Z(f, \chi)| \leq \int_{\mathbb{Q}_p^\times} |f(x)| |x|^\text{Re}(s) d^\times x$$

$$= \int_{\mathbb{Q}_p^\times} \chi_Z(x) |x|^\text{Re}(s) d^\times x$$

$$= \int_{\mathbb{Z}_p^\times \setminus \{0\}} |x|^\text{Re}(s) d^\times x$$

$$= \frac{1}{1-p^{-\text{Re}(s)}} \quad \text{(cf. §6, #27).}$$
3: **Lemma** $Z(f, \chi)$ is a holomorphic function of $s$ in the strip $\text{Re}(s) > 0$.

4: **Notation** Put

$$\gamma = \chi^{-1} | \cdot |_p.$$ 

The integral defining $Z(f, \chi)$ is absolutely convergent if $\text{Re}(1-s) > 0$, i.e., if $l - \text{Re}(s) > 0$ or still, if $\text{Re}(s) < 1$.

5: **Lemma** Let $f, g \in B(Q_p)$ and suppose that $0 < \text{Re}(s) < 1$ — then

$$Z(f, \chi)Z(g, \chi) = Z(f, \chi)Z(g, \chi).$$

[Simply follow verbatim the argument employed in §11, #5.]

Fix $\phi \in B(Q_p)$ and put

$$\rho(\chi) = \frac{Z(\phi, \chi)}{Z(\hat{\phi}, \chi)}.$$ 

Then $\rho(\chi)$ is independent of the choice of $\phi$ and $\forall f \in B(Q_p)$, the functional equation

$$Z(f, \chi) = \rho(\chi)Z(\hat{f}, \chi)$$

obtains.

6: **Lemma** $\rho(\chi)$ is a meromorphic function of $s$ (cf. infra).

7: **Application** $\forall f \in B(Q_p)$, $Z(f, \chi)$ admits a meromorphic continuation to the whole $s$-plane.

8: **Definition** Write

$$L(\chi) = \begin{cases} (1 - \chi(p))^{-1} & (\chi \text{ unramified}) \\ 1 & (\chi \text{ ramified}) \end{cases}$$
3.

There remains the computation of $\rho(\chi)$, the simplest situation being when $\chi$ is unramified, say $\chi = |.|_p^s$, in which case we take $\phi_0(x) = \chi_p(x)\chi_{Z_p}(x)$:

$$Z(\phi_0, \chi) = \int_{\mathcal{O}^\times} \phi_0(x) \chi(x) dx$$

$$= \int_{\mathcal{O}^\times} \chi_p(x) \chi_{Z_p}(x) |x|^s dx$$

$$= \int_{\mathcal{O}^\times} \chi_p(x) |x|^s dx$$

$$= \frac{1}{1-p^{-s}} \text{ (cf. §6, #27)}$$

$$= \frac{1}{1-|p|^s}$$

$$= \frac{1}{1-x(p)} = L(\chi).$$

To finish the determination, it is necessary to explicate the Fourier transform $\hat{\phi}_0$ of $\phi_0$ (cf. §10, #11):

$$\hat{\phi}_0(t) = \int_{\mathcal{O}^\times} \phi_0(x) \chi_p(tx) dx$$

$$= \int_{\mathcal{O}^\times} \chi_p(x) \chi_{Z_p}(x) \chi_p(tx) dx$$

$$= \int_{\mathcal{O}^\times} \chi_p(x) \chi_p(tx) dx$$
4.

\[ = \int_{\mathbb{Z}_p} X_p((1+t)x) \, dx \]

\[ = x_{\mathbb{Z}_p}(t). \]

Therefore

\[ Z(\hat{\phi}_0, \chi) = \int_{0}^{x_{\mathbb{Z}_p}(x)} \hat{\phi}_0(x) \chi(x) \, dx \]

\[ = \int_{0}^{x_{\mathbb{Z}_p}(x)} x_{\mathbb{Z}_p} \, dx \]

\[ = \int_{\mathbb{Z}_p - \{0\}} |x|^{1-s} \, dx \]

\[ = \frac{1}{1-p^{-(1-s)}} \quad \text{(cf. §6, #27)} \]

\[ = \frac{1}{1-|p|^{1-s}} \]

\[ = \frac{1}{1-\chi(p)} = L(\chi). \]

And finally

\[ \rho(\chi) = \frac{Z(\hat{\phi}_0, \chi)}{Z(\hat{\phi}, \chi)} = \frac{L(\chi)}{L(\chi)} \]

or still,

\[ \rho(\chi) = \frac{1-p^{-(1-s)}}{1-p^{-s}}. \]

9: REMARK The function

\[ \frac{1-p^{-(1-s)}}{1-p^{-s}} \]
has a simple pole at \( s = 0 \) with residue

\[
\frac{p^{-1} \log p}{p}
\]

and there are no other singularities.

Suppose now that \( \chi \) is ramified of degree \( n \geq 1: x = |.|^S_p \chi \) (cf. §9, #6) and take \( \phi_n(x) = \chi_p(x) x^{-n} z_p \):

\[
Z(\phi_n, \chi) = \int_{0}^{x} \phi_n(x) \chi(x) d^x x
\]

\[
= \int_{0}^{x} \chi_p(x) x^{-n} z_p \chi(x) d^x x
\]

\[
= \int_{0}^{x} \chi_p(x) x^{-n} z_p \chi(x) d^x x
\]

\[
= \sum_{k=-n}^{\infty} \int_{Z_p}^x \chi_p(p^k u) |p^k u|^S_p \chi(u) d^x u
\]

\[
= \sum_{k=-n}^{\infty} p^{-ks} \int_{Z_p}^x \chi_p(p^k u) \chi(u) d^x u.
\]

10: **Lemma** If \( |v|_p \neq p^n \), then

\[
\int_{Z_p}^x \chi_p(vu) \chi(u) d^x u = 0.
\]

Since \( |_p^k|_p = p^{-k} \), \( Z(\phi_n, \chi) \) reduces to
Let \( E = \{ e_i : i \in I \} \) be a system of coset representatives for \( \mathbb{Z}_p^x/U_{p,n} \) -- then by assumption, \( \chi \) is constant on the cosets mod \( U_{p,n} \), hence

\[
\int_{\mathbb{Z}_p^x} \chi_p(p^{-n}u) \chi(u) \, d^x u
\]

\[
= \sum_{i=1}^{r} \chi(e_i) \int_{e_i U_{p,n}} \chi_p(p^{-n}u) \, d^x u
\]

But

\[
u \in e_i U_{p,n} \Rightarrow p^{-n}u \in p^{-n}e_i + \mathbb{Z}_p
\]

\[
\Rightarrow \chi_p(p^{-n}u) = \chi_p(p^{-n}e_i + x) \quad (x \in \mathbb{Z}_p)
\]

\[
= \chi_p(p^{-n}e_i).
\]

Therefore

\[
\int_{\mathbb{Z}_p^x} \chi_p(p^{-n}u) \chi(u) \, d^x u
\]

\[
= \sum_{i=1}^{r} \chi(e_i) \chi_p(p^{-n}e_i) \int_{e_i U_{p,n}} \, d^x u
\]

\[
= \tau(\chi) \int_{U_{p,n}} \, d^x u
\]

if

\[
\tau(\chi) = \sum_{i=1}^{r} \chi(e_i) \chi_p(p^{-n}e_i).
\]
And

\[ \int_{U_{p,n}} d^x u = \int_{1+p^nz_p} d^x u \]
\[ = \frac{p}{p-1} \int_{1+p^nz_p} \frac{du}{|u|_p} \]
\[ = \frac{p}{p-1} \int_{1+p^nz_p} du \]
\[ = \frac{p}{p-1} \int_{p^nz_p} du \]
\[ = \frac{p}{p-1} p^{-n} = \frac{p^{1-n}}{p-1} \cdot \]

So in the end

\[ z(\phi_n, \chi) = \tau(\chi) \frac{p^{1+n(s-1)}}{p-1} \cdot \]

Next

\[ \hat{\phi}_n(t) = \int_0^t \phi_n(x) \chi_p(tx) dx \]
\[ = \int_0^t \chi_p(x) \chi_{p^{-n}z_p}(x) \chi_p(tx) dx \]
\[ = \int_{p^{-n}z_p} \chi_p(x) \chi_p(tx) dx \]
\[ = \int_{p^{-n}z_p} \chi_p((1+t)x) dx \]
\[ = \text{vol}_{d\chi} (p^{-n}z_p) \chi_{p^n z_p}(t) \]
\[ = p^n \chi_{p^n z_p}(t) \cdot \]
Therefore

\[ z(\phi_n, \chi) = \int_{0}^{x_p} \phi_n(x) \chi(x) \, dx \]

\[ = \int_{0}^{x_p} p^n \chi(x) \chi(x)^{-1} (x) |x|_{p}^{-1} \, dx \]

\[ = p^n \int_{p^nZ_{p}^{-1}} \chi(x) |x|_{p}^{-1-s} \, dx \]

\[ = p^n \int_{p^nZ_{p}^{-1}} \chi(x) \, dx \]

\[ = p^n \int_{1+p^nZ_{p}} \chi(-x) \, dx \]

\[ = p^n \chi^{-1} x \int_{1+p^nZ_{p}} \chi(x) \, dx \]

\[ = p^n \chi^{-1} \int_{1+p^nZ_{p}} \chi(x) \, dx \]

\[ = p^n \chi^{-1} \int_{U_{p,n}} \chi(x) \, dx \]

\[ = p^n \chi^{-1} \frac{p^{1-n}}{p^{1-l}} \]

\[ = \frac{p}{p^{-l}} \chi^{-1}. \]

[Note: \( \chi(-1) = \pm 1 \):

\[ 1 = (-1)(-1) \Rightarrow 1 = \chi(-1)\chi(-1) = \chi(-1)^2 \].

Assembling the data then gives

\[ \rho(\chi) = \frac{z(\phi_n, \chi)}{z(\phi_n, \chi)} \]
9.

\[ \tau(\chi) \frac{p^{1+n(s-1)}}{p-1} \]
\[ = \frac{p}{p-1} \chi(-1) \]
\[ = \tau(\chi) \frac{p^{1+n(s-1)}}{p-1} \frac{p-1}{p\chi(-1)} \]
\[ = \tau(\chi) \chi(-1) p^{n(s-1)} \]
\[ = \tau(\chi) \chi(-1) p^{n(s-1)} \frac{1}{\chi} \]
\[ = \tau(\chi) \chi(-1) p^{n(s-1)} \frac{L(\chi)}{L(\chi)} . \]

11: THEOREM

\[ \rho(\chi) = \varepsilon(\chi) \frac{L(\chi)}{L(\chi)} , \]
where
\[ \varepsilon(\chi) = 1 \]
if \( \chi \) is unramified and
\[ \varepsilon(\chi) = \rho(\chi) \]
if \( \chi \) is ramified of degree \( n \geq 1 \).

12: LEMMA Suppose that \( \chi \) is ramified of degree \( n \geq 1 \) — then
\[ \varepsilon(\chi) \varepsilon(\chi) = \chi(-1) . \]

PROOF \( \forall f \in B(0_p) \),
10.

\[ Z(f, \chi) = \varepsilon(\chi) Z(\hat{f}, \chi) \]

\[ = \varepsilon(\chi) \varepsilon(\chi) \]

But \( \chi = \chi \), hence

\[ Z(\hat{f}, \chi) = \int \hat{f}(x) \chi(x) dx \]

\[ = \int \chi(-x) \chi(x) dx \]

\[ = \int \chi(x) \chi(-x) dx \]

\[ = \chi(-1) \int f(x) \chi(x) dx \]

\[ = \chi(-1) Z(f, \chi) . \]

13: APPLICATION

\[ \tau(\chi) \tau(\chi) = p^n \chi(-1). \]

[In fact,

\[ \varepsilon(\chi) \varepsilon(\chi) \]

\[ = \tau(\chi) p^{n(s-1)} \chi(-1) \tau(\chi) p^{n(1-s-1)} \chi(-1) \]

\[ = \tau(\chi) \tau(\chi) p^{-n} = \chi(-1) \]

\[ => \]

\[ \tau(\chi) \tau(\chi) = p^n \chi(-1). \]
14: **Lemma** Suppose that \( \chi \) is ramified of degree \( n \geq 1 \) -- then

\[
\varepsilon(\chi) = \chi(-1) \overline{\varepsilon(\chi)}.
\]

**Proof** \( \forall f \in B(\mathcal{O}_p), \)

\[
\begin{align*}
\hat{Z}(\hat{f}, \chi) &= \int_{\mathcal{O}_p^\times} \hat{f}(x) \chi(x) d^\times x \\
&= \int_{\mathcal{O}_p^\times} \hat{f}(-x) \chi(x) d^\times x \quad \text{(cf. 10.12)} \\
&= \int_{\mathcal{O}_p^\times} \hat{f}(x) \chi(-x) d^\times x \\
&= \chi(-1) \int_{\mathcal{O}_p^\times} \hat{f}(x) \chi(x) d^\times x \\
&= \chi(-1) \hat{Z}(\hat{f}, \chi).
\end{align*}
\]

But \( \chi = \overline{\chi} \), hence

\[
\begin{align*}
\overline{\hat{Z}(\hat{f}, \chi)} &= \hat{Z}(\hat{f}, \chi) \\
&= \varepsilon(\chi) \overline{\hat{Z}(\hat{f}, \chi)} \\
&= \varepsilon(\chi) \hat{Z}(\hat{f}, \chi) \\
&= \varepsilon(\chi) \chi(-1) \overline{\hat{Z}(\hat{f}, \chi)} \\
&= \varepsilon(\chi) \chi(-1) \overline{\hat{Z}(\hat{f}, \chi)}.
\end{align*}
\]

On the other hand,

\[
\begin{align*}
\overline{\hat{Z}(\hat{f}, \chi)} &= \overline{\varepsilon(\chi) \hat{Z}(\hat{f}, \chi)} \\
&= \overline{\varepsilon(\chi)} \overline{\hat{Z}(\hat{f}, \chi)}.
\end{align*}
\]
12.

Therefore

\[ \varepsilon(\chi)(-1) = \overline{\varepsilon(\chi)} \]

\[ \Rightarrow \]

\[ \varepsilon(\chi) = \chi(-1)\overline{\varepsilon(\chi)}. \]

15: APPLICATION

\[ \tau(\chi) = \chi(-1)\overline{\tau(\chi)}. \]

[In fact,]

\[ \varepsilon(\chi) = \tau(\chi)p^{n(s-1)}\chi(-1) \]

\[ = \chi(-1)\overline{\varepsilon(\chi)} \]

\[ = \chi(-1)\overline{\tau(\chi)}p^{n(s-1)}\chi(-1) \]

\[ = \chi(-1)\overline{\tau(\chi)}p^{n(s-1)}\chi(-1) \]

\[ \Rightarrow \]

\[ \tau(\chi) = \chi(-1)\overline{\tau(\chi)}. \]

16: DEFINITION Let \( \chi \in \hat{Z}_p^x \) be a nontrivial unitary character — then its root number \( W(\chi) \) is prescribed by the relation

\[ W(\chi) = \varepsilon(|.|_p^{1/2} \chi). \]

[Note: If \( \chi \) is trivial, then \( W(\chi) = 1 \).]

17: LEMMA

\[ |W(\chi)| = 1. \]
PROOF Put $\chi = |.\frac{1}{p} \chi \rangle$ then

$$\varepsilon(\chi)\varepsilon(\chi)^{-1} = \chi(-1) \quad (\text{cf. #12})$$

$$\Rightarrow$$

$$\varepsilon(\chi)^{-1} = \varepsilon(\chi)\chi(-1)^{-1}$$

$$= \varepsilon(\chi)\chi(-1)$$

$$= \varepsilon(\chi)\chi(-1) \quad (\chi = \chi)$$

$$= \chi(-1)\varepsilon(\chi)\chi(-1) \quad (\text{cf. #14})$$

$$= \chi(-1)^2\varepsilon(\chi)$$

$$= \varepsilon(\chi)$$

$$\Rightarrow$$

$$|\varepsilon(\chi)| = 1 \Rightarrow |\omega(\chi)| = 1.$$

17: APPLICATION

$$|\tau(\cdot,\frac{1}{p} \chi \rangle| = p^{n/2}.$$

[In fact,

$$1 = |\omega(\chi)| = |\tau(\cdot,\frac{1}{p} \chi \rangle|p^{n(\frac{1}{2} - 1)}.$$

18: EXERCISE AD LIBITUM Show that the theory expounded above for $Q_p$ can be carried over to any finite extension $K$ of $Q_p$.  

§13. RESTRICTED PRODUCTS

Recall:

1: FACT Suppose that $X_i$ ($i \in I$) is a nonempty Hausdorff space — then the product $\prod_{i \in I} X_i$ is locally compact iff each $X_i$ is locally compact and all but a finite number of the $X_i$ are compact.

Let $X_i$ ($i \in I$) be a family of nonempty locally compact Hausdorff spaces and for each $i \in I$, let $K_i \subset X_i$ be an open-compact subspace.

2: DEFINITION The **restricted product**

$$\prod_{i \in I} (X_i; K_i)$$

consists of those $x = \{x_i\}$ in $\prod_{i \in I} X_i$ such that $x_i \in K_i$ for all but a finite number of $i \in I$.

3: N.B.

$$\prod_{i \in I} (X_i; K_i) = \bigcup_{S \subset I} \prod_{i \in S} U_i \times \prod_{i \in I \setminus S} K_i,$$

where $S \subset I$ is finite.

4: DEFINITION A **restricted open rectangle** is a subset of $\prod_{i \in I} (X_i; K_i)$ of the form

$$\prod_{i \in S} U_i \times \prod_{i \in I \setminus S} K_i,$$

where $S \subset I$ is finite and $U_i \subset X_i$ is open.
5: LEMMA The intersection of two restricted open rectangles is a restricted open rectangle.

Therefore the collection of restricted open rectangles is a basis for a topology on \( \prod_{i \in I} (X_i : K_i) \), the restricted product topology.

6: LEMMA If \( I \) is finite, then

\[
\prod_{i \in I} X_i = \prod_{i \in I} (X_i : K_i)
\]

and the restricted product topology coincides with the product topology.

7: LEMMA If \( I = I_1 \cup I_2 \), with \( I_1 \cap I_2 = \emptyset \), then

\[
\prod_{i \in I} (X_i : K_i) \approx \left( \prod_{i \in I_1} (X_i : K_i) \right) \times \left( \prod_{i \in I_2} (X_i : K_i) \right),
\]

the restricted product topology on the left being the product topology on the right.

8: LEMMA The inclusion \( \prod_{i \in I} (X_i : K_i) \rightarrow \prod_{i \in I} X_i \) is continuous but the restricted product topology coincides with the relative topology only if \( X_i = K_i \) for all but a finite number of \( i \in I \).

9: LEMMA \( \prod_{i \in I} (X_i : K_i) \) is a Hausdorff space.

PROOF Taking into account #8, this is because

1. A subspace of a Hausdorff space is Hausdorff;
2. Any finer topology on a Hausdorff space is Hausdorff.

10: LEMMA \( \prod_{i \in I} (X_i : K_i) \) is a locally compact Hausdorff space.
3.

PROOF Let \( x \in \prod_{i \in I} (X_i : K_i) \) -- then there exists a finite set \( S \subset I \) such that \( x_i \in K_i \) if \( i \notin S \). Next, for each \( i \in S \), choose a compact neighborhood \( U_i \) of \( x_i \).

This done, consider
\[
\prod_{i \in S} U_i \times \prod_{i \notin S} K_i,
\]
a compact neighborhood of \( x \).

From this point forward, it will be assumed that \( X_i = G_i \) is a locally compact abelian group and \( K_i \subset G_i \) is an open-compact subgroup.

11: NOTATION
\[
G = \prod_{i \in I} (G_i : K_i).
\]

12: LEMMA \( G \) is a locally compact abelian group.

Given \( i \in I \), there is a canonical arrow
\[
in_i : G_i \to G,
\]
namely
\[
x \mapsto (\ldots, 1, l, x, 1, l, \ldots).
\]

13: LEMMA \( in_i \) is a closed embedding.

PROOF Take \( S = \{i\} \) and pass to
\[
G_i \times \prod_{j \neq i} K_j,
\]
an open, hence closed subgroup of \( G \). The image \( in_i(G_i) \) is a closed subgroup of
4.

\[ G_i \times \prod_{j \neq i} K_j \]

in the product topology, hence in the restricted product topology.

Therefore \( G_i \) can be regarded as a closed subgroup of \( G \).

**14: LEMMA**

1. Let \( \chi \in \tilde{G} \) -- then \( \chi_i = \chi \circ \text{in}_i = \chi|_{G_i} \in \tilde{G}_i \) and \( \chi|_{K_i} = 1 \) for all but a finite number of \( i \in I \), so for each \( x \in G \),

\[ \chi(x) = \chi(\{x_i\}) = \prod_{i \in I} \chi_i(x_i). \]

2. Given \( i \in I \), let \( \chi_i \in \tilde{G}_i \) and assume that \( \chi_i|_{K_i} = 1 \) for all but a finite number of \( i \in I \) -- then the prescription

\[ \chi(x) = \chi(\{x_i\}) = \prod_{i \in I} \chi_i(x_i) \]

defines a \( \chi \in \tilde{G} \).

These observations also apply if \( \tilde{G} \) is replaced by \( \hat{G} \), in which case more can be said.

**15: THEOREM** As topological groups,

\[ \hat{G} = \prod_{i \in I} (\hat{G}_i : K_i^1). \]

[Note: Recall that

\[ K_i^1 = \{x_i \in \hat{G}_i : \chi_i|_{K_i} = 1\} \] (cf. §7, #32)

and a tacit claim is that \( K_i^1 \) is an open-compact subgroup of \( \hat{G} \). To see this,
quote §7, #34 to get

\[ \hat{K}_i \cong \hat{G}/K_i^1, \ K_i^1 \cong G/K_i. \]

Then

- \( K_i \) compact \( \Rightarrow \) \( \hat{K}_i \) discrete \( \Rightarrow \) \( \hat{G}/K_i^1 \) discrete \( \Rightarrow \) \( K_i^1 \) open

- \( K_i \) open \( \Rightarrow \) \( G/K_i \) discrete \( \Rightarrow \) \( G/K_i \) compact \( \Rightarrow \) \( K_i^1 \) compact.

Let \( \mu_i \) be the Haar measure on \( G_i \) normalized by the condition

\[ \mu_i(K_i) = 1. \]

16: LEMMA There is a unique Haar measure \( \mu_G \) on \( G \) such that for every finite subset \( S \subset I \), the restriction of \( \mu_G \) to

\[ G_S := \prod_{i \in S} G_i \times \prod_{i \notin S} K_i \]

is the product measure.

Suppose that \( f_i \) is a continuous, integrable function on \( G_i \) such that \( f_i |_{K_i} = 1 \) for all \( i \) outside some finite set and let \( f \) be the function on \( G \) defined by

\[ f(x) = f([x_i]) = \prod_{i} f_i(x_i). \]

Then \( f \) is continuous. Proof: The \( G_S \) are open and cover \( G \) and on each of them \( f \) is continuous.

17: LEMMA Let \( S \subset I \) be a finite subset of \( I \) -- then
\[ \int_{G} f(x) \, d\mu_{G}(x) = \prod_{i \in S} \int_{G_i} f_i(x_i) \, d\mu_{G_i}(x_i). \]

18: APPLICATION If

\[ \sup_{S} \prod_{i \in S} \int_{G_i} |f_i(x_i)| \, d\mu_{G_i}(x_i) < \infty, \]

then \( f \) is integrable on \( G \) and

\[ \int_{G} f(x) \, d\mu_{G}(x) = \prod_{i \in I} \int_{G_i} f_i(x_i) \, d\mu_{G_i}(x_i). \]

19: EXAMPLE Take \( f_i = \chi_{K_i} \) (which is continuous, \( K_i \) being open-compact) --

then \( \hat{f_i} = \chi_{K_i} \). Setting

\[ f = \prod_{i \in I} f_i, \]

it thus follows that \( \forall \chi \in \hat{G} \),

\[ \hat{f}(\chi) = \prod_{i \in I} \hat{f_i}(\chi_i). \]

Working within the framework of §7, #45, let \( \mu_{\hat{G}_i} \) be the Haar measure on \( \hat{G}_i \) per Fourier inversion.

20: LEMMA

\[ \mu_{\hat{G}_i}(K_i) = 1 \]

PROOF Since \( \chi_{K_i} \in \text{INV}(G_i), \forall x_i \in G_i, \)

\[ \chi_{K_i}(x_i) = \int_{\hat{G}_i} \hat{\chi}_{K_i}(x_i) \hat{x_i}(x_i) \, d\mu_{\hat{G}_i}(x_i). \]
Now set $x_i = 1$ to get

$$1 = \int_{K_i^1} \mu_{\hat{G}}(x_i) \, d\mu_{\hat{G}}(x_i).$$

Let $\mu_{\hat{G}}$ be the Haar measure on $\hat{G}$ constructed as in #16 (i.e., replace $G$ by $\hat{G}$, bearing in mind #20).

**21: LEMMA** $\mu_{\hat{G}}$ is the Haar measure on $\hat{G}$ figuring in Fourier inversion per $\mu_G$.

**PROOF** Take

$$f = \prod_{i \in I} f_i,$$

where $f_i = \chi_{K_i}$ (cf. #19) -- then

$$\hat{f}(x) = \int_{\hat{G}} \hat{f}(\chi) \overline{\chi(x)} \, d\mu_{\hat{G}}(\chi)$$

$$= \prod_{i \in I} \int_{G_i} \hat{f}_i(\chi_i) \overline{\chi_i(x_i)} \, d\mu_{\hat{G}}(x_i)$$

$$= \prod_{i \in I} f_i(x_i) = f(\{x_i\}) = f(x).$$
§14. ADELES AND IDELES

1. DEFINITION The set of finite adeles is the restricted product

\[ A_{\text{fin}} = \prod_p (\mathbb{Q}_p : \mathbb{Z}_p) . \]

2. DEFINITION The set of adeles is the product

\[ A = A_{\text{fin}} \times \mathbb{R} . \]

3. LEMMA \( A \) is a locally compact abelian group (under addition).

4. N.B. \( A \) is a subring of \( \prod_p \mathbb{Q}_p \times \mathbb{R} \).

The image of the diagonal map

\[ \mathbb{Q} \to \prod_p \mathbb{Q}_p \times \mathbb{R} \]

lies in \( A \), so \( \mathbb{Q} \) can be regarded as a subring of \( A \).

5. LEMMA \( \mathbb{Q} \) is a discrete subspace of \( A \).

PROOF To establish the discreteness of \( \mathbb{Q} \subset A \), one need only exhibit a neighborhood \( U \) of 0 in \( A \) such that \( \mathbb{Q} \cap U = \{0\} \). To this end, consider

\[ U = \prod_p \mathbb{Z}_p \times ] - \frac{1}{2} \frac{1}{2} [ . \]

If \( x \in \mathbb{Q} \cap U \), then \( |x|_p \leq 1 \forall p \). But \( \bigcap_p (\mathbb{Q} \cap \mathbb{Z}_p) = \mathbb{Z} \), so \( x \in \mathbb{Z} \). And further, \( |x|_\infty < \frac{1}{2} \), hence finally \( x = 0 \).

6. FACT Let \( G \) be a locally compact group and let \( \Gamma \subset G \) be a discrete
2.

subgroup -- then $\Gamma$ is closed in $G$ and $G/\Gamma$ is a locally compact Hausdorff space.

7: THEOREM The quotient $A/Q$ is a compact Hausdorff space.

PROOF Since $Q \subset A$ is a discrete subgroup, $Q$ must be closed in $A$ and the quotient $A/Q$ must be Hausdorff. As for the compactness, it suffices to show that the compact set $\prod_p \mathbb{Z}_p \times [0,1]$ contains a set of representatives of $A/Q$ because this implies that the projection

$$\prod_p \mathbb{Z}_p \times [0,1] \to A/Q$$

is surjective, hence that $A/Q$ is the continuous image of a compact set. So let $x \in A$ -- then there is a finite set $S$ of primes such that $p \notin S \Rightarrow x_p \in \mathbb{Z}_p$. For $p \in S$, write

$$x_p = f(x_p) + [x_p],$$

thus $[x_p] \in \mathbb{Z}_p$ and if $q \neq p$ is another prime,

$$|f(x_p)|_q = \left| \sum_{n=1} a_n p^n \right|_q \leq \sup\{|a_n p^n|_q\} \leq 1.$$

Agreeing to denote $f(x_p)$ by $r_p$, write

$$x = (x - r_p) + r_p.$$

Then $r_p$ is a rational number and per $x - r_p$, $S$ reduces to $S - \{p\}$. Proceed from here by iteration to get

$$x = y + r,$$

where $\forall p$, $y_p \in \mathbb{Z}_p$, and $r \in \mathbb{Q}$. At infinity,

$$x_\infty = y_\infty + r \quad (r_\infty = r)$$
and there is a unique \( k \in \mathbb{Z} \) such that

\[
y_\infty = (y_\infty - k) + k
\]

with \( 0 \leq y_\infty - k < 1 \). Accordingly,

\[
y = y + r = (y - k) + k + r.
\]

And

\[
\forall p, \ (y - k)_p = y_p - k = y_p - k \in \mathbb{Z}_p,
\]

while

\[
x_\infty = (y_\infty - k) + k + r.
\]

It therefore follows that \( x \) can be written as the sum of an element in 

\[
\prod_{p} \mathbb{Z}_p \times [0,1]
\]

and a rational number, the contention.

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8: DEFINITION The topological group \( \mathbb{A}/\mathbb{Q} \) is called the adele class group.

9: DEFINITION Let \( G \) be a locally compact group and let \( \Gamma \subset G \) be a discrete subgroup -- then a fundamental domain for \( G/\Gamma \) is a Borel measurable subset \( D \subset G \) which is a system of representatives for \( G/\Gamma \).

10: LEMA The set

\[
D = \prod_{p} \mathbb{Z}_p \times [0,1]
\]

is a fundamental domain for \( \mathbb{A}/\mathbb{Q} \).

PROOF The claim is that every \( x \in \mathbb{A} \) can be written uniquely as \( d + r \), where \( d \in D, \ r \in \mathbb{Q} \). The proof of \#7 settles existence, thus the remaining issue is uniqueness: \( d_1 + r_1 = d_2 + r_2 \Rightarrow d_1 = d_2, \ r_1 = r_2 \). To see this, consider
4.

\[ \rho = d_1 - d_2 = r_2 - r_1 \in (D-D) \cap Q. \]

- \forall p, \rho = \rho_p \in D_p - D_p = D_p = Z_p

\[ \Rightarrow \rho \in \cap_{p \in Q} (Q \cap Z_p) = Z. \]

- \rho = \rho_\infty \in D_\infty - D_\infty = ]-1,1[.

Therefore

\[ \rho \in Z \cap ]-1,1[ \Rightarrow \rho = 0. \]

11: REMARK Q is dense in \( A_{\text{fin}} \).

[The point is that Z is dense in \( \prod_p Z_p \).]

12: DEFINITION The set of \textit{finite ideles} is the restricted product

\[ I_{\text{fin}} = \prod_p (Q_p : Z_p^\times). \]

13: DEFINITION The set of \textit{ideles} is the product

\[ I = I_{\text{fin}} \times R^\times. \]

14: LEMMA I is a locally compact abelian group (under multiplication).

Algebraically, I can be identified with \( A^\times \) but there is a topological issue since when endowed with the relative topology, \( A^\times \) is not a topological group:

Multiplication is continuous but inversion is not continuous.

15: LEMMA Equip \( A \times A \) with the product topology and define

\[ \phi: I \to A \times A \]
by

\[ \phi(x) = (x, \frac{1}{x}). \]

Endow the image \( \phi(I) \) with the relative topology — then \( \phi \) is a topological isomorphism of \( I \) onto \( \phi(I) \).

The image of the diagonal map

\[ Q^\times \to \prod_p Q_p \times \mathbb{R}^\times \]

lies in \( I \), so \( Q^\times \) can be regarded as a subgroup of \( I \).

16: **Lemma** \( Q^\times \) is a discrete subspace of \( I \).

**Proof** \( Q \) is a discrete subspace of \( A \) (cf. §5), hence \( Q \times Q \) is a discrete subspace of \( A \times A \), hence \( \phi(Q^\times) \) is a discrete subspace of \( \phi(I) \).

Consequently, \( Q^\times \) is a closed subgroup of \( I \) and the quotient \( I/Q^\times \) is a locally compact Hausdorff space but, as opposed to the adelic situation, it is not compact (see below).

17: **Definition** The topological group \( I/Q^\times \) is called the idele class group.

18: **Notation** Given \( x \in I \), put

\[ |x|_A = \prod_{p < \infty} |x_p|_p. \]

Extend the definition of \( |\cdot|_A \) to all of \( A \) by setting \( |x|_A = 0 \) if \( x \in A - A^\times \).

19: **Lemma** \( \forall x \in Q^\times, \ |x|_A = 1 \) (cf. §1, #21).
20: **Lemma** The homomorphism

\[ |\cdot|_A : I \to \mathbb{R}^x_{>0} \]

is continuous and surjective.

**Proof** Omitting the verification of continuity, fix \( t \in \mathbb{R}^x_{>0} \) and let \( x \) be the idele specified by

\[ x_p = 1 \quad (p < \infty), \quad x_\infty = t. \]

Then \( |x|_A = t \).

21: **Scholium** The idele class group \( \mathbb{I}/\mathbb{Q}^x \) is not compact.

22: **Notation** Let

\[ I^1 = \ker |\cdot|_A. \]

23: **N.B.** \( x \in I^1 \Rightarrow x_\infty \in \mathbb{Q}^x \).

24: **Theorem** The quotient \( I^1/\mathbb{Q}^x \) is a compact Hausdorff space, in fact

\[ I^1/\mathbb{Q}^x \cong \prod_p \mathbb{Z}_p^x, \]

hence

\[ \prod_p \mathbb{Z}_p^x \times \{1\} \]

is a fundamental domain for \( I^1/\mathbb{Q}^x \).

**Proof** The arrow

\[ \prod_p \mathbb{Z}_p^x \to I^1/\mathbb{Q}^x \]

that sends \( x \) to \((x,1)\mathbb{Q}^x\) is an isomorphism of topological groups.
[In obvious notation, the inverse is the map

\[ x = (x_{\text{fin}}, x_{\infty}) + \frac{1}{x_{\infty}} x_{\text{fin}}. \]

25: REMARK \( \forall p, Z_p^x \) is totally disconnected. But a product of totally disconnected spaces is totally disconnected, thus \( \prod_p Z_p^x \) is totally disconnected, thus \( L^{1}/Q^x \) is totally disconnected.

26: N.B. \( \prod_p Z_p^x \times R_{>0}^x \) is a fundamental domain for \( L/Q^x \).

[Note: If \( r \in Q \) and if \( |r|_p = 1 \forall p \), then \( r = \pm 1 \).]

27: LEMMA

\[ I = L^{1} \times R_{>0}^x. \]

PROOF The arrow

\[ I \rightarrow L^{1} \times R_{>0}^x \]

that sends \( x \) to \( (\tilde{x}, |x|_A) \), where

\[
(\tilde{x})_p = \begin{cases} x_p & (p < \infty) \\ x_{\infty} & (p = \infty), \end{cases}
\]

is an isomorphism of topological groups.

28: LEMMA There is a disjoint decomposition
\[ I_{\text{fin}} = \prod_{q \in Q_{>0}^x} q(\prod_{p} Z_p^x). \]

PROOF The right hand side is obviously contained in the left hand side. To go the other way, fix an \( x \in I_{\text{fin}} \) -- then \( |x|_A \in Q_{>0}^x \). Moreover, \( |x|_A x \in I_{\text{fin}} \) and

\[ \forall p, \quad |x|_{A_p} = 1 \quad \text{(for } x_p = p^k u \ (u \in Z_p^x) \Rightarrow |x|_A = p^{-k} r \ (r \in Q^x, \ r \text{ coprime to } p)\), \]

hence

\[ |x|_{A_p} \in \prod_{p} Z_p^x. \]

Now write

\[ x = |x|^{-1}_A (|x|_A x) \]

to conclude that

\[ x \in \prod_{p} Z_p^x \quad (q = |x|^{-1}_A). \]

29: LEMMA There is a disjoint decomposition

\[ I_{\text{fin}} \cap \prod_{p} Z_p^x = \bigcup_{n \in N} \prod_{p} Z_p^x. \]

Normalize the Haar measure \( d^x \) on \( I_{\text{fin}} \) by assigning the open-compact subgroup \( \prod_{p} Z_p^x \) total volume 1.

30: EXAMPLE Suppose that \( \text{Re}(s) > 1 \) -- then

\[ \int_{I_{\text{fin}} \cap \prod_{p} Z_p^x} |x|_A^s d^x. \]
\[ = \sum_{n \in \mathbb{N}} \int_{n(\prod_p \mathbb{Z}_p)} |x|_A^s d^x \]

\[ = \sum_{n \in \mathbb{N}} \int_{\prod_p \mathbb{Z}_p} |nx|_A^s d^x \]

\[ = \sum_{n \in \mathbb{N}} n^{-s} \text{vol}_{d^x} (\prod_p \mathbb{Z}_p) \]

\[ = \sum_{n \in \mathbb{N}} n^{-s} = \zeta(s). \]

[Note: Let \( x \in \prod_p \mathbb{Z}_p \):

\[ \Rightarrow \forall p, \ |x_p|_p = 1 \]

\[ \Rightarrow |nx|_A = \prod_p |nx_p|_p \]

\[ = \prod_p |n|_p \cdot |x_p|_p \]

\[ = \prod_p |n|_p \]

\[ = \prod_p |n|_p \cdot n \cdot \frac{1}{n} \]

\[ = 1 \cdot \frac{1}{n} = n^{-1}. \]

The idelic absolute value \(|\cdot|_A\) can be interpreted measure theoretically.
31: NOTATION Write

$$dx_A = \prod_{p \leq \infty} dx_p$$

for the Haar measure $\mu_A$ on $A$ (cf. §13, #16).

Consider a function of the form $f = \prod_{p \leq \infty} f_p$, where $\forall p$, $f_p$ is a continuous, integrable function on $Q_p$, and for all but a finite number of $p$, $f_p = \chi_{\mathbb{Z}_p}$ -- then

$$\int_A f(x) dx_A = \prod_{p \leq \infty} \int_{Q_p} f_p(x_p) dx_p$$

(cf. §13, #18),

it being understood that $Q_\infty = \mathbb{R}$.

32: LEMMA Let $M \subset A$ be a Borel set with $0 < \mu_A(M) < \infty$ -- then $\forall x \in I$,

$$\frac{\mu_A(xM)}{\mu_A(M)} = |x|_A.$$  

PROOF Take $M = D = \prod_p \mathbb{Z}_p \times [0,1[$ (cf. #10):

$$\mu_A(xM) = \prod_p \mu_{Q_p}(x_p \mathbb{Z}_p) \times \mu_{\mathbb{R}}(\mathbb{x}_\infty[0,1[)$$

$$= \prod_p |x_p|_p \mu_{Q_p}(\mathbb{Z}_p) \times |x_\infty|_\infty \mu_{\mathbb{R}}([0,1[)$$

$$= \prod_p |x_p|_p \times |x_\infty|_\infty$$

$$= \prod_{p \leq \infty} |x_p|_p = |x|_A.$$  

[Note: Needless to say, multiplication by an idele $x$ is an automorphism of $A$, thus transforms $\mu_A$ into a positive constant multiple of itself, the multiplier being $|x|_A$.]

§15. GLOBAL ANALYSIS

By definition,

$$A = A_{\text{fin}} \times \mathbb{R}.$$ 

Therefore

$$\hat{A} \simeq \hat{A}_{\text{fin}} \times \mathbb{R}. $$

And

$$A_{\text{fin}} = \prod_p (\mathbb{Q}_p : \mathbb{Z}_p) \implies \hat{A}_{\text{fin}} = \prod_p (\hat{\mathbb{Q}}_p : \hat{\mathbb{Z}}_p) \quad (\text{cf. §13, #15}).$$

Put

$$x_0 = \prod_{p \leq \infty} x_p,$$

where

$$\chi_0(x) = \exp(-2\pi\sqrt{-1}x) \quad (x \in \mathbb{R}) \quad (\text{cf. §8, #27}).$$

Then

$$\chi_0 \in \hat{A}.$$ 

Given $t \in A$, define $\chi_{0,t} \in \hat{A}$ by the rule

$$\chi_{0,t}(x) = \chi_0(tx).$$

Then the arrow

$$\Xi_0 : A \to \hat{A}$$

that sends $t$ to $\chi_{0,t}$ is an isomorphism of topological groups (cf. §8, #24).
Recall now that \( \forall q \in \mathbb{Q} \),

\[
\chi_q(q) = 1 \quad \text{(cf. §8, #28)}.
\]

Accordingly, \( \chi_q \) passes to the quotient and defines a unitary character of the adele class group \( \mathbb{A}/\mathbb{Q} \). So, \( \forall q \in \mathbb{Q}, \chi_q,q \) is constant on the cosets of \( \mathbb{A}/\mathbb{Q} \), thus it too determines an element of \( \mathbb{A}/\mathbb{Q} \).

Equip \( \mathbb{Q} \) with the discrete topology.

1: **THEOREM** The induced map

\[
\varepsilon_q : \mathbb{Q} : q \rightarrow \mathbb{A}/\mathbb{Q}
\]

\[
q \mapsto \chi_q,q
\]

is an isomorphism of topological groups.

**PROOF** Form \( \mathbb{Q}^{\perp} \subset \mathbb{A}^{\perp} \), the closed subgroup of \( \hat{\mathbb{A}} \) consisting of those \( \chi \) that are trivial on \( \mathbb{Q} \) -- then \( \mathbb{Q} \subset \mathbb{Q}^{\perp} \) and \( \mathbb{A}/\mathbb{Q} \simeq \mathbb{Q}^{\perp} \). But \( \mathbb{A}/\mathbb{Q} \) is compact, thus its unitary dual \( \hat{\mathbb{A}/\mathbb{Q}} \) is discrete, thus \( \mathbb{Q}^{\perp} \) is discrete. The quotient \( \mathbb{Q}^{\perp}/\mathbb{Q} \subset \mathbb{A}/\mathbb{Q} \) (\( \mathbb{A} \simeq \hat{\mathbb{A}} \)) is therefore discrete and closed, hence discrete and compact, hence finite. But \( \mathbb{Q}^{\perp}/\mathbb{Q} \) is a \( \mathbb{Q} \)-vector space, so \( \mathbb{Q}^{\perp}/\mathbb{Q} = \{0\} \) or still, \( \mathbb{Q}^{\perp} = \mathbb{Q} \), which implies that \( \mathbb{Q} \simeq \hat{\mathbb{A}/\mathbb{Q}} \).

2: **N.B.** There are two points of detail that have been tacitly invoked in the foregoing derivation.

- \( \mathbb{Q}^{\perp}/\mathbb{Q} \) in the quotient topology is discrete. **Reason:** Let \( S \) be an arbitrary nonempty subset of \( \mathbb{Q}^{\perp}/\mathbb{Q} \), say \( S = \{xQ : x \in U\} \), \( U \) a subset of \( \mathbb{Q}^{\perp} \) -- then \( U \) is automatically open (\( \mathbb{Q}^{\perp} \) being discrete), thus by the very definition of the quotient
3. topology, $S$ is an open subset of $Q^\perp/Q$.

- The quotient $Q^\perp/Q$ is closed in $A/Q$. Reason: $Q^\perp$ is a closed subgroup of $A$ containing $Q$, so the following generality is applicable: If $G$ is a topological group, if $H$ is a subgroup of $G$, if $F$ is a closed subgroup of $G$ containing $H$, then $\pi(F)$ is closed in $G/H$ ($\pi: G \rightarrow G/H$ the projection).

3: SCHOLIUM

$$Q \cong \hat{A}/\hat{Q} \Rightarrow \hat{Q} \cong \hat{A}/\hat{Q} \cong A/Q.$$  

[Note: Bear in mind that $Q$ carries the discrete topology.]

4: DISCUSSION Explicated, if $\chi \in \hat{Q}$, then there exists a $t \in A$ such that $\chi = \chi_{Q,t}$ and $\chi_{Q,t_1} = \chi_{Q,t_2}$ iff $t_1 - t_2 \in Q$.

5: DEFINITION The Bruhat space $B(A_{\text{fin}})$ consists of all finite linear combinations of functions of the form

$$f = \prod_{p} f_p$$

where $\forall p, f_p \in B(Q_p)$ and $f_p = \chi_{Z_p}$ for all but a finite number of $p$.

6: DEFINITION The Bruhat-Schwartz space $B_\infty(A)$ consists of all finite linear combinations of functions of the form

$$f = \prod_{p} f_p \times f_\infty,$$

where

$$\prod_{p} f_p \in B(A_{\text{fin}}) \text{ and } f_\infty \in S(R).$$
Given an \( f \in \mathcal{B}_\infty(A) \), its Fourier transform is the function \( \hat{f}: A \to \mathbb{C} \) defined by the rule

\[
\hat{f}(t) = \int_A f(x) \chi_{Q, t}(x) \, d\mu_A(x)
\]

\[
= \int_A f(x) \chi_Q(tx) \, d\mu_A(x).
\]

**7: LEMMA If**

\[
f = \prod_p f_p \times f_\infty
\]

is a Bruhat-Schwartz function, then

\[
\hat{f} = \prod_p \hat{f}_p \times \hat{f}_\infty.
\]

**8: REMARK** \( \hat{f}_p \) is computed per §10, #11 but \( \hat{f}_\infty \) is computed per

\[
\chi_\infty(x) = \exp(-2\pi \sqrt{-1} x),
\]

meaning that the sign convention here is the opposite of that laid down in §10 (a harmless deviation).

**9: APPLICATION**

\[f \in \mathcal{B}_\infty(A) \Rightarrow \hat{f} \in \mathcal{B}_\infty(A)\quad (\text{cf. } \S 10, \#16).

**10: N.B.** It is clear that

\[
\mathcal{B}_\infty(A) \subset \text{INV}(A)
\]

and \( \forall f \in \mathcal{B}_\infty(A) \),

\[
\hat{f}(x) = f(-x) \quad (x \in A).
\]
5.

11: LEMMA Given $f \in \mathcal{B}_∞(A)$, the series

$$\sum_{r \in \mathbb{Q}} f(x+r), \sum_{q \in \mathbb{Q}} \hat{f}(x+q)$$

are absolutely and uniformly convergent on compact subsets of $A$.

12: POISSON SUMMATION FORMULA Given $f \in \mathcal{B}_∞(A)$,

$$\sum_{r \in \mathbb{Q}} f(r) = \sum_{q \in \mathbb{Q}} \hat{f}(q).$$

The proof is not difficult but there are some measure-theoretic issues to be dealt with first.

On general grounds,

$$\int_A = \int_{A/Q} \sum_{Q}$$

(cf. §6, #11).

Here the integral $\int_A$ is with respect to the Haar measure $\mu_A$ on $A$ (cf. §14, #31).

Taking $\mu_Q$ to be counting measure, this choice of data fixes the Haar measure $\mu_{A/Q}$ on $A/Q$.

[Note: The restriction of $\mu_A$ to the fundamental domain

$$D = \prod_{\mathfrak{P}} \mathbb{Z}_p \times [0,1[$$

for $A/Q$ (cf. §14, #10) determines $\mu_{A/Q}$ and

$$1 = \mu_A(D) = \mu_{A/Q}(A/Q).$$

If $\phi:Q \to C$, then $\hat{\phi}:\hat{Q} \to C$, i.e., $\hat{\phi}:A/Q \to C$ or still,

$$\hat{\phi}(\chi) = \sum_{r \in \mathbb{Q}} \phi(r) \chi(r).$$
Specialize and suppose that \( \phi \) is the characteristic function of \( \{0\} \), so \( \forall \chi \),

\[
\hat{\phi}(\chi) = \chi(0) = 1.
\]

Therefore \( \hat{\phi} \) is the constant function 1 on \( A/Q \). Pass now to \( \hat{\phi} \), thus \( \hat{\phi} : A/Q \rightarrow \mathbb{C} \)
or still,

\[
\hat{\phi}(x_Q,q) = \int_{A/Q} \phi(x) \chi_{Q,q}(x) d\mu_{A/Q}(x)
\]

\[
= \int_{A/Q} \chi_{Q,q}(x) d\mu_{A/Q}(x)
\]

which is 1 if \( q = 0 \) and is 0 otherwise (cf. §7, #46 \( A/Q \) is compact)), hence

\( \hat{\phi} = \phi \). But \( \phi(r) = \phi(-r) \), thereby leading to the conclusion that the Haar measure

\( \mu_{A/Q} \) on \( A/Q \) is the one singled out by Fourier inversion (cf. §7, #45).

Summary: Per Fourier inversion,

- \( \mu_Q \) is paired with \( \mu_{A/Q} \).
- \( \mu_{A/Q} \) is paired with \( \mu_Q \).

Given \( f \in B_c(A) \), put

\[
F(x) = \sum_{r \in \mathbb{Q}} f(x+r).
\]

Then \( F \) lives on \( A/Q \), so \( \hat{F} \) lives on \( \widehat{A/Q} \approx \mathbb{Q} \):

\[
\hat{F}(q) = \int_{A/Q} F(x) \chi_{Q,q}(x) d\mu_{A/Q}(x)
\]

\[
= \int_{A/Q} F(x) \chi_{Q,q}(x) d\mu_{A/Q}(x)
\]

On the other hand,

\[
\hat{F}(q) = \int_A f(x) \chi_{Q,q}(x) d\mu_A(x)
\]
\[ \begin{align*}
= & \int_A f(x) \chi_Q(qx) \, d\mu_A(x) \\
= & \int_{A/Q} \left( \sum_{r \in Q} f(x+r) \chi_Q(q(x+r)) \right) \, d\mu_{A/Q}(x) \\
= & \int_{A/Q} \left( \sum_{r \in Q} f(x+r) \chi_Q(qx+qr) \right) \, d\mu_{A/Q}(x) \\
= & \int_{A/Q} \left( \sum_{r \in Q} f(x+r) \chi_q(qx) \chi_q(qr) \right) \, d\mu_{A/Q}(x) \\
= & \int_{A/Q} \{ f(x) \chi_Q(qx) \} \, d\mu_{A/Q}(x) \\
= & \hat{F}(q) .
\end{align*} \]

To finish the proof, per Fourier inversion, write

\[ F(x) = \sum_{q \in Q} \hat{F}(q) \chi_Q(qx) \]

and then put \( x = 0 \):

\[ F(0) = \sum_{r \in Q} f(r) = \sum_{q \in Q} \hat{F}(q) = \sum_{q \in Q} \hat{f}(q) . \]

13: THEOREM Let \( x \in I \) -- then \( \forall f \in B_\infty(A) \),

\[ \sum_{r \in Q} f(rx) = \frac{1}{|x|_A} \sum_{q \in Q} \hat{f}(qx^{-1}) . \]

PROOF Work with \( f_x \in B_\infty(A) \) (\( f_x(y) = f(xy) \)):

\[ \sum_{r \in Q} f_x(r) = \sum_{q \in Q} \hat{f}_x(q) . \]
But

\[ \hat{f}_x(q) = \int_A f_x(y) \chi_{Q, q}(y) \, d\mu_A(y) \]

\[ = \int_A f_x(y) \chi_{Q}(qy) \, d\mu_A(y) \]

\[ = \int_A f(xy) \chi_{Q}(qey^{-1}y) \, d\mu_A(y) \]

\[ = \frac{1}{|x|} \int_A f(y) \chi_{Q}(qy^{-1}y) \, d\mu_A(y) \]

\[ = \frac{1}{|x|} \hat{f}(qy^{-1}). \]


§16. FUNCTIONAL EQUATIONS

Let

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\Re(s) > 1) \]

be the Riemann zeta function — then \( \zeta(s) \) can be meromorphically continued into the whole \( s \)-plane with a simple pole as \( s = 1 \) and satisfies there the functional equation

\[ \pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma((1-s)/2) \zeta(1-s). \]

1. REMARK The product \( \pi^{-s/2} \Gamma(s/2) \) was denoted by \( \Gamma_{R}(s) \) in §11, #8.

There are many proofs of the functional equation satisfied by \( \zeta(s) \). Of these, we shall single out two, one "classical", the other "modern".

To proceed in the classical vein, start with

\[ \Gamma(s) = \int_{0}^{\infty} e^{-x}s \frac{dx}{x} \quad (\Re(s) > 1). \]

Then by change of variable,

\[ \pi^{-s/2} \Gamma(s/2)n^{-s} = \int_{0}^{\infty} e^{-n\pi x}x^{s/2} \frac{dx}{x}. \]

So, upon summing from \( n = 1 \) to \( \infty \):

\[ \pi^{-s/2} \Gamma(s/2) \zeta(s) = \int_{0}^{\infty} \psi(x)x^{s/2} \frac{dx}{x}, \]

where

\[ \psi(x) = \sum_{n=1}^{\infty} e^{-n^{2} \pi x}. \]

Put now

\[ \theta(x) = 1 + 2\psi(x) = \sum_{n \in \mathbb{Z}} e^{-n^{2} \pi x}. \]
2. \textbf{LEMMA} \hspace{1cm} 

\[ \theta \left( \frac{1}{x} \right) = \sqrt{x} \theta (x). \]

Therefore

\[ \psi \left( \frac{1}{x} \right) = - \frac{1}{2} + \frac{1}{2} \theta \left( \frac{1}{x} \right) \]

\[ = - \frac{1}{2} + \frac{\sqrt{x}}{2} \theta (x) \]

\[ = - \frac{1}{2} + \frac{\sqrt{x}}{2} + \sqrt{x} \psi (x). \]

One may then write

\[ \pi^{-s/2} \Gamma(s/2) \zeta(s) = \int_{0}^{\infty} \psi (x) x^{s/2} \, \frac{dx}{x} \]

\[ = \int_{0}^{1} \psi (x) x^{s/2} \, \frac{dx}{x} + \int_{1}^{\infty} \psi (x) x^{s/2} \, \frac{dx}{x} \]

\[ = \int_{1}^{\infty} \psi \left( \frac{1}{x} \right) x^{-s/2} \, \frac{dx}{x} + \int_{1}^{\infty} \psi (x) x^{s/2} \, \frac{dx}{x} \]

\[ = \int_{1}^{\infty} \left( - \frac{1}{2} + \frac{\sqrt{x}}{2} + \sqrt{x} \psi (x) \right) x^{-s/2} \, \frac{dx}{x} + \int_{1}^{\infty} \psi (x) x^{s/2} \, \frac{dx}{x} \]

\[ = \frac{1}{s-1} - \frac{1}{s} + \int_{1}^{\infty} \psi (x) \left( x^{s/2} + x^{(1-s)/2} \right) \, \frac{dx}{x}. \]

The last integral is convergent for all values of \( s \) and thus defines a holomorphic function. Moreover, the last expression is unchanged if \( s \) is replaced by \( 1 - s \). I.e.:

\[ \pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma((1-s)/2) \zeta(1-s). \]
The modern proof of this relation uses the adele-idele machinery.

Thus let

\[ \phi(x) = e^{-\pi x^2} \prod_{p} \chi_{\mathbb{Z}_p}(x_p)(x \in A). \]

Then if \( \text{Re}(s) > 1 \),

\[
\int_{I} \phi(x) |x|^s d^x x
\]

\[
= \int_{R^x} \lim_{t \to \infty} t^s \frac{dt}{t} \cdot \prod_{p} \int_{Q_p^x} \chi_{\mathbb{Z}_p}(x_p)|x_p|^s d^x x_p
\]

\[
= \pi^{-s/2} \Gamma(s/2) \cdot \prod_{p} \int_{p \cdot (0)} \chi_{\mathbb{Z}_p}(x_p)|x_p|^s d^x x_p
\]

\[
= \pi^{-s/2} \Gamma(s/2) \cdot \prod_{p} \frac{1}{1-p^{-s}} \quad (\text{cf. } \S 6, \#26)
\]

\[
= \pi^{-s/2} \Gamma(s/2) \zeta(s).
\]

To derive the functional equation, we shall calculate the integral

\[
\int_{I} \phi(x) |x|^s d^x x
\]

in another way. To this end, put

\[ D^x = \prod_{p} \mathbb{Z}_p^x \times \mathbb{R}_0^x, \]

a fundamental domain for \( I/Q^x \) (cf. \S 14, \#26), so

\[ I = \bigcup_{r \in Q^x} rD^x \quad (\text{disjoint union}). \]

Therefore

\[
\int_{I} \phi(x) |x|^s d^x x
\]
4.

\[ \sum_{r \in Q^\times} \int_{D^x} \phi(x) |x| A d^x x \]

\[ = \int_{D^x} \sum_{\tilde{r} \in Q^\times} \phi(\tilde{r}x) |x| A d^x x \]

\[ = \int_{D^x} \sum_{|x|_A \leq 1} \phi(\tilde{r}x) |x| A d^x x \]

\[ + \int_{D^x} \sum_{|x|_A \geq 1} \phi(\tilde{r}x) |x| A d^x x. \]

To proceed further, recall that \( \hat{\phi} = \phi \) (\( \Rightarrow \hat{\phi}(0) = \phi(0) = 1 \)), hence (cf. \S 15, #13)

\[ 1 + \sum_{r \in Q^\times} \phi(rx) = \frac{1}{|x|_A} + \frac{1}{|x|_A} \sum_{\tilde{q} \in Q^{-1}} \phi(q^{-1}x). \]

Accordingly,

\[ \int_{D^x} \sum_{|x|_A \leq 1} \phi(rx) |x| A d^x x \]

\[ = \int_{D^x} (\frac{1}{|x|_A} + \sum_{|x|_A} \phi(q^{-1}) |x| A d^x x \]

\[ = \int_{D^x} (|x|_A^{s-1} - |x|_A^s) d^x x + \int_{D^x} \sum_{|x|_A \geq 1} \phi(qx) |x| A d^x x. \]
5.

But

\[ \int_{D_x} (|x|_A^{s-1} - |x|_A^s) \, dx \]

\[ |x|_A \leq 1 \]

\[ = \int_0^1 (t^{s-1} - t) \, \frac{dt}{t} = \frac{1}{s-1} - \frac{1}{s}. \]

So, upon assembling the data, we conclude that

\[ \int_D \phi(x) |x|_A^{s-1} \, dx \]

\[ = \frac{1}{s-1} - \frac{1}{s} + \int_{D_x} \sum_{q \in \mathbb{Q}^x} \phi(qx) \, \left( |x|_A^s + |x|_A^{1-s} \right) \, dx. \]

Since the second expression is invariant under the transformation \( s \to 1-s \), the functional equation for \( \zeta(s) \) follows once again.

3: REMARK Consider

\[ \int_{D_x} \sum_{q \in \mathbb{Q}^x} \phi(qx) \]

Then from the definitions,

\[ x \in D^x \Rightarrow x_p \in \mathbb{Z}_p^x \] \& \( qx_p \in \mathbb{Z}_p \)

\[ \Rightarrow q \in \mathbb{Z}. \]

Matters thus reduce to

\[ 2 \int_{2}^{\infty} \sum_{n=1}^{\infty} e^{-n^2 \pi t^2} \left( t^s + t^{1-s} \right) \frac{dt}{t} \]

or still,

\[ \int_1^{\infty} \psi(t) \left( t^{s/2} + t^{(1-s)/2} \right) \frac{dt}{t}, \]

the classical expression.
1.

§17. GLOBAL ZETA FUNCTIONS

Structurally, there is a short exact sequence

\[ 1 \to I^{1/\mathbb{Q}^\times} \to I/\mathbb{Q}^\times \to R^\times_{>0} \to 1 \quad \text{(cf. §14, #27)} \]

and \( I^{1/\mathbb{Q}^\times} \) is compact (cf. §14, #24).

1: DEFINITION Given \( f \in \mathcal{B}_\infty(A) \) and a unitary character \( \omega: I/\mathbb{Q}^\times \to T \), the global zeta function attached to the pair \((f, \omega)\) is

\[
Z(f, \omega, s) = \int_I f(x) \omega(x) |x|_A^{s} \, dx \quad \text{(Re}(s) > 1).
\]

2: EXAMPLE In the notation of §16, take

\[
f(x) = \phi(x) = e^{-\pi x^2} \prod_p \chi_p(x_p) \quad (x \in A)
\]

and let \( \omega = 1 \) -- then as shown there

\[
Z(f, 1, s) = \pi^{-s/2} \Gamma(s/2)\zeta(s).
\]

3: LEMMA \( Z(f, \omega, s) \) is a holomorphic function of \( s \) in the strip \( \text{Re}(s) > 1 \).

4: THEOREM \( Z(f, \omega, s) \) can be meromorphically continued into the whole \( s \)-plane and satisfies the functional equation

\[
Z(f, \omega, s) = Z(\hat{f}, \overline{\omega}, 1-s).
\]

[Note:

\[
f \in \mathcal{B}_\infty(A) \Rightarrow \hat{f} \in \mathcal{B}_\infty(A) \quad \text{(cf. §15, #9)}.
\]

The proof is a computation, albeit a lengthy one.
To begin with,
\[ I \approx R_{>0}^x \times I^1 \]  (cf. §14, #27).

Therefore
\[
Z(f, \omega, s) = \int_I f(x) \omega(x) |x|^s_A d^x
\]
\[
= \int_{R_{>0}^x \times I^1} f(tx) \omega(tx) |tx|^s_A \frac{dt}{t} d^x
\]
\[
= \int_0^\infty \left( \int_{I^1} f(tx) \omega(tx) |tx|^s_A d^x \right) \frac{dt}{t}.
\]

5: NOTATION Put
\[
Z_{\mu}(f, \omega, s) = \int_{I^1} f(tx) \omega(tx) |tx|^s_A d^x.
\]

6: LEMMA
\[
Z_{\mu}(f, \omega, s) + \hat{f}(0) \int_{I^1/Q^x} \omega(tx) |tx|^s_A d^x
\]
\[
= Z_{\mu}(\hat{f}, \tilde{\omega}, 1-s) + \hat{f}(0) \int_{I^1/Q^x} \tilde{\omega}(t^{-1}x) |t^{-1}x|^1 A d^x.
\]

PROOF Write
\[
\int_{I^1} f(tx) \omega(tx) |tx|^s_A d^x
\]
\[
= \int_{I^1/Q^x} \left( \sum_{r \in Q^x} f(\rho tx) \omega(\rho tx) |\rho tx|^s_A d^x \right)
\]
\[
= \int_{I^1/Q^x} \left( \sum_{r \in Q^x} f(\rho tx) \omega(tx) |tx|^s_A d^x \right).
\]
Then

\[
Z_t(f, \omega, s) + f(0) \int_{I^{1/\mathbb{Q}^x}} \omega(tx)|tx|_A^s d^x
\]

\[
= \int_{I^{1/\mathbb{Q}^x}} (\sum_{r \in \mathbb{Q}} f(rt)) \omega(tx)|tx|_A^s d^x
\]

\[
= \int_{I^{1/\mathbb{Q}^x}} \left( \sum_{q \in \mathbb{Q}} \hat{f}(qt^-1x^-1) \right) \omega(tx)|tx|_A^s d^x \quad \text{(cf. §15, #13)}
\]

\[
= \int_{I^{1/\mathbb{Q}^x}} \left( \sum_{q \in \mathbb{Q}} \hat{f}(qt^-1x^-1) \right) \bar{\omega}(t^-1x)|t^-1x|_A^{1-s} d^x \quad (x + x^-1)
\]

\[
= \int_{I^{1/\mathbb{Q}^x}} \left( \sum_{q \in \mathbb{Q}} \hat{f}(qt^-1x^-1) \right) \omega(tx)|tx|_A^{1-s} d^x
\]

\[
= \int_{I^{1/\mathbb{Q}^x}} \left( \sum_{q \in \mathbb{Q}} \hat{f}(qt^-1x^-1) \bar{\omega}(qt^-1x)|qt^-1x|_A^{1-s} d^x \right.
\]

\[
\left. + \hat{f}(0) \int_{I^{1/\mathbb{Q}^x}} \bar{\omega}(t^-1x)|t^-1x|_A^{1-s} d^x \right)
\]

\[
= \int_{I^{1/\mathbb{Q}^x}} \hat{f}(t^-1x) \bar{\omega}(t^-1x)|t^-1x|_A^{1-s} d^x
\]

\[
\left. + \hat{f}(0) \int_{I^{1/\mathbb{Q}^x}} \bar{\omega}(t^-1x)|t^-1x|_A^{1-s} d^x \right)
\]

\[
= Z_{t^-1}(\hat{f}, \bar{\omega}, 1-s) + \hat{f}(0) \int_{I^{1/\mathbb{Q}^x}} \bar{\omega}(t^-1x)|t^-1x|_A^{1-s} d^x.
\]

Return to $Z(f, \omega, s)$ and break it up as follows:
4.

\[ Z(f, \omega, s) = \int_0^1 z_t(f, \omega, s) \, \frac{dt}{t} + \int_1^\infty z_t(f, \omega, s) \, \frac{dt}{t}. \]

7: **LEMMA** The integral

\[ \int_1^\infty z_t(f, \omega, s) \, \frac{dt}{t} \]

is a holomorphic function of \( s \).

[It can be expressed as]

\[ \int_1^\infty f(x)\omega(x) \left| x \right|_A \, dx. \]

This leaves

\[ \int_0^1 z_t(f, \omega, s) \, \frac{dt}{t}, \]

which can thus be represented as

\[ \int_0^1 (z_{-1} \hat{f}, \omega, 1-s) \]

\[ - f(0) \int_1^{1/Q} \omega(tx) \left| tx \right|_A \, dx \]

\[ + \hat{f}(0) \int_1^{1/Q} \omega(t^{-1}x) \left| t^{-1}x \right|_A \, dx \]

To carry out the analysis, subject

\[ \int_0^1 z_{-1} \hat{f}, \omega, 1-s) \, \frac{dt}{t} \]

to the change of variable \( t \rightarrow t^{-1} \), thereby leading to

\[ \int_1^\infty z_t(\hat{f}, \omega, 1-s) \, \frac{dt}{t}, \]

a holomorphic function of \( s \) (cf. #7 supra).
It remains to discuss

\[ R(f, \omega, s) = \int_0^1 (- f(0) \int_{I^{1/Q^X}} \omega(tx) |tx|_A^s \, dx + \hat{f}(0) \int_{I^{1/Q^X}} \bar{\omega}(t^{-1}x) |t^{-1}x|_A^{1-s} \, dx \, dt) \frac{dt}{t} \]

\[ = \int_0^1 (- f(0) \omega(t) |t| \int_{I^{1/Q^X}} \omega(x) \, dx + \hat{f}(0) \bar{\omega}(t^{-1}) |t^{-1}|^{1-s} \int_{I^{1/Q^X}} \bar{\omega}(x) \, dx \, dt) \frac{dt}{t} , \]

there being two cases.

1. \( \omega \) is nontrivial on \( I^1 \). Since \( I^{1/Q^X} \) is compact (cf. §14, #24), the integrals

\[ \int_{I^{1/Q^X}} \omega(x) \, dx, \int_{I^{1/Q^X}} \bar{\omega}(x) \, dx \]

must vanish (cf. §7, #46). Therefore \( R(f, \omega, s) = 0 \), hence

\[ Z(f, \omega, s) = \int_1^\infty Z_t(f, \omega, s) \frac{dt}{t} + \int_1^\infty Z_t(\hat{f}, \bar{\omega}, 1-s) \frac{dt}{t} , \]

a holomorphic function of \( s \).

2. \( \omega \) is trivial on \( I^1 \). Let \( \phi:R^{X}_{>0} + I/1^1 \) be the isomorphism per §14, #27 -- then \( \omega \circ \phi:R^{X}_{>0} + T \) is a unitary character of \( R^{X}_{>0} \), thus for some \( w \in R \), \( \omega \circ \phi = |.|^{-\sqrt{-1}} w \), so

\[ \omega = |.|^{-\sqrt{-1}} w \circ \phi^{-1} \Rightarrow \omega(x) = |x|_A^{-\sqrt{-1}} w . \]

Therefore

\[ R(f, \omega, s) = - f(0) \text{vol}(I^{1/Q^X}) \int_0^1 t^{-\sqrt{-1}} w + s-1 \, dt \]
6. 

\[ + \hat{f}(0) \text{vol}(I^1/Q^X) \int_0^1 t^{-\sqrt{-1}w} e^{s-2} \, dt \]

\[ = - f(0) \text{vol}(I^1/Q^X) + \hat{f}(0) \text{vol}(I^1/Q^X) \]

\[ - \sqrt{-1} w + s \]

\[ - \sqrt{-1} w + s - 1 \]

a meromorphic function that has a simple pole at 

\[ s = \sqrt{-1} w \text{ with residue } - f(0) \text{vol}(I^1/Q^X) \text{ if } f(0) \neq 0 \]

\[ s = \sqrt{-1} w + 1 \text{ with residue } \hat{f}(0) \text{vol}(I^1/Q^X) \text{ if } \hat{f}(0) \neq 0. \]

8: N.B. To explicate vol(I^1/Q^X), use the machinery of §16: In the notation of #2 above,

\[ Z(f,w,s) = - \frac{1}{s} + \frac{1}{s^{-1}} + \ldots \]

\[ \Rightarrow \text{vol}(I^1/Q^X) = 1. \]

[Note: Here, \( w = 0 \) and \( f(0) = 1, \hat{f}(0) = 1 \).]

That \( Z(f,\omega,s) \) can be meromorphically continued into the whole \( s \)-plane is now manifest. As for the functional equation, we have

\[ Z(f,\omega,s) = \int_1^{\infty} Z_t(f,\omega,s) \frac{dt}{t} \]

\[ + \int_1^{\infty} Z_t(\hat{f},\bar{\omega},1-s) \frac{dt}{t} \]

\[ + R(f,\omega,s) \]

\[ = \int_1^{\infty} (\int_1^{\infty} f(tx)\omega(tx) |tx|^{\omega} d^{\omega}x) \frac{dt}{t} \]

\[ + \int_1^{\infty} (\int_1^{\infty} \hat{f}(tx)\bar{\omega}(tx) |tx|^{1-s} d^{\omega}x) \frac{dt}{t} \]

\[ + R(f,\omega,s). \]
And we also have

\[ Z(f, \omega, 1-s) = \int_1^\infty z_t(f, \omega, 1-s) \, dt \]

\[ + \int_1^\infty z_t(f, \omega, 1 - (1-s)) \, dt \]

\[ + R(f, \omega, 1-s) \]

\[ = \int_1^\infty z_t(f, \omega, 1-s) \, dt \]

\[ + \int_1^\infty z_t(f, \omega, s) \, dt \]

\[ + R(f, \omega, 1-s) \]

\[ = \int_1^\infty \left( \int_1^\infty \hat{f}(tx) \omega(tx) |tx|^{1-s} \, dx \right) \, dt \]

\[ + R(f, \omega, 1-s). \]

The first of these terms can be left as is (since it already figures in the formula for \( Z(f, \omega, s) \)). Recalling that

\[ \hat{f}(x) = f(-x) \quad (x \in A) \quad (cf. \ \S 15, \ #10), \]

the second term becomes

\[ \int_1^\infty \left( \int_1^\infty f(-tx) \omega(tx) |tx|^s \, dx \right) \, dt \]
or still,

\[ \int_1^\infty \left( \int_1^1 f(tx) \omega(-tx) \left| \frac{tx}{A} \right|^s d^x x \right) \frac{dt}{t} \]

\[ = \int_1^\infty \left( \int_1^1 f(tx) \omega(-tx) \left| \frac{tx}{A} \right|^s d^x x \right) \frac{dt}{t} . \]

But by hypothesis, \( \omega \) is trivial on \( 0^x \), hence

\[ \omega(-tx) = \omega((-1)tx) = \omega(-1)\omega(tx) = \omega(tx) , \]

and we end up with

\[ \int_1^\infty \left( \int_1^1 f(tx) \omega(tx) \left| \frac{tx}{A} \right|^s d^x x \right) \frac{dt}{t} \]

which likewise figures in the formula for \( Z(f, \omega, s) \). Finally, if \( \omega \) is trivial on \( 1^1 \), then

\[
R(f, \bar{\omega}, 1-s) = - \frac{\hat{f}(0)}{\sqrt{-1} w + 1-s} + \frac{\hat{f}(0)}{\sqrt{-1} w + (1-s)-1}
\]

\[ = \frac{f(0)}{\sqrt{-1} w - s} - \frac{\hat{f}(0)}{\sqrt{-1} w + 1-s} \]

\[ = - \frac{f(0)}{\sqrt{-1} w + s} + \frac{\hat{f}(0)}{\sqrt{-1} w + s-1} \]

\[ = R(f, \omega, s) . \]

On the other hand, if \( \omega \) is nontrivial on \( 1^1 \), then \( \bar{\omega} \) is nontrivial on \( 1^1 \) and

\[ R(f, \omega, s) = 0, R(f, \bar{\omega}, 1-s) = 0. \]
18. LOCAL ZETA FUNCTIONS [BIS]

To be in conformity with the global framework laid down in §17, we shall reformulate the local theory of §11 and §12.

1: DEFINITION Given \( f \in S(\mathbb{R}) \) and a unitary character \( \omega: \mathbb{R}^\times \to T \), the local zeta function attached to the pair \((f, \omega)\) is

\[
Z(f, \omega, s) = \int_{\mathbb{R}^\times} f(x) \omega(x) |x|^s \, dx \quad (\Re(s) > 0).
\]

2: THEOREM There exists a meromorphic function \( \rho(\omega, s) \) such that \forall f,

\[
\rho(\omega, s) = \frac{Z(f, \omega, s)}{Z(f, \bar{\omega}, 1-s)}.
\]

Decompose \( \omega \) as a product:

\[
\omega(x) = (\text{sgn } x)^\sigma |x|^{-\sqrt{-1} w} \quad (\sigma \in \{0, 1\}, w \in \mathbb{R}).
\]

3: DEFINITION Write (cf. §11, #9)

\[
L(\omega, s) = \begin{cases} 
\Gamma_R (s - \sqrt{-1} w) & (\sigma = 0) \\
\Gamma_R (s - \sqrt{-1} w + 1) & (\sigma = 1).
\end{cases}
\]

4: FACT

\[
\rho(\omega, s) = \frac{L(\omega, s)}{L(\omega, 1-s)} \quad (\sigma = 0) \quad \rho(\omega, s) = -\sqrt{-1} \frac{L(\omega, s)}{L(\omega, 1-s)} \quad (\sigma = 1).
\]
5: REMARK The complex case can be discussed analogously but it will not be needed in the sequel.

6: DEFINITION Given \( f \in B(Q_{\mathbb{P}}) \) and a unitary character \( \omega: Q_{\mathbb{P}}^\times + 1 \), the local zeta function attached to the pair \( (f, \omega) \) is

\[
Z(f, \omega, s) = \int_{Q_{\mathbb{P}}^\times} f(x) \omega(x) |x|_p^s \, \, dx \quad (\text{Re}(s) > 0).
\]

7: THEOREM There exists a meromorphic function \( \rho(\omega, s) \) such that \( \forall f, \)

\[
\rho(\omega, s) = \frac{Z(f, \omega, s)}{Z(\hat{f}, \hat{\omega}, 1-s)}.
\]

Decompose \( \omega \) as a product:

\[
\omega(x) = \omega(x) |x|_p^{-\frac{1}{2}} w \quad (\omega \in Z_{\mathbb{P}}^{\times}, w \in R).
\]

8: DEFINITION Write (cf. §12, #8)

\[
L(\omega, s) = \begin{cases} 
(1 - \omega(p)p^{-s})^{-1} & (\omega = 1) \\
1 & (\omega \neq 1).
\end{cases}
\]

[Note: If \( \omega = 1 \), then

\[
\omega(p) = |p|_p^{-1} w = p^{-1} w.\]

9: FACT \( \omega = 1 \)

\[
\rho(\omega, s) = \frac{L(\omega, s)}{L(\hat{\omega}, 1-s)} = \frac{1 - \omega(p)p^{-(1-s)}}{1 - \omega(\hat{p})p^{-s}}.
\]
10: FACT \((\omega \neq 1)\)

\[
\rho(\omega, s) = \tau(\omega) \omega(-1) p^{n(s + \sqrt{-1} -1)},
\]

where

\[
\tau(\omega) = \sum_{i=1}^{r} \omega(e_i) \chi_p(p^{-n} e_i)
\]

and \(\deg \omega = n \geq 1\).

APPENDIX

It can happen that

\[
Z(f, \omega, s) \equiv 0.
\]

To illustrate, suppose that \(\omega(-1) = -1\) and \(f(x) = f(-x)\). Working with \(Q_p^x\) (the story for \(R^x\) being the same), we have

\[
Z(f, \omega, s) = \int_{Q_p^x} f(x) \omega(x) |x|^s \, d^x
\]

\[
= \int_{Q_p^x} f(-x) \omega(-x) |x|^s \, d^x
\]

\[
= \omega(-1) \int_{Q_p^x} f(x) \omega(x) |x|^s \, d^x
\]

\[
= -Z(f, \omega, s).
\]
Let \( \omega : \mathbb{I}/\mathbb{Q}^\times \to \mathbb{T} \) be a unitary character.

1: **LEMMA** There is a unique unitary character \( \omega \) of \( \mathbb{I}/\mathbb{Q}^\times \) of finite order and a unique real number \( w \) such that

\[
\omega = \frac{\omega}{\sqrt{-1}} w.
\]

[Note: To say that \( \omega \) is of finite order means that there exists a positive integer \( n \) such that \( \omega(x)^n = 1 \) for all \( x \in \mathbb{I} \).]

2: **N.B.**

\[
\omega = \prod_p \omega_p \times \omega_\infty,
\]

where

\[
\omega_p = \frac{\omega_p}{\sqrt{-1}} w
\]

and

\[
\omega_\infty = (\text{sgn})^\frac{\omega}{\sqrt{-1}} w.
\]

3: **DEFINITION**

\[
L(\omega, s) = \prod_p L(\omega_p, s) \times L(\omega_\infty, s).
\]

4: **RAPPEL**

\[
L(\omega_p, s) = \begin{cases} 
(1 - \omega_p(p)p^{-s})^{-1} & (\omega_p = 1) \\
1 & (\omega_p \neq 1)
\end{cases}
\]

(cf. §18, #8).
[Note: The set $S_\omega$ of primes for which $\omega_p \neq 1$ is finite.]

5: SUBLEMMA

\[ |x| < 1 \Rightarrow \log (1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}. \]

Therefore

\[ |x| > 1 \Rightarrow \log \frac{1}{1-x} = \log 1 - \log (1-x^{-1}) \]

\[ = - \sum_{k=1}^{\infty} \frac{x^{-k}}{k}. \]

6: N.B.

\[ \log f(z) = \log |f(z)| + \sqrt{-1} \arg f(z) \]

\[ \Rightarrow \]

\[ \text{Re } \log f(z) = \log |f(z)|. \]

7: LEMMA The product

\[ \prod_{p} L(\omega_p, s) \]

is absolutely convergent provided $\text{Re}(s) > 1$.

PROOF Ignoring $S_\omega$ (a finite set), it is a question of estimating

\[ \prod_{p} \frac{1}{|1 - \omega_p(p)p^{-s}|}. \]
3.

So take its logarithm and consider

\[ \sum \log \left( \frac{1}{|1 - \omega_p(p)p^{-s}|} \right) \]

\[ = \sum \text{Re} \log \left( \frac{1}{1 - \omega_p(p)p^{-s}} \right) \]

\[ = \text{Re} \sum \frac{\omega_p(p)^k p^{-ks}}{k} \]

The claim then is that the series

\[ \sum \frac{\omega_p(p)^k p^{-ks}}{k} \]

is absolutely convergent. But

\[ \sum \left| \frac{\omega_p(p)^k p^{-ks}}{k} \right| \]

\[ = \sum \frac{p^{-k} \text{Re}(s)}{k} \]

which is bounded by

\[ \sum \frac{p^{-k} \text{Re}(s)}{k} \]

\[ \sum \frac{p^{-k(1+\delta)}}{k} \quad (\text{Re}(s) = 1 + \delta) \]
4.

\[ \sum \sum_{p, k = 1}^{\infty} p^{-k(1+\delta)} \]

\[ = \sum p^{-k(1+\delta)} \]

\[ = \sum \frac{1}{p p^{(1+\delta)} (1 - p^{-(1+\delta)})} \]

\[ \leq 2 \sum \frac{1}{p p^{1+\delta}} < \infty. \]

8: EXAMPLE Take \( \omega = 1 \) -- then

\[ L(\omega, s) = \prod \frac{1}{p} \frac{1}{1 - p^{-s}} \times \Gamma \left( \frac{s}{2} \right) \]

\[ = \pi^{-s/2} \Gamma(s/2) \zeta(s). \]

9: LEMMA \( L(\omega, s) \) is a holomorphic function of \( s \) in the strip \( \text{Re}(s) > 1 \).

10: LEMMA \( L(\omega, s) \) admits a meromorphic continuation to the whole \( s \)-plane (see below).

Owing to §17, #4, \( \forall f \in B_{\infty}(A), \)

\[ Z(f, \omega, s) = Z(\hat{f}, \omega, 1-s). \]
To exploit this, assume that

\[ f = \prod_{p} f_{p} \times f_{\infty}, \]

where \( \forall \, p, \, f_{p} \in \mathcal{B}(Q_{p}) \) and \( f_{p} = \chi_{Z_{p}} \) for all but a finite number of \( p \), while \( f_{\infty} \in S(\mathbb{R}) \) — then

\[ Z(f,\omega,s) \]

\[ = \int_{I} f(x)\omega(x) |x|^{s} d^{\times}x \]

\[ = \prod_{p} \int_{Q_{p}} f_{p}(x)\omega_{p}(x) |x_{p}|^{s} d^{\times}x_{p} \times \int_{R} f_{\infty}(x_{\infty})\omega_{\infty}(x_{\infty}) |x_{\infty}|^{s} d^{\times}x_{\infty} \]

\[ = \prod_{p} Z(f_{p},\omega_{p},s) \times Z(f_{\infty},\omega_{\infty},s) \]

and analogously for \( Z(\hat{f},\omega,1-s) \).

Therefore

\[ 1 = \frac{Z(f,\omega,s)}{Z(\hat{f},\omega,1-s)} \]

\[ = \prod_{p} \frac{Z(f_{p},\omega_{p},s)}{Z(\hat{f}_{p},\omega_{p},1-s)} \times \frac{Z(f_{\infty},\omega_{\infty},s)}{Z(\hat{f}_{\infty},\omega_{\infty},1-s)} \]

\[ = \prod_{p} \rho(\omega_{p},s) \times \rho(\omega_{\infty},s) \]

\[ = \prod_{p \in S_{\omega}} \rho(\omega_{p},s) \times \prod_{p \in S_{\omega}} \rho(\omega_{p},s) \times \rho(\omega_{\infty},s) \]
\[
= \prod_{p \in S} \frac{L(w, s)}{L(\bar{w}, 1-s)} \times \prod_{p \in S} \frac{\rho(p, s)}{L(p, 1-s)} \times \frac{L(\omega, s)}{L(\bar{\omega}, 1-s)}
\]
\[
= \prod_{p \in S} \rho(p, s) \times \prod_{p \in S} \frac{L(p, s)}{L(\bar{p}, 1-s)} \times \frac{L(\omega, s)}{L(\bar{\omega}, 1-s)}
\]
\[
= \prod_{p \in S} \rho(p, s) \times \frac{\prod_{p} L(p, s) \times L(\omega, s)}{\prod_{p} L(\bar{p}, 1-s) \times L(\bar{\omega}, 1-s)}
\]
\[
= \prod_{p \in S} \rho(p, s) \times \frac{L(w, s)}{L(\bar{w}, 1-s)}
\]
\[
= \prod_{p \in S} \varepsilon(p, s) \times \frac{L(w, s)}{L(\bar{w}, 1-s)} \quad \text{(cf. \#12, \#11)}
\]
\[
= \varepsilon(w, s) \times \frac{L(w, s)}{L(\bar{w}, 1-s)}
\]

where
\[
\varepsilon(w, s) = \prod_{p \in S} \varepsilon(p, s).
\]

11: THEOREM
\[
L(\bar{w}, 1-s) = \varepsilon(w, s)L(w, s).
\]

12: EXAMPLE Take \( \omega = 1 \) (cf. \#8) -- then \( \varepsilon(w, s) = 1 \) and
\[
L(\bar{w}, 1-s) = L(\omega, s)
\]
translates into

\[ \pi^{-\frac{1}{2}}(1-s)/(1-s/2)\zeta(1-s) = \pi^{-s/2}(s/2)\zeta(s) \quad (\text{cf. §16}). \]

Make the following explicit choice for

\[ f = \prod_p f_p \times f_\infty. \]

- If \( w_p = 1 \), let

\[ f_p(x_p) = \chi_p(x_p)\chi_{2p}(x_p). \]

Then

\[ Z(f_p, w_p, s) = L(\omega_p, s). \]

- If \( w_p \neq 1 \) and \( \deg \omega_p = n \geq 1 \), let

\[ f_p(x_p) = \chi_p(x_p)\chi_{n-p}(x_p). \]

Then

\[ Z(f_p, \omega_p, s) = \tau(\omega_p) \frac{1 + n(s + \sqrt{-1} w - 1)}{p - 1} L(\omega_p, s). \]

At infinity, take

\[ f_\infty(x_\infty) = e^{-\pi x_\infty^2} (\sigma = 0) \text{ or } f_\infty(x_\infty) = x_\infty e^{-\pi x_\infty^2} (\sigma = 1). \]

Then

\[ Z(f_\infty, \omega_\infty, s) = L(\omega_\infty, s). \]

13: NOTATION Put

\[ H(\omega, s) = \prod_{p \in S(\omega)} \tau(\omega_p) \frac{1 + n(s + \sqrt{-1} w - 1)}{p - 1}. \]
8.

14: N.B. $H(\omega, s)$ is a never zero entire function of $s$.

15: **Lemma**

$$Z(f, \omega, s) = H(\omega, s)L(\omega, s).$$

Since $Z(f, \omega, s)$ is a meromorphic function of $s$ (cf. §17, #4), it therefore follows that $L(\omega, s)$ is a meromorphic function of $s$.

Working now within the setting of §17, we distinguish two cases per $\omega$.

1. $\omega$ is nontrivial on $I^1$, hence $\omega \neq 1$ and in this situation, $Z(f, \omega, s)$ is a holomorphic function of $s$, hence the same is true of $L(\omega, s)$.

2. $\omega$ is trivial on $I^1$ — then $\omega = \left. \sqrt{-1} \right|_{\mathbb{A}} W$ and there are simple poles at

$$s = \sqrt{-1} \omega \text{ with residue } -f(0) \text{ if } f(0) \neq 0$$

$$s = \sqrt{-1} \omega + 1 \text{ with residue } \hat{f}(0) \text{ if } \hat{f}(0) \neq 0.$$ 

But $\forall \ p$, $\omega_p = \left. \sqrt{-1} \right|_{\mathbb{P}} W$ ($\Rightarrow \omega_p = 1$), so $f_p(0) = 1$. And likewise $f'_\infty(0) = 1$ ($\sigma = 0$).

Conclusion: $f(0) = 1$. As for the Fourier transforms, $\hat{f}_p = \chi_{\mathbb{P}} \Rightarrow \hat{f}_p(0) = 1$.

Also $\hat{f}_\infty = f'_\infty$ ($\sigma = 0$) $\Rightarrow \hat{f}_\infty(0) = 1$. Conclusion: $\hat{f}(0) = 1$. The respective residues are therefore $-1$ and $1$.

16: **Theorem** Suppose that $\omega_{1,p} = \omega_{2,p}$ for all but finitely many $p$ and $\omega_{1,\infty} = \omega_{2,\infty}$ — then $\omega_1 = \omega_2$.

**Proof** Put $\omega = \omega_1^{-1} \omega_2$, thus $\omega_p = 1$ for all $p$ outside a finite set $S$ of primes, so

$$L(\omega, s) = \prod_p L(\omega_p, s) \times L(\omega_\infty, s)$$
where $\alpha_p = \omega_p(p)$ if $\omega_p = 1$ and $\alpha_p = 0$ if $\omega_p \neq 1$, and each factor

$$\frac{1 - p^{-s}}{1 - \alpha_p p^{-s}}$$

is nonzero at $s = 0$ and $s = 1$. Therefore $L(\omega, s)$ has a simple pole at $s = 0$ and $s = 1$. Consider the decomposition

$$\omega = \omega_1 \cdot \sqrt{-1} w \quad \text{(cf §19, #1).}$$

Then $\omega = 1$ since otherwise $L(\omega, s)$ would be holomorphic, which it isn't. But then from the theory, $L(\omega, s)$ has simple poles at

- $s = \sqrt{-1} w$ with residue $-1$
- $s = \sqrt{-1} w + 1$ with residue $1$,

thereby forcing $w = 0$, which implies that $\omega = 1$, i.e., $\omega_1 = \omega_2$.

[Note: In the end, $\omega_p = 1 \forall p$, hence

$$\prod_{p \in S} \frac{1 - p^{-s}}{1 - \alpha_p p^{-s}} = \prod_{p \in S} \frac{1 - p^{-s}}{1 - p^{-s}} = 1,$$

as it has to be.]
§20. **FINITE CLASS FIELD THEORY**

Given a finite field $\mathbb{F}_q$ of characteristic $p$ (thus $q$ is an integral power of $p$), then in $\mathbb{F}_p^\times$,

$$\mathbb{F}_q = \{x : x^q = x\}.$$

1: **LEMMA** The multiplicative group

$$\mathbb{F}_q^\times = \{x : x^{q-1} = 1\}$$

is cyclic of order $q - 1$.

2: **NOTATION**

$$\mathbb{F}_q^n = \{x : x^{q^n} = x\} \quad (n \geq 1).$$

3: **LEMMA** $\mathbb{F}_q^n$ is a Galois extension of $\mathbb{F}_q$ of degree $n$.

4: **LEMMA** $\text{Gal}(\mathbb{F}_q^n/\mathbb{F}_q)$ is a cyclic group of order $n$ generated by the element $\sigma_{q,n}$, where

$$\sigma_{q,n}(x) = x^{q^n} \quad (x \in \mathbb{F}_q^n).$$

5: **LEMMA** The $\mathbb{F}_q^n$ are finite abelian extensions of $\mathbb{F}_q$ and they comprise all the finite extensions of $\mathbb{F}_q$, hence the algebraic closure $\bigcup \mathbb{F}_q^n$ is $\mathbb{F}_q^{ab}$.

6: **THEOREM** There is a 1-to-1 correspondence between the finite abelian
extensions of $F_q$ and the subgroups of $\mathbb{Z}$ of finite index which is given by

$$F_q \leftrightarrow n\mathbb{Z} \quad (n \geq 1).$$

Schematically:

$$F_q \subset F_q^2 \subset F_q^4$$

$$n \quad n$$

$$F_q^3 \subset F_q^6$$

$$n$$

$$F_q^9$$

$$\leftrightarrow$$

$$\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z}$$

$$\cup \quad \cup$$

$$3\mathbb{Z} \supset 6\mathbb{Z}$$

$$\cup$$

$$9\mathbb{Z}.$$

The "class field" aspect of all this is the existence of a canonical homomorphism

$$\mathrm{rec}_q : \mathbb{Z} \to \mathrm{Gal}(F_q^{ab}/F_q).$$

7: NOTATION Define

$$\sigma_q \in \mathrm{Gal}(F_q^{ab}/F_q)$$

by

$$\sigma_q(x) = x^q.$$
8: N.B. Under the arrow of restriction

$$\text{Gal}(F^\text{ab}/F_q) \to \text{Gal}(F^n_q/F_q),$$

$\sigma_q$ is sent to $\sigma_{q,n}$.

9: DEFINITION

$$\text{rec}_q(k) = \sigma_q^k \quad (k \in \mathbb{Z}).$$

10: LEMMA The identification

$$\mathbb{Z}/n\mathbb{Z} \cong \text{Gal}(F^n_q/F_q)$$

is the arrow $k \mapsto \sigma_q^k$.

On general grounds,

$$\text{Gal}(F^\text{ab}/F_q) = \lim\limits_\leftarrow \text{Gal}(F^n_q/F_q).$$

[Note: The open subgroups of $\text{Gal}(F^\text{ab}_q/F_q)$ are the $\text{Gal}(F^\text{ab}_q/F_{q^n})$ and $\text{Gal}(F^\text{ab}_q/F_q)/\text{Gal}(F^\text{ab}_q/F_{q^n}) \cong \text{Gal}(F^n_q/F_q).$]

Therefore

$$\text{Gal}(F^\text{ab}_q/F_q) \approx \lim\limits_\leftarrow \mathbb{Z}/n\mathbb{Z},$$

another realization of the RHS being $\prod\limits_p \mathbb{Z}_p$ which if invoked leads to

$$\sigma_q \leftrightarrow (1,1,1,\ldots).$$
11: N.B. The composition

\[ \text{rec}_q: \mathbb{Z} \rightarrow \text{Gal}(F_q^{ab}/F_q) \cong \lim_{\rightarrow} \mathbb{Z}/n\mathbb{Z} \]

coincides with the canonical map

\[ k \mapsto (k \mod n)_n. \]

12: REMARK Give \( \mathbb{Z} \) the discrete topology -- then

\[ \text{rec}_q: \mathbb{Z} \rightarrow \text{Gal}(F_q^{ab}/F_q) \]

is continuous and injective but it is not a homeomorphism (\( \text{Gal}(F_q^{ab}/F_q) \) is compact).

[Note: The image \( \text{rec}_q(\mathbb{Z}) \) is the cyclic subgroup \( <\sigma_q> \) generated by \( \sigma_q \). And:

- \( <\sigma_q> = \text{Gal}(F_q^{ab}/F_q) \)
- \( \overline{<\sigma_q>} = \text{Gal}(F_q^{ab}/F_q'). \)]

13: SCHOLIUM The finite abelian extensions of \( F_q \) correspond 1-to-1 with the open subgroups of \( \text{Gal}(F_q^{ab}/F_q) \).

[Quote the appropriate facts from infinite Galois theory.]

14: SCHOLIUM The open subgroups of \( \text{Gal}(F_q^{ab}/F_q) \) correspond 1-to-1 with the open subgroups of \( \mathbb{Z} \) of finite index.

[Given an open subgroup \( U \subset \text{Gal}(F_q^{ab}/F_q) \), send it to \( \text{rec}_q^{-1}(U) \subset \mathbb{Z} \) (discrete topology). Explicated:

\[ \text{rec}_q^{-1}(\text{Gal}(F_q^{ab}/F_q)) = n\mathbb{Z}. \] ]
The norm map

\[ N_{F_{q^n}/F_q} : F_{q^n}^\times \to F_q^\times \]

is surjective.

[Let \( x \in F_{q^n}^\times \):

\[ N_{F_{q^n}/F_q} (x) = \prod_{i=0}^{n-1} (\sigma_{q^n,q})^{i} x \]

\[ = \prod_{i=0}^{n-1} x q^i \]

\[ = \sum_{i=0}^{n-1} q^i \]

\[ = x \]

\[ = x(q^n-1)/(q-1). \]

Specialize now and take for \( x \) a generator of \( F_{q^n}^\times \), hence \( x \) is of order \( q^n-1 \), hence \( N_{F_{q^n}/F_q} (x) \) is of order \( q-1 \), hence is a generator of \( F_q^\times \).]
§21. LOCAL CLASS FIELD THEORY

Let $K$ be a local field -- then there exists a unique continuous homomorphism

$$\text{rec}_K: K^\times \to \text{Gal}(K^{ab}/K),$$

the so-called reciprocity map, that has the properties delineated in the results that follow.

1: CHART

<table>
<thead>
<tr>
<th>finite field $K$</th>
<th>$\mathbb{Z}$</th>
<th>$\text{Gal}(K^{ab}/K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>local field $K$</td>
<td>$K^\times$</td>
<td>$\text{Gal}(K^{ab}/K)$</td>
</tr>
</tbody>
</table>

2: CONVENTION An abelian extension is a Galois extension whose Galois group is abelian.

3: SCHOLIUM The finite abelian extensions $L$ of $K$ correspond 1-to-1 with the open subgroups of $\text{Gal}(K^{ab}/K)$:

$$L \leftrightarrow \text{Gal}(K^{ab}/L).$$

[Note: $\text{Gal}(L/K)$ is a homomorphic image of $\text{Gal}(K^{ab}/K)$:

$$\text{Gal}(L/K) \approx \text{Gal}(K^{ab}/K)/\text{Gal}(K^{ab}/L).$$]

4: LEMMA Suppose that $L$ is a finite extension of $K$ -- then

$$N_{L/K}: L^\times \to K^\times$$

is continuous, sends open sets to open sets, and closed sets to closed sets.
5: **Lemma** Suppose that $L$ is a finite extension of $K$ -- then

$$[K^\times : N_{L/K}(L^\times)] \leq [L : K].$$

6: **Lemma** Suppose that $L$ is a finite extension of $K$ -- then

$$[K^\times : N_{L/K}(L^\times)] = [L : K]$$

iff $L/K$ is abelian.

7: **Notation** Given a finite abelian extension $L$ of $K$, denote the composition

$$K^\times \xrightarrow{\text{rec}_K} \text{Gal}(K^{ab}/K) \xrightarrow{\pi_{L/K}} \text{Gal}(L/K)$$

by $\langle . , L/K \rangle$, the norm residue symbol.

8: **Theorem** Suppose that $L$ is a finite abelian extension of $K$ -- then

the kernel of $\langle . , L/K \rangle$ is $N_{L/K}(L^\times)$, hence

$$K^\times / N_{L/K}(L^\times) \approx \text{Gal}(L/K).$$

9: **Example** Take $K = \mathbb{R}$, thus $K^{ab} = \mathbb{C}$ and

$$N_{\mathbb{C}/\mathbb{R}}(C^\times) = \mathbb{R}^\times_{>0}.$$ 

Moreover,

$$\text{Gal}(\mathbb{C}/\mathbb{R}) = \{ \text{id}_\mathbb{C}, \sigma \},$$

where $\sigma$ is the complex conjugation. Define now

$$\text{rec}_R : \mathbb{R}^\times \to \text{Gal}(\mathbb{R}^{ab}/\mathbb{R})$$

by stipulating that

$$\text{rec}_R(\mathbb{R}^\times_{>0}) = \text{id}_\mathbb{C}, \ \text{rec}_R(\mathbb{R}^\times_{<0}) = \sigma.$$
10: EXAMPLE Take $K = C$ -- then $K^{ab} = C = K$ and matters in this situation are trivial.

11: THEOREM The arrow

$$L \rightarrow N_{L/K}(L^\times)$$

is a bijection between the finite abelian extensions of $K$ and the open subgroups of finite index of $K^\times$.

12: THEOREM The arrow $U \rightarrow \text{rec}_K^{-1}(U)$ is a bijection between the open subgroups of $\text{Gal}(K^{ab}/K)$ and the open subgroups of finite index of $K^\times$.

From this point forward, it will be assumed that $K$ is non-archimedean, hence is a finite extension of $\mathbb{Q}_p$ for some $p$ (cf. §5, #13).

13: LEMMA $\text{rec}_K$ is injective and its image is a proper, dense subgroup of $\text{Gal}(K^{ab}/K)$.

14: LEMMA

$$(K^\times, L/K) = \text{Gal}(L/K_{ur}),$$

where $K_{ur}$ is the largest unramified extension of $K$ contained in $L$ (cf. §5, #33).

[Note: The image

$$(1 + p^i, L/K) = G^i (i \geq 1),$$

the $i^{th}$ ramification group in the upper numbering (conventionally, one puts $G^0 = \text{Gal}(L/K_{ur})$]
and refers to it as the inertia group.)

Working within \( K^{\text{sep}} \), the extension \( K^{\text{ur}} \) generated by the finite unramified extensions of \( K \) is called the maximal unramified extension of \( K \). This is a Galois extension and

\[
\text{Gal}(K^{\text{ur}}/K) \cong \text{Gal}(F_{q}^{\text{ab}}/F_{q}),
\]

where \( F_{q} = R/P \) (cf. §5, #19).

15: REMARK The finite unramified extensions \( L \) of \( K \) correspond 1-to-1 with the finite extensions of \( R/P = F_{q} \) and

\[
\text{Gal}(L/K) \cong \text{Gal}(F_{q}^{n}/F_{q}) \quad (n = [L:K]).
\]

16: LEMMA \( K^{\text{ur}} \) is the field obtained by adjoining to \( K \) all roots of unity having order prime to \( p \).

17: APPLICATION \( K^{\text{ur}} \) is a subfield of \( K^{\text{ab}} \).

[Cyclotomic extensions are Galois and abelian.]

18: THEOREM There is a commutative diagram

\[
\begin{array}{ccc}
K^{x} & \xrightarrow{\text{rec}_K} & \text{Gal}(K^{\text{ab}}/K) \\
\downarrow{\text{V}_K} & & \downarrow \\
Z & \xrightarrow{\text{rec}_q} & \text{Gal}(F_{q}^{\text{ab}}/F_{q})
\end{array}
\]
the vertical arrow on the right being the composition

$$\text{Gal}(k_{\text{ur}}/K) \rightarrow \text{Gal}(k_{\text{ur}}/K)/\text{Gal}(k_{\text{ur}}/K)'$$

$$\approx \text{Gal}(k_{\text{ur}}/K)$$

$$\approx \text{Gal}(F_{q^{n}}/F_{q}).$$

[Note: \( \forall a \in K^{\times}, \)

\[ \text{mod}_{K}(a) = q^{\text{ord}_{k}(a)}. \]

19: N.B. The image of

$$\text{rec}_{K}(\pi) | k_{\text{ur}} \in \text{Gal}(k_{\text{ur}}/K)$$

in \( \text{Gal}(F_{q^{n}}/F_{q}) \) is \( \sigma_{q} \) (cf. §20, #7).

[Note: If \( L \) is a finite unramified extension of \( K \) and if \( \tilde{\sigma}_{q,n} \) is the generator of \( \text{Gal}(L/K) \) which is the lift of the generator \( \sigma_{q,n} \) of \( \text{Gal}(F_{q^{n}}/F_{q}) \)

\((n = [L:K])\), then

\( (\pi,L/K) = \tilde{\sigma}_{q,n}. \)]

20: Functoriality Suppose that \( L > K \) is a finite extension of \( K \) -- then the diagram

\[ \begin{array}{ccc}
L^{\times} & \xrightarrow{\text{rec}_{L}} & \text{Gal}(L_{\text{ab}}/L) \\
N_{L/K} & \downarrow & \downarrow \text{res} \\
K^{\times} & \xrightarrow{\text{rec}_{K}} & \text{Gal}(k_{\text{ab}}/K)
\end{array} \]

commutes.
21: **DEFINITION** Given a Hausdorff topological group $G$, let $G^*$ be its commutator subgroup, and put $G^{ab} = G/G^*$ -- then $G^{ab}$ is a closed normal subgroup of $G$ and $G^{ab}$ is abelian, the topological abelianization of $G$.

22: **EXAMPLE**

$$\text{Gal}(K^{sep}/K)^{ab} = \text{Gal}(K^{ab}/K).$$

23: **CONSTRUCTION** Let $G$ be a Hausdorff topological group and let $H$ be a closed subgroup of finite index -- then the **transfer** homomorphism $\tau: G^{ab} \to H^{ab}$ is defined as follows: Choose a section $s: H \backslash G \to G$ and for $x \in G$, put

$$\tau(xG^*) = \prod_{\alpha \in H \backslash G} h_{x, \alpha} \pmod{H^*},$$

where $h_{x, \alpha} \in H$ is defined by

$$s(\alpha)x = h_{x, \alpha}s(\alpha x).$$

24: **EXAMPLE** Suppose that $L \supset K$ is a finite extension of $K$ -- then $L^{sep}$ is a closed subgroup of finite index (viz. $[L:K]$), hence there is a transfer homomorphism

$$\tau: \text{Gal}(K^{ab}/K) \to \text{Gal}(L^{ab}/L).$$

25: **THEOREM** The diagram

$$
\begin{array}{ccc}
L^x & \xrightarrow{\text{rec}_L} & \text{Gal}(L^{ab}/L) \\
\uparrow & & \uparrow \tau \\
K^x & \xrightarrow{\text{rec}_K} & \text{Gal}(K^{ab}/K)
\end{array}
$$

commutes.
§22. WEIL GROUPS: THE ARCHIMEDEAN CASE

1. DEFINITION Put \( W_C = \mathbb{C}^\times \), call it the Weil group of \( C \), and leave it at that.

2. DEFINITION Put

\[ W_R = \mathbb{C}^\times \cup J\mathbb{C}^\times \] (disjoint union) (\( J \) a formal symbol),

where \( J^2 = -1 \) and \( JzJ^{-1} = \overline{z} \) (obvious topology on \( W_R \)). Accordingly, there is a nonsplit short exact sequence

\[ 1 \to \mathbb{C}^\times \to W_R \to \text{Gal}(\mathbb{C}/\mathbb{R}) \to 1, \]

the image of \( J \) in \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) being complex conjugation.

[Note: \( H^2(\text{Gal}(\mathbb{C}/\mathbb{R}),\mathbb{C}^\times) \) is cyclic of order 2, thus up to equivalence of extensions of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) by \( \mathbb{C}^\times \) per the canonical action of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) on \( \mathbb{C}^\times \), there are two possibilities:

1. A split extension

\[ 1 \to \mathbb{C}^\times \to E \to \text{Gal}(\mathbb{C}/\mathbb{R}) \to 1. \]

2. A nonsplit extension

\[ 1 \to \mathbb{C}^\times \to E \to \text{Gal}(\mathbb{C}/\mathbb{R}) \to 1. \]

The Weil group \( W \) is a representative of the second situation which is why we took \( J^2 = -1 \) (rather than \( J^2 = +1 \)).]

3. LEMMA The commutator subgroup \( W_R^\circ \) of \( W_R \) consists of all elements of the form \( JzJ^{-1}z^{-1} = \frac{\overline{z}}{z} \), i.e., \( W_R^\circ = S \), thus is closed.
Let \( pr: W_R \rightarrow R^x \) be the map sending \( J \) to \(-1\) and \( z \) to \(|z|^2\).

4: LEMMA \( S \) is the kernel of \( pr \) and \( pr \) is surjective.

5: LEMMA The arrow
\[ pr^{ab}: W_R^{ab} \rightarrow R^x \]
induced by \( pr \) is an isomorphism.

6: REMARK The inverse \( R^x \rightarrow W_R^{ab} \) of \( pr^{ab} \) is characterized by the conditions
\[
-1 \rightarrow JW_R^* \\
x \rightarrow \sqrt{x} W_R^* \quad (x > 0).
\]

7: NOTATION Define
\[ ||\cdot||: W_R \rightarrow R_{>0}^x \]
by the prescription
\[ ||z|| = zz \quad (z \in \mathbb{C}), \quad ||J|| = 1. \]

8: N.B. \( ||\cdot|| \) drops to a continuous homomorphism \( W_R^{ab} \rightarrow R_{>0}^x \).

9: DEFINITION A representation of \( W_R \) is a continuous homomorphism
\( \rho: W_R \rightarrow GL(V) \), where \( V \) is a finite dimensional complex vector space.

10: EXAMPLE If \( s \in \mathbb{C}, \) then the assignment \( w \rightarrow ||w||^s \) is a 1-dimensional
3.

representation of $W_R$, i.e., is a character.

11: N.B. If $\chi$ is a character of $R^x$, then $\chi \circ \text{pr}$ is a character of $W_R$ and all such have this form.

[For any $\rho \in \tilde{W}_R$,

$$\rho(z) = \rho(JzJ^{-1}) = \rho(J)\rho(z)\rho(J)^{-1} = \rho(z).$$

Therefore

$$1 = \rho(-1) \quad (\text{cf. } \S\, 7, \#12).$$

But

$$\rho(-1) = \rho(J^2) = \rho(J)^2,$$

so $\rho(J) = \pm 1$. This said, the characters of $R^x$ are described in $\S\, 7$, $\#11$, thus the 1-dimensional representations of $W_R$ are parameterized by a sign and a complex number $s$:

- $(+,s):\rho(z) = |z|^s, \rho(J) = +1$
- $(-,s):\rho(z) = |z|^s, \rho(J) = -1.$

Let $V$ be a finite dimensional complex vector space.

12: DEFINITION A linear transformation $T:V \to V$ is semisimple if every $T$-invariant subspace has a complementary $T$-invariant subspace.

13: FACT $T$ is semisimple iff $T$ is diagonalizable, i.e., in some basis $T$ is represented by a diagonal matrix.

[Bear in mind that $C$ is algebraically closed...]
14: DEFINITION A representation \( \rho : \mathbb{W}_R \to \text{GL}(V) \) is **semisimple** if \( \forall w \in \mathbb{W}_R, \rho(w) : V \to V \) is semisimple.

15: DEFINITION A representation \( \rho : \mathbb{W}_R \to \text{GL}(V) \) is **irreducible** if \( V \neq 0 \) and the only \( \rho \)-invariant subspaces are 0 and \( V \).

The irreducible 1-dimensional representations of \( \mathbb{W}_R \) are its characters (which, of course, are automatically semisimple).

16: LEMMA If \( \rho : \mathbb{W}_R \to \text{GL}(V) \) is a semisimple irreducible representation of \( \mathbb{W}_R \) of dimension \( > 1 \), then \( \dim V = 2 \).

PROOF There is a nonzero vector \( v \in V \) and a character \( \chi : \mathbb{C}^\times \to \mathbb{C}^\times \) such that \( \forall z \in \mathbb{C}^\times, \rho(z)v = \chi(z)v. \)

Since the span \( S \) of \( v, \rho(J)v \) is a \( \rho \)-invariant subspace, the assumption of irreducibility implies that \( \dim V = 2 \).

[To check the \( \rho \)-invariance of \( S \), note that

\[
\begin{pmatrix}
\rho(z)\rho(J)v = \rho(zJ)v = \rho(J\overline{z})v = \rho(J)\rho(\overline{z})v = \rho(J)\chi(\overline{z})v \\
\rho(J)\rho(J)v = \rho(J^2)v = \rho(-1)v = \chi(-1)v.
\end{pmatrix}
\]

Given an integer \( k \) and a complex number \( s \), define a character \( \chi_{k,s} : \mathbb{C}^\times \to \mathbb{C}^\times \) by the prescription

\[
\chi_{k,s}(z) = (\frac{z}{|z|})^k(|z|^2)^s
\]

and let \( \rho_{k,s} = \text{ind} \chi_{k,s} \) be the representation of \( \mathbb{W}_R \) which it induces.
17: LEMMA $\rho_{k,s}$ is 2-dimensional.

18: LEMMA $\rho_{k,s}$ is semisimple.

19: LEMMA $\rho_{k,s}$ is irreducible iff $k \neq 0$.

20: DEFINITION Let

$$\begin{align*}
\rho_1 : W_R &\to GL(V_1) \\
\rho_2 : W_R &\to GL(V_2)
\end{align*}$$

be representations of $W_R$ -- then $(\rho_1, V_1)$ is equivalent to $(\rho_2, V_2)$ if there exists an isomorphism $f : V_1 \to V_2$ such that $\forall w \in W_R'$,

$$f \circ \rho_1(w) = \rho_2(w) \circ f.$$ 

21: LEMMA $\rho_{k_1,s_1}$ is equivalent to $\rho_{k_2,s_2}$ iff $k_1 = k_2$, $s_1 = s_2$ or $k_1 = -k_2$, $s_1 = s_2$.

22: THEOREM Every 2-dimensional semisimple irreducible representation of $W_R$ is equivalent to a unique $\rho_{k,s}$ ($k > 0$).

23: N.B. Therefore the equivalence classes of 2-dimensional semisimple irreducible representations of $W_R$ are parameterized by the points of $N \times \mathbb{C}$.

24: DEFINITION A representation $\rho : W_R \to GL(V)$ is completely reducible if $V$ is the direct sum of a collection of irreducible $\rho$-invariant subspaces.
25: **Lemma** Let $\rho : W \to GL(V)$ be a semisimple representation -- then $\rho$ is completely reducible.

**Proof** The characters of $C^L$ are of the form $z \mapsto z^{\mu}z^{\nu}$ with $\mu, \nu \in C$, $\mu - \nu \in Z$ and $V$ is the direct sum of subspaces $V_{\mu, \nu}$, where $\rho(z)|_{V_{\mu, \nu}} = z^{\mu}z^{\nu} \text{id}_{V_{\mu, \nu}}$. Claim:

$$\rho(J)_{\mu, \nu} = V_{\nu, \mu}.$$ 

Proof: $\forall \nu \in V_{\mu, \nu}$,

$$\rho(z) \rho(J)_{\nu} = \rho(JzJ^{-1})\rho(J)_{\nu}$$

$$= \rho(J)\rho(\overline{z})\rho(J^{-1})\rho(J)_{\nu}$$

$$= \rho(J)\rho(\overline{z})_{\nu}$$

$$= \rho(J)z^{\nu}z^{\nu}$$

$$= \rho(J)z^{\nu}z^{\nu}$$

$$= z^{\nu}z^{\nu}\rho(J)_{\nu}.$$ 

Proceeding:

- $\mu = \nu$ Choose a basis of eigenvectors for $\rho(J)$ on $V_{\mu, \mu}$ -- then the span of each eigenvector is a 1-dimensional $\rho$-invariant subspace.

- $\mu \neq \nu$ Choose a basis $v_1, \ldots, v_r$ for $V_{\mu, \nu}$ and put $v_i' = \rho(J)v_i$ ($1 \leq i \leq r$) -- then $Cv_i \oplus Cv_i'$ is a 2-dimensional $\rho$-invariant subspace and the direct sum

$$\bigoplus_{i=1}^{r} (Cv_i \oplus Cv_i').$$
equals

\[ V_{\mu, \nu} \oplus V_{\nu, \mu} \]

26: REMARK Suppose that \( \rho: W_R \rightarrow \text{GL}(V) \) is a representation -- then

\[ J^2 = -1 \Rightarrow (-1)J \cdot J = 1 \]

\[ \Rightarrow (-1)J = J^{-1} \]

\[ \Rightarrow \rho(J)^{-1} = \rho(J^{-1}) \]

\[ = \rho((-1)J) \]

\[ = \rho(-1)\rho(J). \]

On the other hand, if \( J^2 = 1 \) (the split extension situation (cf. #2)), then

\[ \text{id}_V = \rho(1) \]

\[ = \rho(J^2) = \rho(J)\rho(J) \]

\[ \Rightarrow \]

\[ \rho(J)^{-1} = \rho(J). \]
Let $K$ be a non-archimedean local field.

1. **NOTATION** Put

\[
G_K = \text{Gal}(K^{\text{sep}}/K) \\
G^\text{ab}_K = \text{Gal}(K^{\text{ab}}/K).
\]

2. **N.B.** Every character of $G_K$ factors through $G^\text{ab}_K$, hence gives rise to a character of $G^\text{ab}_K$.

To study the characters of $G^\text{ab}_K$, precompose with the reciprocity map

\[
\text{rec}_K : K^\times \to G^\text{ab}_K, \text{ thus}
\]

\[
\begin{array}{ccc}
(G^\text{ab}_K)^\sim & \to & (K^\times)^\sim \\
\chi_K : & & \\
\chi & \mapsto & \chi \circ \text{rec}_K.
\end{array}
\]

3. **LEMMA** $\chi_K$ is a homomorphism.

4. **LEMMA** $\chi_K$ is injective.

**PROOF** Suppose that

\[
\chi_K(\chi) = \chi \circ \text{rec}_K
\]

is trivial -- then $\chi|\text{Im } \text{rec}_K = 1$. But $\text{Im } \text{rec}_K$ is dense in $G^\text{ab}_K$ (cf. §21, #13), so by continuity, $\chi \equiv 1$. 
5: LEMMA $\chi_K$ is not surjective.

PROOF $G_K^{ab}$ is compact abelian and totally disconnected. Therefore $(G_K^{ab})^\sim = (G_K)^\sim$ and every $\chi$ is unitary and of finite order (cf. §7, #7 and §8, #2), thus the $\chi_K(x)$ are unitary and of finite order. But there are characters of $K^\times$ for which this is not the case.

6: N.B. The failure of $\chi_K$ to be surjective will be remedied below (cf. #19).

The kernel of the arrow

$$\text{Gal}(K^{sep}/K) \to \text{Gal}(K^{ur}/K)$$

of restriction is $\text{Gal}(K^{sep}/K^{ur})$ and there is an exact sequence

$$1 \to \text{Gal}(K^{sep}/K^{ur}) \to \text{Gal}(K^{sep}/K) \to \text{Gal}(K^{ur}/K) \to 1.$$ 

Identify

$$\text{Gal}(K^{ur}/K)$$

with

$$\text{Gal}(F_q^{ab}/F_q)$$

and put

$$W(F_q^{ab}/F_q) = \langle q \rangle \quad (\text{discrete topology}).$$

7: DEFINITION The Weil group $W(K^{sep}/K)$ is the inverse image of $W(F_q^{ab}/F_q)$ in $\text{Gal}(K^{sep}/K)$, i.e., the elements in $\text{Gal}(K^{sep}/K)$ which induce an integral power of $q$.  

8: NOTATION Abbreviate $W(K^{sep}/K)$ to $W_K$, hence $W_K \subset G_K$.

Setting

$$I_K = \text{Gal}(K^{sep}/K)$$

(the inertia group),

there is an exact sequence

$$1 \to I_K \to W_K \to W(F^\text{ab}_q/F_q) \to 1.$$

(Note: Fix an element $\sigma_q \in W_K$ which maps to $\sigma_q$ -- then structurally, $W_K$ is the disjoint union

$$\bigcup_{n \in \mathbb{Z}} (\sigma_q)^n I_K.$$)

Topologize $W_K$ by taking for a neighborhood basis at the identity the

$$\text{Gal}(K^{sep}/L) \cap I_K,$$

where $L$ is a finite Galois extension of $K$.

9: REMARK $I_K$ has the relative topology per the inclusion $I_K \to G_K$ and any splitting $Z \to W_K$ induces an isomorphism $W_K \cong I_K \times Z$ of topological groups, where $Z$ has the discrete topology.

10: LEMMA $W_K$ is a totally disconnected locally compact group.

(Note: $W_K$ is not compact... .)

11: LEMMA The inclusion $W_K \to G_K$ is continuous and has a dense image.
4.

12: **LEMMA** \( \mathcal{L} \) is open in \( \mathcal{W}_K \).

13: **LEMMA** \( \mathcal{L} \) is a maximal compact subgroup of \( \mathcal{W}_K \).

Suppose that \( \mathcal{L} \supset K \) is a finite extension of \( K \) -- then \( G_L \subseteq G_K \) is the subgroup of \( G_K \) fixing \( \mathcal{L} \), hence
\[
\mathcal{W}_L \subseteq G_L \subseteq G_K.
\]

14: **LEMMA**
\[
\mathcal{W}_L = G_L \cap \mathcal{W}_K \subseteq \mathcal{W}_K
\]
is open and of finite index in \( \mathcal{W}_K \), it being normal in \( \mathcal{W}_K \) iff \( \mathcal{L}/K \) is Galois.

15: **THEOREM** The arrow
\[
\mathcal{L} \to \mathcal{W}_L
\]
is a bijection between the finite extensions of \( K \) and the open subgroups of finite index of \( \mathcal{W}_K \).

[By contrast, the arrow
\[
\mathcal{L} \to \text{Gal}(K^{\text{sep}}/\mathcal{L})
\]
is a bijection between the finite extensions of \( K \) and the open subgroups of \( G_K \).]

16: **LEMMA**
\[
\mathcal{W}_K^{ab} = \mathcal{G}_K^{ab}
\]

17: **APPLICATION** The homomorphism \( \mathcal{W}_K^{ab} \to \mathcal{G}_K^{ab} \) is 1-to-1.
5.

18: THEOREM The image of $\text{rec}_K: K^\times \to G_{K}^{ab}$ is $W_{K}^{ab}$ and the induced map $K^\times \to W_{K}^{ab}$ is an isomorphism of topological groups (cf. §21, #13).

The characters of $W_{K}$ "are" the characters of $W_{K}^{ab}$, so we have:

19: SCHOLIUM There is a bijective correspondence between the characters of $W_{K}$ and the characters of $K^\times$ or still, there is a bijective correspondence between the 1-dimensional representations of $W_{K}$ and the 1-dimensional representations of $GL_1(K)$.

Suppose that $L \supset K$ is a finite Galois extension of $K$ -- then $G_L \subset G_K$ and

$$G_K/G_L \cong \text{Gal}(L/K)$$

is finite of cardinality $[L:K]$. Since $W_K$ is dense in $G_K$, it follows that the image of the arrow

$$W_K \to G_K/G_L$$

$$w \to wG_L$$

is all of $G_K/G_L$, its kernel being those $w \in W_K$ such that $w \in G_L$, i.e., its kernel is $G_L \cap W_K$ or still, is $W_L$.

20: LEMMA

$$W_K/W_L \cong G_K/G_L \cong \text{Gal}(L/K).$$

21: LEMMA $\overline{W}_L$ is a normal subgroup of $W_K$. 


[Bearing in mind that $W_L$ is a normal subgroup of $W_K$, if $\alpha, \beta \in W_L^*$ and if $\gamma \in W_K^*$, then]

\[\gamma \alpha \beta^{-1} \gamma^{-1} = (\gamma \alpha \gamma^{-1})(\gamma \beta \gamma^{-1})(\gamma \alpha^{-1} \gamma^{-1})(\gamma \beta^{-1} \gamma^{-1}).\]

There is an exact sequence

\[1 \to W_L \to W_K \to (W_K/W_L)/(W_L/W_L) \to 1\]

or still, there is an exact sequence

\[1 \to W_L \to W_K \to W_K/W_L \to 1.\]

22: **NOTATION** Put

\[W(L,K) = W_K^{W_K}.\]

23: **SCHOLIUM** There is an exact sequence

\[1 \to W_L^{ab} \to W(L,K) \to W_K/W_L \to 1\]

and a diagram

\[
\begin{array}{ccc}
W_L^{ab} & \longrightarrow & W(L,K) \\
\text{rec}_L & & \\
1 \longrightarrow L^x & \longrightarrow & W_K/W_L \approx \text{Gal}(L/K) \to 1.
\end{array}
\]

24: **NOTATION** Given $w \in W_K$, let $|w|$ denote the effect on $w$ of passing
from $W_K$ to $R_{>0}^x$ via the arrows

$$
W_K \longrightarrow \hat{W}_K^{ab} \xrightarrow{\text{rec}_K^{-1}} K^x \xrightarrow{\text{mod}_K} R_{>0}^x.
$$

25: **Lemma** $||.||: W_K \rightarrow R_{>0}^x$ is a continuous homomorphism and its kernel is $I_K$.

[Under the arrow

$$W_K \rightarrow \hat{W}_K,$$

$I_K$ drops to

$$\text{Gal}(K^{ab}/K^{ur}) \subset \hat{W}_K^{ab}.$$

Consider now the arrow

$$\text{rec}_K: K^x \rightarrow \hat{W}_K^{ab}.$$

Then $R^x$ is sent to $\text{Gal}(K^{ab}/K^{ur})$ and a prime element $\pi \in R$ is sent to an element $\overline{\sigma}_q$ in $\hat{W}_K^{ab}$ whose image in $W(F^{ab}/qq')$ is $\sigma_{qq'}$. And

$$\hat{W}_K^{ab} = \bigcup_{n \in \mathbb{Z}} (\overline{\sigma}_q)^n \text{Gal}(K^{ab}/K^{ur}).$$

26: **Definition** A representation of $W_K$ is a continuous homomorphism $\rho: W_K \rightarrow \text{GL}(V)$, where $V$ is a finite dimensional complex vector space.

27: **Lemma** A homomorphism $\rho: W_K \rightarrow \text{GL}(V)$ is continuous per the usual topology on $\text{GL}(V)$ iff it is continuous per the discrete topology on $\text{GL}(V)$.

[GL(V) has no small subgroups.]
28: SCHOLIUM  The kernel of every representation of $W_K$ is trivial on an open subgroup $J$ of $I_K$. Conversely, if $\rho: W_K \to \text{GL}(V)$ is a homomorphism which is trivial on an open subgroup $J$ of $I_K$, then the inverse image of any subset of $\text{GL}(V)$ is a union of cosets of $J$, hence is open, hence $\rho$ is continuous, so by definition is a representation of $W_K$.

29: EXAMPLE  Suppose that $L \rightarrow K$ is a finite Galois extension of $K$ -- then

$$W_L \cap I_K = G_L \cap W_K \cap I_K$$

$$= G_L \cap I_K$$

is an open subgroup of $I_K$. But

$$W_K/W_L \cong \text{Gal}(L/K)$$

(cf. #20).

Therefore every homomorphism $\text{Gal}(L/K) \to \text{GL}(V)$ lifts to a homomorphism $W_K \to \text{GL}(V)$ which is trivial on an open subgroup of $I_K$, hence is a representation of $W_K$.

30: N.B.  Representations of $W_K$ arising in this manner are said to be of Galois type.

31: LEMMA  A representation of $W_K$ is of Galois type iff it has finite image.

32: EXAMPLE  $||.||$ is a character of $W_K$ but as a representation, is not of Galois type.

33: LEMMA Let $\rho: W_K \to \text{GL}(V)$ be a representation -- then the image $\rho(I_K)$ is finite.
9. 

PROOF Suppose that $J$ is an open subgroup of $I_K$ on which $\rho$ is trivial. Since $I_K$ is compact and $J$ is open, the quotient $I_K/J$ is finite, thus $\rho(I_K) = \rho(I_K/J)$ is finite.

34: DEFINITION A representation $\rho:W_K \to \text{GL}(V)$ is irreducible if $V \neq 0$ and the only $\rho$-invariant subspaces are 0 and $V$.

35: THEOREM Given an irreducible representation $\rho$ of $W_K$, there exists an irreducible representation $\tilde{\rho}$ of $W_K$ and a complex parameter $s$ such that $\rho \cong \tilde{\rho} \otimes |.|^s$.

36: LEMMA Let $\rho:W_K \to \text{GL}(V)$ be a representation — then $V$ is the sum of its irreducible $\rho$-invariant subspaces iff every $\rho$-invariant subspace has a $\rho$-invariant complement.

37: DEFINITION Let $\rho:W_K \to \text{GL}(V)$ be a representation — then $\rho$ is semisimple if it satisfies either condition of the preceding lemma.

38: N.B. Irreducible representations are semisimple.

39: THEOREM Let $\rho:W_K \to \text{GL}(V)$ be a representation — then the following conditions are equivalent.

1. $\rho$ is semisimple.

2. $\rho(\tilde{\sigma}_{q})$ is semisimple.

3. $\rho(w)$ is semisimple $\forall w \in W_K$. 
§24. THE WEIL-DELINE GROUP

1: DEFINITION The Weil-Deligne group $W_k^*$ is the semidirect product $\mathbb{C} \rtimes W_k$, the multiplication rule being

$$(z_1,w_1)(z_2,w_2) = (z_1 + |w_1|z_2,w_1w_2).$$

[Note: The identity in $W_k^*$ is $(0,e)$ and the inverse of $(z,w)$ is $(-|w|^{-1}z,w^{-1})$:]

$$(z,w)(-|w|^{-1}z,w^{-1})$$

$$= (z + |w|(-|w|^{-1}z),ww^{-1})$$

$$= (z - z,e) = (0,e).]$$

2: N.B. The topology on $W_k^*$ is the product topology.

3: DEFINITION A Deligne representation of $W_k$ is a triple $(\rho,V,N)$, where $\rho:W_k \rightarrow GL(V)$ is a representation of $W_k$ and $N:V \rightarrow V$ is a nilpotent endomorphism of $V$ subject to the relation

$$\rho(w)N\rho(w)^{-1} = |w|N \quad (w \in W_k).$$

[Note: $N = 0$ is admissible so every representation of $W_k$ is a Deligne representation.]

4: EXAMPLE Take $V = \mathbb{C}^n$, hence $GL(V) = GL_n(\mathbb{C})$. Let $e_0,e_1,\ldots,e_{n-1}$ be the usual basis of $V$. Define $\rho$ by the rule

$$\rho(w)e_i = |w|^{i}e_i \quad (w \in W_k, 0 \leq i \leq n-1)$$
and define \( N \) by the rule
\[
N_{e_i} = e_{i+1} \quad (0 \leq i \leq n-2), \quad N_{e_{n-1}} = 0.
\]

Then the triple \((\rho, V, N)\) is a Deligne representation of \( W_K \), the \( n \)-dimensional special representation, denoted \( \text{sp}(n) \).

5: DEFINITION A representation of \( W_D \) is a continuous homomorphism \( \rho_1 : W_D \to \text{GL}(V) \) whose restriction to \( C \) is complex analytic, where \( V \) is a finite dimensional complex vector space.

6: LEMMA Every Deligne representation \((\rho, V, N)\) of \( W_K \) gives rise to a representation \( \rho_1 : W_D \to \text{GL}(V) \) of \( W_D \).

PROOF Put
\[
\rho_1(z, w) = \exp(zN) \rho(w).
\]
Then
\[
\rho_1(z_1, w_1) \rho_1(z_2, w_2)
\]
\[
= \exp(z_1 N) \rho(w_1) \exp(z_2 N) \rho(w_2)
\]
\[
= \exp(z_1 N) \rho(w_1) \exp(z_2 N) \rho(w_1^{-1}) \rho(w_1) \rho(w_2)
\]
\[
= \exp(z_1 N) \exp(z_2 |w_1| |N|) \rho(w_1 w_2)
\]
\[
= \exp(z_1 N + z_2 |w_1| |N|) \rho(w_1 w_2)
\]
\[
= \exp((z_1 + |w_1| z_2) N) \rho(w_1 w_2)
\]
3.

\[ = \rho'(z_1 + |w_1|z_2,w_1w_2) \]

\[ = \rho'((z_1,w_1)(z_2,w_2)). \]

[Note: The continuity of \( \rho' \) is manifest as is the complex analyticity of its restriction to \( \mathcal{C} \).]

One can also go the other way but this is more involved.

7: RAPPEL If \( T: V \to V \) is unipotent, then

\[ \log T = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} (T - I)^n \]

is nilpotent.

8: SUBLEMMA Let \( \rho': W \to GL(V) \) be a representation of \( W \) -- then \( \forall \ z \neq 0 \), \( \rho'(z,e) \) is unipotent.

9: SUBLEMMA Let \( \rho': W \to GL(V) \) be a representation of \( W \) -- then \( \forall \ z \neq 0 \),

\[ \log \rho'(z,e) \]

is nilpotent and

\[ (\log \rho'(z,e))/z \quad (z \neq 0) \]

is independent of \( z \).

10: LEMMA Every representation \( \rho': W \to GL(V) \) of \( W \) gives rise to a Deligne representation \( (\rho, V, N) \) of \( W \).

PROOF Put

\[ \rho = \rho'|0 \times W, \quad N = \log \rho'(1,e). \]
Then \( \forall w \in W_k \),

\[
\rho(w)N\rho(w)^{-1} = \rho(w)\log \rho'(1,e)\rho(w)^{-1}
\]

\[
= \rho(w) \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\rho'(1,e) - 1)^n \right) \rho(w)^{-1}
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\rho(w)\rho'(1,e)\rho(w)^{-1} - 1)^n.
\]

And

\[
\rho(w)\rho'(1,e)\rho(w)^{-1}
\]

\[
= \rho'(0,w)\rho'(1,e)\rho'(0,w^{-1})
\]

\[
= \rho'((0,w)(1,e)(0,w^{-1}))
\]

\[
= \rho'((|w|,w)(0,w^{-1}))
\]

\[
= \rho'(|w|,e).
\]

Therefore

\[
\rho(w)N\rho(w)^{-1}
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\rho'(|w|,e) - 1)^n
\]

\[
= \log \rho'(|w|,e)
\]

\[
= |w| \left( \log \rho'(|w|,e) / |w| \right)
\]

\[
= |w| \log \rho'(1,e)
\]

\[
= |w| N.
\]
11: OPERATIONS

- **Direct Sum:** Let \((\rho_1, V_1, N_1), (\rho_2, V_2, N_2)\) be Deligne representations -- then their direct sum is the triple

\[ (\rho_1 \oplus \rho_2, V_1 \oplus V_2, N_1 \oplus N_2). \]

- **Tensor Product:** Let \((\rho_1, V_1, N_1), (\rho_2, V_2, N_2)\) be Deligne representations -- then their tensor product is the triple

\[ (\rho_1 \otimes \rho_2, V_1 \otimes V_2, N_1 \otimes I_2 + I_1 \otimes N_2). \]

- **Contragredient:** Let \((\rho, V, N)\) be a Deligne representation -- then its contragredient is the triple

\[ (\rho^\vee, V^\vee, - N^\vee). \]

[Note: \(V^\vee\) is the dual of \(V\) and \(N^\vee\) is the transpose of \(N\) (thus \(\forall f \in V^\vee, N^\vee(f) = f \circ N\)).]

12: REMARK The definitions of \(\oplus, \otimes, \vee\) when transcribed to the "prime picture" are the usual representation-theoretic formalities applied to the group \(W_K\).

13: N.B. Let

\[
\begin{pmatrix}
(\rho_1, N_1, V_1) \\
(\rho_2, N_2, V_2)
\end{pmatrix}
\]

be Deligne representations of \(W_K\) -- then a morphism

\[ (\rho_1, N_1, V_1) \rightarrow (\rho_2, N_2, V_2) \]
is a linear map \( T: V_1 \to V_2 \) such that

\[
T_{\rho_1}(w) = \rho_2(w) T \quad (w \in W_K)
\]

and \( TN_1 = N_2 T \).

[Note: If \( T \) is a linear isomorphism, then the Deligne representations

\[
\begin{pmatrix}
(\rho_1, N_1, V_1) \\
(\rho_2, N_2, V_2)
\end{pmatrix}
\]

are said to be isomorphic.]

14: DEFINITION Suppose that \((\rho, V, N)\) is a Deligne representation of \( W_K \) --
then a subspace \( V_0 \subset V \) is an invariant subspace if it is invariant under \( \rho \) and \( N \).

15: LEMMA The kernel of \( N \) is an invariant subspace.

PROOF If \( NV = 0 \), then \( \forall w \in W_K \),

\[
N_{\rho}(w)v = \|w^{-1}\|\rho(w)Nv = 0.
\]

16: DEFINITION A Deligne representation \((\rho, V, N)\) of \( W_K \) is indecomposable
if \( V \) cannot be written as a direct sum of proper invariant subspaces.

17: EXAMPLE Consider \( sp(n) \) -- then it is indecomposable.

[If \( C^\mathbb{N} = S \oplus T \) was a nontrivial decomposition into proper invariant subspaces,
then both \[
\begin{pmatrix}
S \cap \text{Ker } N \\
T \cap \text{Ker } N
\end{pmatrix}
\]
would be nontrivial.]
18: DEFINITION A Deligne representation \((\rho,V,N)\) of \(W_K\) is **semisimple** if \(\rho\) is semisimple (cf. §23, #37).

19: EXAMPLE Consider \(sp(n)\) -- then it is semisimple.

20: LEMMA Let \(\pi\) be an irreducible representation of \(W_K\) -- then \(sp(n) \otimes \pi\) is semisimple and indecomposable.

[Note: Recall that \(\pi\) is identified with \((\pi,0)\).]

21: THEOREM Every semisimple indecomposable Deligne representation of \(W_K\) is equivalent to a Deligne representation of the form \(sp(n) \otimes \pi\), where \(\pi\) is an irreducible representation of \(W_K\) and \(n\) is a positive integer.

22: THEOREM Let \((\rho,N,V)\) be a semisimple Deligne representation of \(W_K\) -- then there is a decomposition

\[
(\rho,V,N) = \bigoplus_{i=1}^{s} sp(n_i) \otimes \pi_i,
\]

where \(\pi_i\) is an irreducible representation of \(W_K\) and \(n_i\) is a positive integer. Furthermore, if

\[
(\rho,V,N) = \bigoplus_{j=1}^{t} sp(n'_j) \otimes \pi'_j
\]

is another such decomposition, then \(s = t\) and after a renumbering of the summands, \(\pi_i \cong \pi'_i\) and \(n_i = n'_i\).

APPENDIX

Instead of working with

\[
WD_K = C \times | W_K'|
\]
some authorities work with

\[ SL(2, \mathbb{C}) \times W_K, \]

the rationale for this being that the semisimple representations of the two groups are the "same".

Given \( w \in W_K \), let

\[
h_w = \begin{bmatrix}
|w|^{1/2} & 0 \\
0 & |w|^{-1/2}
\end{bmatrix}
\]

and identify \( z \in \mathbb{C} \) with

\[
\begin{bmatrix}
1 & z \\
0 & 1
\end{bmatrix}.
\]

Then

\[
h_w \begin{bmatrix}
1 & z \\
0 & 1
\end{bmatrix} h_w^{-1} = \begin{bmatrix}
1 & |w|z \\
0 & 1
\end{bmatrix}.
\]

But conjugation by \( h_w \) is an automorphism of \( SL(2, \mathbb{C}) \), thus one can form the semi-direct product \( SL(2, \mathbb{C}) \times| W_K, \) the multiplication rule being

\[
(X_1, w_1)(X_2, w_2) = (X_1 h_{w_1} X_2 h_{w_1}^{-1}, w_1 w_2).
\]
LEMMA The arrow

$$(X,w) \rightarrow (Xw',w)$$

from

$$\text{SL}(2,\mathbb{C}) \times W_K$$

is an isomorphism of groups.

DEFINITION A representation of $\text{SL}(2,\mathbb{C}) \times W_K$ is a continuous homomorphism

$$\rho: \text{SL}(2,\mathbb{C}) \times W_K \rightarrow \text{GL}(V)$$

(V a finite dimensional complex vector space) such that the restriction of $\rho$ to $\text{SL}(2,\mathbb{C})$ is complex analytic.

N.B. $\rho$ is semisimple iff its restriction to $W_K$ is semisimple.

[The restriction of $\rho$ to $\text{SL}(2,\mathbb{C})$ is necessarily semisimple.]

The finite dimensional irreducible representations of $\text{SL}(2,\mathbb{C})$ are parameterized by the positive integers:

$$n \longleftrightarrow \text{sym}(n), \dim \text{sym}(n) = n.$$

THEOREM The isomorphism classes of semisimple Deligne representations of $W_K$ are in a 1-to-1 correspondence with the isomorphism classes of semisimple representations of $\text{SL}(2,\mathbb{C}) \times W_K$.

To explicate matters, start with a semisimple indecomposable Deligne representation of $W_K$, say $\text{sp}(n) \otimes \pi$, and assign to it the external tensor product $\text{sym}(n) \otimes \pi$, hence in general

$$\bigotimes_{i=1}^{s} \text{sp}(n_i) \otimes \pi_i \otimes \bigotimes_{i=1}^{s} \text{sym}(n_i) \otimes \pi_i.$$