ABELIAN THEORY

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1. GROUP SCHEMES

1: NOTATION \( \text{SCH} \) is the category of schemes, \( \text{RNG} \) is the category of commutative rings with unit.

Fix a scheme \( S \) -- then the category \( \text{SCH}/S \) of schemes over \( S \) (or of \( S \)-schemes) is the category whose objects are the morphisms \( X \rightarrow S \) of schemes and whose morphisms

\[
\text{Mor}(X \rightarrow S, Y \rightarrow S)
\]

are the morphisms \( X \rightarrow Y \) of schemes with the property that the diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
S & \longrightarrow & S
\end{array}
\]

commutes.

[Note: Take \( S = \text{Spec}(\mathbb{Z}) \) -- then

\( \text{SCH}/S = \text{SCH} \).]

2: N.B. If \( S = \text{Spec}(A) \) (\( A \) in \( \text{RNG} \)) is an affine scheme, then the terminology is "schemes over \( A \)" (or "\( A \)-schemes") and one writes \( \text{SCH}/A \) in place of \( \text{SCH}/\text{Spec}(A) \).

3: NOTATION Abbreviate \( \text{Mor}(X \rightarrow S, Y \rightarrow S) \) to \( \text{Mor}_S(X,Y) \) (or to \( \text{Mor}_A(X,Y) \) if \( S = \text{Spec}(A) \)).

4: REMARK The \( S \)-scheme \( \text{id}_S: S \rightarrow S \) is a final object in \( \text{SCH}/S \).
5: THEOREM SCH/S has pullbacks:

\[
\begin{array}{c}
X \times_S Y \longrightarrow Y \\
\downarrow \downarrow \\
X \longrightarrow S.
\end{array}
\]

[Note: Every diagram admits a unique filler

\[
(u,v)_S : Z \rightarrow X \times_S Y
\]

such that

\[
\begin{align*}
p \circ (u,v)_S &= u \\
q \circ (u,v)_S &= v.
\end{align*}
\]

6: FORMALITIES Let \( X, Y, Z \) be objects in SCH/S -- then

\[
X \times_S S \simeq X,
\]

\[
X \times_S Y \simeq Y \times_S X,
\]

and

\[
(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z).
\]
7: REMARK If $X, Y, X', Y'$ are objects in $\text{SCH}/S$ and if $u: X \to X', \, v: Y \to Y'$ are $S$-morphisms, then there is a unique morphism $u \times_S v$ (or just $u \times v$) rendering the diagram

![Diagram](image-url)

commutative.

[Spelled out, $u \times_S v = (u \circ p, v \circ q)_S$.]

8: BASE CHANGE Let $u: S' \to S$ be a morphism in $\text{SCH}$.

- If $X \to S$ is an $S$-object, then $X \times_S S'$ is an $S'$-object via the projection $X \times_S S' \to S'$, denoted $u^*(X)$ or $X_{(S')}$, and called the base change of $X$ by $u$.

- If $X \to S, \, Y \to S$ are $S$-objects and if $f: (X \to S) \to (Y \to S)$ is an $S$-morphism, then

$$f \times_S \text{id}_{S'}, \quad X \times_S S' \to Y \times_S S'$$
is a morphism of $S'$-objects, denoted $u^*(f)$ or $f_{(S')}$ and called the base change of $f$ by $u$.

These considerations thus lead to a functor

$$u^*: \text{SCH}/S \rightarrow \text{SCH}/S'$$
called the base change by $u$.

9: N.B. If $u': S'' \rightarrow S'$ is another morphism in SCH, then the functors $(u \circ u')^*$ and $(u')^* \circ u$ from $\text{SCH}/S$ to $\text{SCH}/S''$ are isomorphic.

10: LEMMA Let $u: S' \rightarrow S$ be a morphism in SCH. Suppose that $T' \rightarrow S'$ is an $S'$-object — then $T'$ can be viewed as an $S$-object $T$ via postcomposition with $u$ and there are canonical mutually inverse bijections

$$\text{Mor}_S(T', X_{(S')}) \rightarrow \text{Mor}_S(T, X)$$

functorial in $T'$ and $X$.

11: NOTATION Each $S$-scheme $X \rightarrow S$ determines a functor

$$(\text{SCH}/S)^{\text{OP}} \rightarrow \text{SET},$$
viz. the assignment

$$T \rightarrow \text{Mor}_S(T, X) \cong X_S(T),$$
the set of $T$-valued points of $X$.

[Note: In terms of category theory,

$$X_S(T) = h_X \circ g(T \rightarrow S).]$$
12: **Lemma** To give a morphism \( (X \to S) \xrightarrow{f} (Y \to S) \) in \( \text{SCH}/S \) is equivalent to giving for all \( S \)-schemes \( T \) a map
\[
f(T): X_S(T) \to Y_S(T)
\]
which is functorial in \( T \), i.e., for all morphisms \( u: T' \to T \) of \( S \)-schemes the diagram
\[
\begin{array}{ccc}
X_S(T) & \xrightarrow{f(T)} & Y_S(T) \\
\downarrow X_S(u) & & \downarrow Y_S(u) \\
X_S(T') & \xrightarrow{f(T')} & Y_S(T')
\end{array}
\]
commutes.

13: **Definition** A **group scheme** over \( S \) (or an **\( S \)-group**) is an object \( G \) of \( \text{SCH}/S \) and \( S \)-morphisms
\[
m: G \times_S G \to G \quad \text{("multiplication")}
\]
\[
e: S \to G \quad \text{("unit")}
\]
\[
i: G \to G \quad \text{("inversion")}
\]
such that the diagrams
\[
\begin{array}{ccc}
G \times_S G & \xrightarrow{m \times \text{id}_G} & G \times_S G \\
\downarrow \text{id}_G \times m & & \downarrow m \\
G \times_S G & \xrightarrow{m} & G
\end{array}
\]
6.

\[
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {$G \times_S S$};
\node (B) at (2,0) {$G \times_S G$};
\node (C) at (0,-1) {$G$};
\node (D) at (2,-1) {$G$};
\node (E) at (1,0) {$m$};
\node (F) at (1,-1) {$id_G$};
\node (G) at (2,-2) {$S$};
\node (H) at (0,-2) {$id_G$};
\node (I) at (1,-2) {$e$};
\draw[->] (A) to (B);
\draw[->] (C) to (D);
\draw[->] (E) to (F);
\draw[->] (G) to (H);
\draw[->] (I) to (D);
\end{tikzpicture}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {$G$};
\node (B) at (2,0) {$G \times_S G$};
\node (C) at (0,-1) {$S$};
\node (D) at (2,-1) {$G$};
\node (E) at (1,0) {$m$};
\node (F) at (1,-1) {$e$};
\draw[->] (A) to (B);
\draw[->] (C) to (D);
\draw[->] (E) to (D);
\end{tikzpicture}
\end{array}
\]

Remark: To say that \((G; m, e, i)\) is a group scheme over \(S\) amounts to saying that \(G\) is a group object in \(\text{SCH}/S\).

15: Lemma Let \(G\) be an \(S\)-scheme -- then \(G\) gives rise to a group scheme over \(S\) iff for all \(S\)-schemes \(T\), the set \(G_S(T)\) carries the structure of a group which is functorial in \(T\) (i.e., for all \(S\)-morphisms \(T' \to T\), the induced map \(G_S(T) \to G_S(T')\) is a homomorphism of groups).

16: Remark It suffices to define functorial group structures on the \(G_S(A)\), where \(\text{Spec}(A) \to S\) is an affine \(S\)-scheme.

[This is because morphisms of schemes can be "glued".]

17: Lemma Let \(u: S' \to S\) be a morphism in \(\text{SCH}\). Suppose that \((G; m, e, i)\) is
a group scheme over $S$ -- then

$$(G \times_S S', m_{(S')}, e_{(S')}, i_{(S')})$$

is a group scheme over $S'$.

[Note: For every $S'$-object $T' \to S'$,

$$(G \times_S S')_{S'}(T') = G_{S}(T)'$$

where $T$ is the $S$-object $T' \to S' \to S$.]

18: THEOREM If $(X, \mathcal{O}_X)$ is a locally ringed space and if $A$ is a commutative ring with unit, then there is a functorial set-theoretic bijection

$$\text{Mor}(S, \text{Spec}(A)) \cong \text{Mor}(A, \Gamma(X, \mathcal{O}_X)).$$

[Note: The "Mor" on the LHS is in the category of locally ringed spaces and the "Mor" on the RHS is in the category of commutative rings with unit.]

19: EXAMPLE Take $S = \text{Spec}(Z)$ and let

$$A^n = \text{Spec}(Z[t_1, \ldots, t_n]).$$

Then for every scheme $X$,

$$\text{Mor}(X, A^n) \cong \text{Mor}(Z[t_1, \ldots, t_n], \Gamma(X, \mathcal{O}_X))$$

$$\cong \Gamma(X, \mathcal{O}_X)^n (\phi + (\phi(t_1), \ldots, \phi(t_n))).$$

Therefore $A^n$ is a group object in $\text{SCH}$ called affine $n$-space.

[Note: Here $\Gamma(X, \mathcal{O}_X)$ is being viewed as an additive group, hence the underlying multiplicative structure is being ignored.]
20: N.B. Given any scheme $S$,

$$A^n_S = A^n \times_S S$$

is an $S$-scheme and for every morphism $S' \to S$,

$$A^n_S \times_S S' \simeq A^n \times_S S \times_S S' \simeq A^n_{S'}.$$ 


22: NOTATION Given $A$ in $\text{RNG}$, denote

$$G_\alpha \times_Z \text{Spec}(A)$$

by $G_\alpha \otimes A$ or still, by $G_{\alpha, A}$.

23: N.B.

$$G_{\alpha, A} = \text{Spec}(\mathbb{Z}[t]) \times_Z \text{Spec}(A)$$

$$= \text{Spec}(\mathbb{Z}[t] \otimes A) = \text{Spec}(A[t]).$$

24: LEMMA $G_{\alpha, A}$ is a group object in $\text{SCH}/A$.

There are two other "canonical" examples of group objects in $\text{SCH}/A$.

- $G_{m, A} = \text{Spec}(A[u, v]/(uv - 1))$

which assigns to an $A$-scheme $X$ the multiplicative group $\Gamma(X, 0_X)^\times$ of invertible elements in the ring $\Gamma(X, 0_X)$.

- $GL_{n, A} = \text{Spec}(A[t_{11}, \ldots, t_{nn}, \det(t_{ij})^{-1}])$
which assigns to an A-scheme X the group
\[ \text{GL}_{n}(\Gamma(X,\mathcal{O}_X)) \]
of invertible \( n \times n \)-matrices with entries in the ring \( \Gamma(X,\mathcal{O}_X) \).

25: **DEFINITION** If \( G \) and \( H \) are \( S \)-groups, then a homomorphism from \( G \) to \( H \) is a morphism \( f:G \to H \) of \( S \)-schemes such that for all \( S \)-schemes \( T \) the induced map \( f(T):G_S(T) \to H_S(T) \) is a group homomorphism.

26: **EXAMPLE** Take \( S = \text{Spec}(A) \) -- then
\[ \text{det}_A:G_{n,A} \to G_{m,A} \]
is a homomorphism.

27: **DEFINITION** Let \( G \) be a group scheme over \( S \) -- then a subscheme (resp. an open subscheme, resp. a closed subscheme) \( H \subset G \) is called an \( S \)-subgroup scheme (resp. an open \( S \)-subgroup scheme, resp. a closed \( S \)-subgroup scheme) if for every \( S \)-scheme \( T \), \( H_S(T) \) is a subgroup of \( G_S(T) \).

28: **EXAMPLE** Given a positive integer \( n \), \( \mu_{n,A} \) is the group object in \( \text{SCH}/A \) which assigns to an A-scheme X the multiplicative subgroup of \( \Gamma(X,\mathcal{O}_X)^X \) consisting of those \( \phi \) such that \( \phi^n = 1 \), thus
\[ \mu_{n,A} = \text{Spec}(A[t]/(t^n-1)) \]
and \( \mu_{n,A} \) is a closed A-subgroup of \( G_{m,A} \).

29: **EXAMPLE** Fix a prime number \( p \) and suppose that \( A \) has characteristic \( p \).
Given a positive integer $n$, $\alpha_{n, A}$ is the group object in $\text{SCH}/A$ which assigns to an $A$-scheme $X$ the additive subgroup of $\Gamma(X, \mathcal{O}_X)$ consisting of those $\phi$ such that $\phi^p^n = 0$, thus

$$\alpha_{n, A} = \text{Spec}(A[t]/(t^p^n))$$

and $\alpha_{n, A}$ is a closed $A$-subgroup of $G_a, A$.

30: CONSTRUCTION Let $f: G \to H$ be a homomorphism of $S$-groups. Define $\text{Ker}(f)$ by the pullback square

$$\begin{array}{ccc}
\text{Ker}(f) = S \times_H G & \longrightarrow & G \\
\downarrow & & \downarrow f \\
S & \longrightarrow & H.
\end{array}$$

Then for all $S$-schemes $T$,

$$\text{Mor}_S(T, \text{Ker}(f)) = \text{Ker}(G_S(T) \longrightarrow H_S(T)),$$

so $\text{Ker}(f)$ is an $S$-group.

31: EXAMPLE The kernel of $\text{det}_A$ is $\text{SL}_{n, A}$.

32: N.B. Other kernels are $\mu_{n, A}$ and $\alpha_{n, A}$.

33: CONVENTION If $P$ is a property of morphisms of schemes, then an $S$-group $G$ has property $P$ if this is the case of its structural morphism $G \to S$. 
E.g.: The property of morphisms of schemes being quasi-compact, locally of finite type, separated, étale etc.

34: Lemma Let

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
Y' & \longrightarrow & Y
\end{array}
\]

be a pullback square in \textit{SCH}. Suppose that \( f \) is a closed immersion -- then the same holds for \( f' \).

35: Application Let \( g:Y \rightarrow X \) be a morphism of schemes that has a section \( s:X \rightarrow Y \). Assume: \( g \) is separated -- then \( s \) is a closed immersion.

[The commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow s & & \downarrow \Delta_{Y/X} \\
Y & \longrightarrow & Y \times_X Y \\
\end{array}
\]

is a pullback square in \textit{SCH}. But \( g \) is separated, hence the diagonal morphism \( \Delta_{Y/X} \) is a closed immersion. Now quote the preceding lemma.]

If \( G \rightarrow S \) is a group scheme over \( S \), then the composition

\[
e\quad S \longrightarrow G \longrightarrow S
\]

is \( \text{id}_S \). Proof: \( e \) is an \( S \)-morphism and the diagram
commutes. Therefore $e$ is a section for the structural morphism $G \to S$:

$$
\begin{array}{ccc}
S & \xrightarrow{e} & G \\
\downarrow & & \downarrow \\
S & \xrightarrow{id_S} & S
\end{array}
$$

36: LEMMA Let $G \to S$ be a group scheme over $S$ -- then the structural morphism $G \to S$ is separated iff $e:S \to G$ is a closed immersion.

[To see that "closed immersion" $\Rightarrow$ "separated", consider the pullback square

$$
\begin{array}{ccc}
G & \xrightarrow{\Delta_{G/S}} & S \\
\downarrow & & \downarrow e \\
G \times_S G & \xrightarrow{m \circ (id_C \times i)} & G
\end{array}
$$

37: LEMMA If $S$ is a discrete scheme, then every $S$-group is separated.

38: APPLICATION Take $S = \text{Spec}(k)$, where $k$ is a field -- then the structural morphism $X \to \text{Spec}(k)$ of a $k$-scheme $X$ is separated.
1.

§2. SCH/k

Fix a field $k$.

1: **DEFINITION** A $k$-algebra is an object in RNG and a ring homomorphism $k \rightarrow A$.

2: **NOTATION** $\text{Alg}/k$ is the category whose objects are the $k$-algebras $k \rightarrow A$ and whose morphisms

$$(k \rightarrow A) \rightarrow (k \rightarrow B)$$

are the ring homomorphisms $A \rightarrow B$ with the property that the diagram

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\uparrow & & \uparrow \\
k & \rightarrow & k
\end{array}
$$

commutes.

3: **DEFINITION** Let $A$ be a $k$-algebra -- then $A$ is **finitely generated** if there exists a surjective homomorphism $k[t_1, \ldots, t_n] \rightarrow A$ of $k$-algebras.

4: **DEFINITION** Let $A$ be a $k$-algebra -- then $A$ is **finite** if there exists a surjective homomorphism $k^n \rightarrow A$ of $k$-modules.

5: **N.B.** A finite $k$-algebra is finitely generated.

Recall now that $\text{SCH}/k$ stands for $\text{SCH}/\text{Spec}(k)$.

6: **LEMMA** The functor

$$A \rightarrow \text{Spec}(A)$$
from $(\text{ALG/k})^{\text{OP}}$ to $\text{SCH/k}$ is fully faithful.

7: DEFINITION Let $X \to \text{Spec}(k)$ be a k-scheme -- then $X$ is locally of finite type if there exists an affine open covering $X = \bigcup U_i$ such that for all $i \in I$, $U_i = \text{Spec}(A_i)$, where $A_i$ is a finitely generated k-algebra.

8: DEFINITION Let $X \to \text{Spec}(k)$ be a k-scheme -- then $X$ is of finite type if $X$ is locally of finite type and quasi-compact.

9: LEMA If a k-scheme $X \to \text{Spec}(k)$ is locally of finite type and if $U \subset X$ is an open affine subset, then $\Gamma(U, \mathcal{O}_X)$ is a finitely generated k-algebra.

10: APPLICATION If $A$ is a finitely generated k-algebra, then the k-scheme $\text{Spec}(A) \to \text{Spec}(k)$ is of finite type.

11: LEMA If $X \to \text{Spec}(k)$ is a k-scheme of finite type, then all subschemes of $X$ are of finite type.

12: RAPPEL Let $(X, \mathcal{O}_X)$ be a locally ringed space. Given $x \in X$, denote the stalk of $\mathcal{O}_X$ at $x$ by $\mathcal{O}_{X,x}$ -- then $\mathcal{O}_{X,x}$ is a local ring. And:

- $\mathfrak{m}_x$ is the maximal ideal in $\mathcal{O}_{X,x}$.
- $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field of $\mathcal{O}_{X,x}$.

13: CONSTRUCTION Let $(X, \mathcal{O}_X)$ be a scheme. Given $x \in X$, let $U = \text{Spec}(A)$ be an affine open neighborhood of $x$. Denote by $\mathfrak{p}$ the prime ideal of $A$ corresponding
to \(x\), hence \(O_{X,x} = 0_{U,x} = A_p\) (the localization of \(A\) at \(p\)) and the canonical homomorphism \(A \rightarrow A_p\) leads to a morphism

\[
\text{Spec}(O_{X,x}) = \text{Spec}(A_p) ightarrow \text{Spec}(A) = U \subset X
\]

of schemes (which is independent of the choice of \(U\)).

14: N.B. There is an arrow \(O_{X,x} \rightarrow \kappa(x)\), thus an arrow \(\text{Spec}(\kappa(x)) \rightarrow \text{Spec}(O_{X,x})\), thus an arrow

\[
i_x: \text{Spec}(\kappa(x)) \rightarrow X
\]

whose image is \(x\).

Let \(K\) be any field, let \(f: \text{Spec}(K) \rightarrow X\) be a morphism of schemes, and let \(x\) be the image of the unique point \(p\) of \(\text{Spec}(K)\). Since \(f\) is a morphism of locally ringed spaces, at the stalk level there is a homomorphism

\[
0_{X,x} \rightarrow 0_{\text{Spec}(K),p} = K
\]

of local rings meaning that the image of the maximal ideal \(m_x \subset 0_{X,x}\) is contained in the maximal ideal \((1)\) of \(K\), so there is an induced homomorphism

\[
\iota: \kappa(x) \rightarrow K.
\]

Consequently,

\[
f = i_x \circ \text{Spec}(\iota).
\]

15: SCHOLIUM There is a bijection

\[
\text{Mor}(\text{Spec}(K),X) \rightarrow \{(x,\iota): x \in X, \iota: \kappa(x) \rightarrow K\}.
\]
If $X \to \text{Spec}(k)$ is a $k$-scheme, then for any $x \in X$, there is an arrow
$$\text{Spec}(\kappa(x)) \to X,$$
from which an arrow
$$\text{Spec}(\kappa(x)) \to \text{Spec}(k),$$
or still, an arrow $k \to \kappa(x)$.

16: **Lemma** Let $X \to \text{Spec}(k)$ be a $k$-scheme locally of finite type -- then $x \in X$ is closed iff the field extension $\kappa(x)/k$ is finite.

17: **Application** Let $X \to \text{Spec}(k)$ be a $k$-scheme locally of finite type. Assume: $k$ is algebraically closed -- then
$$\{x \in X : x \text{ closed}\} = \{x \in X : k = \kappa(x)\}$$
$$= \text{Mor}_k(\text{Spec}(k), X) \cong X(k).$$

18: **Definition** A subset $Y$ of a topological space $X$ is dense in $X$ if $\overline{Y} = X$.

19: **Definition** A subset $Y$ of a topological space $X$ is very dense in $X$ if for every closed subset $F \subset X$, $\overline{F \cap Y} = F$.

20: **N.B.** If $Y$ is very dense in $X$, then $Y$ is dense in $X$.

[Take $F = X : \overline{X \cap Y} = \overline{Y} = X$.]

21: **Lemma** Let $X \to \text{Spec}(k)$ be a $k$-scheme locally of finite type -- then
$$\{x \in X : x \text{ closed}\}$$
is very dense in $X$.

22: **Definition** Let $X \to \text{Spec}(k)$ be a $k$-scheme -- then a point $x \in X$ is
called k-rational if the arrow \( k + \kappa(x) \) is an isomorphism.

23: N.B. Sending a k-morphism \( \text{Spec}(k) \to X \) to its image sets up a bijection between the set

\[
X(k) = \text{Mor}_k(\text{Spec}(k),X)
\]

and the set of k-rational points of \( X \).

24: REMARK \( X(k) \) may very well be empty. 

[Consider what happens if \( k'/k \) is a proper field extension.]

Given a k-scheme \( X \to \text{Spec}(k) \) and a field extension \( K/k \), let

\[
X(K) = \text{Mor}_K(\text{Spec}(K),X)
\]

be the set of \( K \)-valued points of \( X \). If \( x: \text{Spec}(K) \to X \) is a \( K \)-valued point with image \( x \in X \), then there are field extensions

\[
k + \kappa(x) + K.
\]

25: N.B. \( \text{Spec}(K) \) is a k-scheme, the structural morphism \( \text{Spec}(K) \to \text{Spec}(k) \) being derived from the arrow of inclusion \( j:k \to K \).

Let \( G = \text{Gal}(K/k) \). Given \( \sigma: K \to K \) in \( G \),

\[
\text{Spec}(\sigma): \text{Spec}(K) \to \text{Spec}(K),
\]

hence

\[
\begin{array}{ccc}
\text{Spec}(\sigma) & \to & \text{Spec}(K) \\
\downarrow & & \downarrow \\
\text{Spec}(K) & \to & \text{Spec}(K) \\
\end{array}
\]

and we put

\[
\sigma \cdot x = x \circ \text{Spec}(\sigma).
\]
6.

- $\sigma \cdot x$ is a $K$-valued point.

There is a commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{\sigma} & K \\
\uparrow j & & \uparrow j \\
k & \xrightarrow{id_k} k
\end{array}
\]

so $\sigma \cdot j = j \circ id_k = j$, and if $\pi : X \to \text{Spec}(k)$ is the structural morphism, there is a commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{x} & X \\
\downarrow \text{Spec}(j) & & \downarrow \pi \\
\text{Spec}(k) & \xrightarrow{j} & \text{Spec}(k)
\end{array}
\]

so $\pi \circ x = \text{Spec}(j)$. The claim then is that the diagram

\[
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{x \circ \text{Spec}(\sigma)} & X \\
\downarrow \text{Spec}(j) & & \downarrow \pi \\
\text{Spec}(k) & \xrightarrow{\text{Spec}(\sigma)} & \text{Spec}(k)
\end{array}
\]

commutes. But

\[
\pi \circ x \circ \text{Spec}(\sigma) = \text{Spec}(j) \circ \text{Spec}(\sigma)
\]

\[
= \text{Spec}(\sigma \circ j)
\]

\[
= \text{Spec}(j).
\]

- The operation

\[
G \times X(K) \to X(K)
\]

\[
(\sigma, x) \mapsto \sigma \cdot x
\]
is a left action of $G$ on $X(K)$.

$$\tau \circ \sigma$$

[Given $\sigma, \tau \in G: \mathbb{K} \longrightarrow \mathbb{K} \longrightarrow \mathbb{K}$, it is a question of checking that

$$(\sigma \circ \tau) \cdot x = \sigma \cdot (\tau \cdot x).$$

But the LHS equals

$$x \circ \text{Spec}(\sigma \circ \tau) = x \circ \text{Spec}(\tau) \circ \text{Spec}(\sigma)$$

while the RHS equals

$$\tau \cdot x \circ \text{Spec}(\sigma) = x \circ \text{Spec}(\tau) \circ \text{Spec}(\sigma).$$]

26: NOTATION Let

$$K^G = \text{Inv}(G)$$

be the invariant field associated with $G$.

27: LEMMA The set $X(K)^G$ of fixed points in $X(K)$ for the left action of $G$ on $X(K)$ coincides with the set $X(K^G)$.

28: APPLICATION If $K$ is a Galois extension of $k$, then

$$X(K)^G = X(k).$$

Take $K = k^{\text{sep}}$, thus now $G = \text{Gal}(k^{\text{sep}}/k)$.

29: DEFINITION Suppose given a left action $G \times S \to S$ of $G$ on a set $S$ -- then $S$ is called a $G$-set if $\forall s \in S$, the $G$-orbit $G \cdot s$ is finite or, equivalently, the stabilizer $G_s \subset G$ is an open subgroup of $G$.

30: EXAMPLE Let $X \to \text{Spec}(k)$ be a $k$-scheme locally of finite type -- then
8.

∀ x ∈ X(k^{sep}), the G-orbit G · x of x in X(k^{sep}) is finite, hence X(k^{sep}) is a G-set.

31: DEFINITION Let X → Spec(k) be a k-scheme — then X is étale if it is of the form

\[ X = \bigsqcup_{i \in I} \text{Spec}(K_i), \]

where I is some index set and where K_i/k is a finite separable field extension.

There is a category \( \mathcal{E}T/k \) whose objects are the étale k-schemes and there is a category G-SET whose objects are the G-sets.

Define a functor

\[ \phi: \mathcal{E}T/k \to \text{G-SET} \]

by associating with each X in \( \mathcal{E}T/k \) the set X(k^{sep}) equipped with its left G-action.

32: LEMMA \( \phi \) is an equivalence of categories.

PROOF To construct a functor

\[ \psi: \text{G-SET} \to \mathcal{E}T/k \]

such that

\[ \psi \circ \phi \simeq \text{id} \quad \text{and} \quad \phi \circ \psi \simeq \text{id}_{\text{G-SET}'} \]

ET/k

\[ \text{G-SET'} \]

take a G-set S and write it as a union of G-orbits, say

\[ S = \bigsqcup_{i \in I} G \cdot s_i. \]

Let K_i ⊃ k be the finite separable field extension inside k^{sep} corresponding to
the open subgroup $G_{s_i} \subset G$ and assign to $S$ the étale $k$-scheme $\bigsqcup_{i \in I} \text{Spec}(K_i)$.

Proceed...

The foregoing equivalence of categories induces an equivalence between the corresponding categories of group objects:

étale group $k$-schemes $\approx$ $G$-groups,

where a $G$-group is a group which is a $G$-set, the underlying left action being by group automorphisms.

33: CONSTRUCTION Given a group $M$, let $M_k$ be the disjoint union

\[
\bigsqcup_M \text{Spec}(k),
\]

the constant group $k$-scheme, thus for any $k$-scheme $X \rightarrow \text{Spec}(k)$,

\[
\text{Mor}_k(X, M_k)
\]

is the set of locally constant maps $X \rightarrow M$ whose group structure is multiplication of functions.

[The terminology is standard but not the best since if $M$ is nontrivial, then \[
\text{Mor}_k(X, M_k) \approx M
\]
only if $X$ is connected.]

34: EXAMPLE For any étale group $k$-scheme $X$,

\[
X \times_k \text{Spec}(k^{\text{sep}}) \approx X(k^{\text{sep}}) \times_k \text{Spec}(k^{\text{sep}}).
\]

[Note: Here (and elsewhere),

\[
\times_k = \times_{\text{Spec}(k)}.
\]
35: **RAPPEL** An A in RING is reduced if it has no nilpotent elements \( \neq 0 \) (i.e., \( \exists a \neq 0 : a^n = 0 \ (\exists n) \)).

36: **DEFINITION** A scheme \( X \) is reduced if for any nonempty open subset \( U \subset X \), the ring \( \Gamma(U, \mathcal{O}_X) \) is reduced.

[Note: This is equivalent to the demand that all the local rings \( \mathcal{O}_{X,x} \) (\( x \in X \)) are reduced.]

37: **DEFINITION** Let \( X \) be a \( k \)-scheme -- then \( X \) is geometrically reduced if for every field extension \( K \supseteq k \), the \( K \)-scheme \( X \times_k \text{Spec}(K) \) is reduced.

38: **LEMMA** If \( X \) is a reduced \( k \)-scheme, then for every separable field extension \( K/k \), the \( K \)-scheme \( X \times_k \text{Spec}(K) \) is reduced.

39: **APPLICATION** Assume: \( k \) is a perfect field -- then every reduced \( k \)-scheme \( X \) is geometrically reduced.

40: **THEOREM** Assume: \( k \) is of characteristic zero. Suppose that \( X \) is a group \( k \)-scheme which is locally of finite type -- then \( X \) is reduced, hence is geometrically reduced.
53. AFFINE GROUP k-SCHEMES

Fix a perfect field $k$.

[Recall that a field $k$ is perfect if every field extension of $k$ is separable (equivalently, $\text{char}(k) = 0$ or $\text{char}(k) = p > 0$ and the arrow $x \to x^p$ is surjective).]

1: DEFINITION An affine group $k$-scheme is a group $k$-scheme of the form $\text{Spec}(A)$, where $A$ is a $k$-algebra.

2: EXAMPLE

$$G_{a,k} = \text{Spec}(k[t])$$

is an affine group $k$-scheme.

3: EXAMPLE

$$G_{m,k} = \text{Spec}(k[t, t^{-1}])$$

is an affine group $k$-scheme.

4: EXAMPLE

$$\mathbb{A}^{n,k} = \text{Spec}(k[t]/(t^n - 1)) \quad (n \in \mathbb{N})$$

is an affine group $k$-scheme.

There is a category $\text{GRP}/k$ whose objects are the group $k$-schemes and whose morphisms are the morphisms $f:X \to Y$ of $k$-schemes such that for all $k$-schemes $T$ the induced map

$$f(T):\text{Mor}_k(T,X) \to \text{Mor}_k(T,Y)$$

is a group homomorphism.
5: NOTATION

\text{AFF-GRP}/k

is the full subcategory of \text{GRP}/k whose objects are the affine group \text{k-schemes}.

6: NOTATION

\text{GRP-ALG}/k

is the category of group objects in \text{ALG}/k and

\text{GRP-}(\text{ALG}/k)^{\text{OP}}

is the category of group objects in \text{(ALG}/k)^{\text{OP}}.

7: LEMMA The functor

\text{A} \to \text{Spec}(A)

from \text{(ALG}/k)^{\text{OP}} to \text{SCH}/k is fully faithful and restricts to an equivalence

\text{GRP-}(\text{ALG}/k)^{\text{OP}} \leftrightarrow \text{AFF-GRP}/k.

8: REMARK An object in \text{GRP-}(\text{ALG}/k)^{\text{OP}} is a \text{k-algebra \text{A} which carries the structure of a commutative Hopf algebra over \text{k:}} \exists \text{k-algebra homomorphisms}

\Delta: A \to A \otimes_k A, \varepsilon: A \to k, S: A \to A

satisfying the "usual" conditions.

9: N.B. There is another way to view matters, viz. any functor \text{ALG}/k \to \text{GRP}

which is representable by a \text{k-algebra serves to determine an affine group \text{k-scheme}}
(and vice versa). From this perspective, a morphism \text{G} \to \text{H} of affine group \text{k-schemes}

is a natural transformation of functors, i.e., a collection of group homomorphisms
3.

$G(A) \to H(A)$ such that if $A \to B$ is a $k$-algebra homomorphism, then the diagram

$$
\begin{array}{c}
G(A) \quad \rightarrow \quad H(A) \\
\downarrow \quad \quad \quad \downarrow \\
G(B) \quad \rightarrow \quad H(B)
\end{array}
$$

commutes.

[Note: Suppose that

$$
\begin{align*}
G &= h^X = \text{Mor}(X, -) \\
H &= h^Y = \text{Mor}(Y, -).
\end{align*}
$$

Then from Yoneda theory,

$$\text{Mor}(G,H) \simeq \text{Mor}(Y,X).$$]

10: EXAMPLE $k[t, t^{-1}]$ represents $G_m, k$ and

$$k[t_{11}, \ldots, t_{nn}, \det(t_{ij})^{-1}]$$

represents $GL_n, k$. Given any $k$-algebra $A$, the determinant is a group homomorphism

$$GL_n, k(A) \to G_m, k(A)$$

and

$$\det_k \in \text{Mor}(GL_n, k, G_m, k).$$

[Note: There is a homomorphism

$$k[t, t^{-1}] \to k[t_{11}, \ldots, t_{nn}, \det(t_{ij})^{-1}]$$

of $k$-algebras that defines $\det_k$. E.g.: If $n = 2$, then the homomorphism in question sends $t$ to $t_{11}t_{22} - t_{12}t_{21}$.]

11: PRODUCTS Let

\[
G = h^X \quad (X \text{ in } \text{ALG}/k) \\
H = h^Y \quad (Y \text{ in } \text{ALG}/k)
\]

be affine group k-schemes. Consider the functor

\[
G \times H : \text{ALG}/k \to \text{GRP}
\]

defined on objects by

\[
A \mapsto G(A) \times H(A).
\]

Then this functor is represented by the k-algebra \(X \otimes_k Y:\)

\[
\text{Mor}(X \otimes_k Y, A) \cong \text{Mor}(X, A) \times \text{Mor}(Y, A)
\]

\[
= G(A) \times H(A).
\]

12: EXAMPLE Take

\[
G = \mathbb{G}_m, \mathbb{R} \\
H = \mathbb{G}_m, \mathbb{R}^ *
\]

Then

\[
(\mathbb{G}_m, \mathbb{R}) \times (\mathbb{G}_m, \mathbb{R}) (R) = \mathbb{R}^* \times \mathbb{R}^* = \mathbb{C}^*
\]

and

\[
(\mathbb{G}_m, \mathbb{R}) \times (\mathbb{G}_m, \mathbb{R}) (\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*.
\]

Let \(k'/k\) be a field extension — then for any k-algebra A, the tensor product

\(A \otimes_k k'\)

is a \(k'\)-algebra, hence there is a functor

\[
\text{ALG}/k \to \text{ALG}/k'
\]
termed extension of the scalars. On the other hand, every $k'$-algebra $B'$ can be regarded as a $k$-algebra $B$, from which a functor

$$\text{ALG}/k' \to \text{ALG}/k$$

termed restriction of the scalars.

13: LEMMA For all $k$-algebras $A$ and for all $k'$-algebras $B'$,

$$\text{Mor}_{k'}(A \otimes_k k', B') = \text{Mor}_k(A, B).$$

14: SCHOLIUM The functor "extension of the scalars" is a left adjoint for the functor "restriction of the scalars".

Let $G$ be an affine group $k$-scheme. Abusing the notation, denote still by $G$ the associated functor

$$\text{ALG}/k \to \text{GRP}.$$  

Then there is a functor

$$G_{k'}: \text{ALG}/k' \to \text{GRP},$$

namely

$$G_{k'}(A') = G(A),$$

where $A$ is $A'$ viewed as a $k$-algebra.

15: LEMMA $G_{k'}$ is an affine group $k'$-scheme and the assignment $G \to G_{k'}$ is functorial:

$$\text{AFF-GRP}/k \to \text{AFF-GRP}/k'.$$

[Note: Suppose that $G = X$ — then

$$\text{Mor}_{k'}(X \otimes_k k', A') \simeq \text{Mor}_k(X, A)$$]
Therefore $G_{k'}$ is represented by $X \otimes_k k'$:

$$G_{k'} = X \otimes_k k'.$$

Matters can also be interpreted "on the other side":

$$G_{k'} = \text{Spec}(X) \times_k \text{Spec}(k') \longrightarrow \text{Spec}(k')$$

$$G = \text{Spec}(X) \longrightarrow \text{Spec}(k).$$

16: **DEFINITION** $G_{k'}$ is said to have been obtained from $G$ by extension of the scalars.

17: **NOTATION** Given an affine group $k'$-scheme $G'$, let $G_{k'}/k$ be the functor

$$\text{ALG}/k \rightarrow \text{GRP}$$

defined by the rule

$$A \rightarrow G'(A \otimes_k k').$$

[Note: If $k' = k$, then $G_{k'}/k = G$.]

18: **THEOREM** Assume that $k'/k$ is a finite field extension — then $G_{k'}/k$ is an affine group $k$-scheme and the assignment $G' \rightarrow G_{k'}/k$ is functorial:

$$\text{AFF-GRP}/k' \rightarrow \text{AFF-GRP}/k.$$

19: **DEFINITION** $G_{k'}/k$ is said to have been obtained from $G'$ by restriction of the scalars.
LEMMA Assume that $k'/k$ is a finite field extension -- then for all affine group $k$-schemes $H$,

$$\text{Mor}_k(H, G_{k'} / k) \cong \text{Mor}_{k'}(H_{k'}, G').$$

SCHOLIUM The functor "restriction of the scalars" is a right adjoint for the functor "extension of the scalars".

Accordingly, there are arrows of adjunction

$$G \rightarrow (G_{k'})_{k'/k}$$

$$(G_{k'/k})_{k'} \rightarrow G'.$$

NOTATION

$$\text{Res}_{k'/k} : \text{AFF-GRP}/k' \rightarrow \text{AFF-GRP}/k$$

is the functor defined by setting

$$\text{Res}_{k'/k}(G') = G_{k'/k}.$$

So, by definition,

$$\text{Res}_{k'/k}(G')(A) = G'(A \otimes_k k').$$

And in particular:

$$\text{Res}_{k'/k}(G')(k) = G'(k \otimes_k k') = G'(k').$$

EXAMPLE Take $G' = A^n_{k'}$, then

$$\text{Res}_{k'/k}(A^n_{k'}) = A^{nd}_k (d = [k':k]).$$
EXAMPLE Take \( k = R, \ k' = C, \ G' = G_m, C' \) and consider
\[
Res_{C/R}(G_m, C).
\]
Then
\[
Res_{C/R}(G_m, C)(R) = C^x
\]
and
\[
Res_{C/R}(G_m, C)(C) = C^x \times C^x.
\]
[Note: \( Res_{C/R}(G_m, C) \)

is not isomorphic to \( G_m, R \) (its group of real points is \( R^x \)).]

LEMMA Let \( k' \) be a finite Galois extension of \( k \) -- then
\[
(Res_{k'/k}(G'))_{k'} \cong \bigoplus_{\sigma \in Gal(k'/k)} \sigma G'.
\]
[Note: \( \forall \sigma \in Gal(k'/k), \) there is a pullback square

\[
\begin{array}{ccc}
\sigma G' & \longrightarrow & \text{Spec} (k') \\
\downarrow & & \downarrow \\
G' & \longrightarrow & \text{Spec} (k')
\end{array}
\]

EXAMPLE Take \( k = R, \ k' = C, \ G' = G_m, C' \) -- then
\[
(Res_{C/R}(G_m, C))_C \cong G_m, C \times \sigma G_m, C
\]
\[
\cong G_m, C \times G_m, C.
\]
Let \( G \) be an affine group \( k \)-scheme.
27: DEFINITION A character of $G$ is an element of

$$X(G) = \text{Mor}_k(G, \mathbb{G}_m, k).$$

Given $\chi \in X(G)$, for every $k$-algebra $A$, there is a homomorphism

$$\chi(A): G(A) \to \mathbb{G}_m, k(A) = A^\times.$$

Given $\chi_1, \chi_2 \in X(G)$, define

$$(\chi_1 + \chi_2)(A) : G(A) \to \mathbb{G}_m, k(A) = A^\times$$

by the stipulation

$$(\chi_1 + \chi_2)(A)(t) = \chi_1(A)(t)\chi_2(A)(t),$$

from which a character $\chi_1 + \chi_2$ of $G$, hence $X(G)$ is an abelian group.

28: EXAMPLE Take $G = \mathbb{G}_m, k$ — then the characters of $G$ are the morphisms $G \to \mathbb{G}_m, k$ of the form

$$t \mapsto t^n \quad (n \in \mathbb{Z}),$$

i.e.,

$$X(G) \cong \mathbb{Z}.$$

29: EXAMPLE Take $G = \mathbb{G}_m, k \times \cdots \times \mathbb{G}_m, k$ ($d$ factors) — then the characters of $G$ are the morphisms $G \to \mathbb{G}_m, k$ of the form

$$(t_1, \ldots, t_d) \mapsto t_1^{n_1} \cdots t_d^{n_d} \quad (n_1, \ldots, n_d \in \mathbb{Z}),$$

i.e.,

$$X(G) \cong \mathbb{Z}^d.$$
10.

30: EXAMPLE Given an abelian group $M$, its group algebra $k[M]$ is canonically a $k$-algebra. Consider the functor $\text{D}(M):\text{ALG}/k \to \text{GRP}$ defined on objects by the rule

$$A \to \text{Mor}(M,A^x).$$

Then $\forall A$,

$$\text{Mor}(M,A^x) \cong \text{Mor}(k[M],A),$$

so $k[M]$ represents $\text{D}(M)$ which is therefore an affine group $k$-scheme. And

$$X(\text{D}(M)) \cong M,$$

the character of $\text{D}(M)$ corresponding to $m \in M$ being the assignment

$$\text{D}(M)(A) = \text{Mor}(M,A^x)$$

$$f \to f(m)$$

$$\longrightarrow A^x = G^m_{m,k}(A).$$

31: NOTATION Given $\chi' \in X(G')$, let $N_{k'/k}(\chi')$ stand for the rule that assigns to each $k$-algebra $A$ the homomorphism

$$G'_{k'/k}(A) \to G^m_{m,k}(A) = A^x$$

defined by the composition

$$G'_{k'/k}(A) \longrightarrow G'(A \otimes_k k')$$

$$G'(A \otimes_k k') \longrightarrow G^m_{m,k'}(A \otimes_k k') = (A \otimes_k k')^x$$

$$(A \otimes_k k')^x \longrightarrow A^x.$$
32: LEMMA The arrow

\[ X' \rightarrow N_{k'/k}(X') \]

is a homomorphism

\[ X(G') \rightarrow X(G_{k'/k}) \]

of abelian groups.

33: THEOREM The arrow

\[ X' \rightarrow N_{k'/k}(X') \]

is bijective, hence defines an isomorphism

\[ X(G') \rightarrow X(G_{k'/k}) \]

of abelian groups.

34: APPLICATION Consider

\[ \text{Res}_{C/R}(G_{m,C}). \]

Then its character group is isomorphic to the character group of \( G_{m,C} \), i.e., to \( \mathbb{Z} \).

Therefore

\[ \text{Res}_{C/R}(G_{m,C}) \]

is not isomorphic to \( G_{m,R} \times G_{m,R} \).
§4. ALGEBRAIC TORI

Fix a field \(k\) of characteristic zero.

1. **DEFINITION** Let \(G\) be an affine group \(k\)-scheme — then \(G\) is **algebraic** if its associated representing \(k\)-algebra \(A\) is finitely generated.

2. **REMARK** It can be shown that every algebraic affine group \(k\)-scheme is isomorphic to a closed subgroup of some \(\text{GL}_{n,k}(\mathbb{F})\).

3. **CONVENTION** The term algebraic \(k\)-group means "algebraic affine group \(k\)-scheme".

4. **N.B.** It is automatic that an algebraic \(k\)-group is reduced (cf. §2, #40), hence is geometrically reduced (cf. §2, #39).

5. **LEMMA** Assume that \(k'/k\) is a finite field extension — then the functor

\[
\text{Res}_{k'/k} : \text{AFF-GRP}/k' \rightarrow \text{AFF-GRP}/k
\]

sends algebraic \(k'\)-groups to algebraic \(k\)-groups.

Given a finite field extension \(k'/k\), let \(\Sigma\) be the set of \(k\)-embeddings of \(k'\) into \(k_\text{sep}\) and identify \(k' \otimes_k k_\text{sep}\) with \((k_\text{sep})^\Sigma\) via the bijection which takes \(x \otimes y\) to the string \((x(y))_{\sigma \in \Sigma}\).

6. **LEMMA** Let \(G'\) be an algebraic \(k'\)-group — then

\[
(G_{k'/k}) \times_k \text{Spec}(k_\text{sep}) \simeq \prod_{\sigma \in \Sigma} \sigma G',
\]
where \( \sigma G' \) is the algebraic \( k^{\text{sep}} \)-group defined by the pullback square

\[
\begin{array}{ccc}
\sigma G' & \rightarrow & \text{Spec}(k^{\text{sep}}) \\
\downarrow & & \downarrow \\
G' & \rightarrow & \text{Spec}(k')
\end{array}
\]

[Note: To review, the LHS is

\[
(\text{Res}_{k'/k}(G'))_{k^{\text{sep}}}
\]

and the Galois group \( \text{Gal}(k^{\text{sep}}/k) \) operates on it through the second factor. On the other hand, to each pair \((\tau, \sigma) \in \text{Gal}(k^{\text{sep}}/k) \times \Sigma\), there corresponds a bijection \( \sigma G' \rightarrow (\tau \circ \sigma)G' \) leading thereby to an action of \( \text{Gal}(k^{\text{sep}}/k) \) on

\[
\prod_{\sigma \in \Sigma} \sigma G'.
\]

The point then is that the identification

\[
(\text{Res}_{k'/k}(G'))_{k^{\text{sep}}} \approx \prod_{\sigma \in \Sigma} \sigma G'
\]

respects the actions, i.e., is \( \text{Gal}(k^{\text{sep}}/k) \)-equivariant.]

7: N.B. Consider the commutative diagram

\[
\begin{array}{ccc}
(\tau \circ \sigma)G' & \rightarrow & \text{Spec}(k^{\text{sep}}) \\
\downarrow & & \downarrow \\
\sigma G' & \rightarrow & \text{Spec}(k^{\text{sep}}) \\
\downarrow & & \downarrow \\
G' & \rightarrow & \text{Spec}(k')
\end{array}
\]
Then the "big" square is a pullback. Since this is also the case of the "small" bottom square, it follows that the "small" upper square is a pullback.

8: DEFINITION A split $k$-torus is an algebraic $k$-group $T$ which is isomorphic to a finite product of copies of $\mathbb{G}_m,k$.

9: EXAMPLE The algebraic $R$-group 

$$\text{Res}_{C/R}(\mathbb{G}_m,C)$$

is not a split $R$-torus (cf. §3, #24 and #34).

10: LEMMA If $T$ is a split $k$-torus, then $X(T)$ is a finitely generated free abelian group.

11: THEOREM The functor 

$$T \to X(T)$$

from the category of split $k$-tori to the category of finitely generated free abelian groups is a contravariant equivalence of categories.

12: N.B. \forall k$-algebra $A$,

$$T(A) \approx \text{Mor}(X(T),A^\times).$$

[Note: Explicated,]

$$T \approx \text{Spec}(k[X(T)])$$ (cf. §3, #30).

Therefore

$$T(A) \approx \text{Mor}(\text{Spec}(A),T)$$

$$\approx \text{Mor}(\text{Spec}(A),\text{Spec}(k[X(T)]))$$

$$\approx \text{Mor}(k[X(T)],A)$$

$$\approx \text{Mor}(X(T),A^\times).$$]
13: **DEFINITION** A $k$-torus is an algebraic $k$-group $T$ such that

$$T \text{_{sep}} = T \times_k \text{Spec}(k^{\text{sep}})$$

is a split $k^{\text{sep}}$-torus.

14: **N.B.** A split $k$-torus is a $k$-torus.

15: **EXAMPLE** Let $k'/k$ be a finite field extension and take $G' = G_{m,k'}$ -- then the algebraic $k$-group $G_{k'/k}$ is a $k$-torus (cf. #6).

16: **DEFINITION** Let $T$ be a $k$-torus -- then a splitting field for $T$ is a finite field extension $K/k$ such that $T_K$ is a split $K$-torus.

17: **THEOREM** Every $k$-torus $T$ admits a splitting field which is minimal (i.e., contained in any other splitting field) and Galois.

18: **NOTATION** Given a $k$-scheme $X$ and a Galois extension $K/k$, the Galois group $\text{Gal}(K/k)$ operates on

$$X_K = X \times_k \text{Spec}(K)$$

via the second term, hence $\sigma \mapsto 1 \otimes \sigma$.

[Note: $1 \otimes \sigma$ is a $k$-automorphism of $X_K$.]

19: **NOTATION** Given $k$-schemes $X,Y$ and a Galois extension $K/k$, the Galois group $\text{Gal}(K/k)$ operates on $\text{Mor}_K(X_K,Y_K)$ by the prescription

$$\sigma f = (1 \otimes \sigma)f(1 \otimes \sigma)^{-1}.$$ 

[Note: If $f \in \text{Mor}_K(X_K,Y_K)$, then the condition $\sigma f = f$ for all $\sigma \in \text{Gal}(K/k)$]
is equivalent to the condition that $f$ is the lift of a $k$-morphism $\phi : X \to Y$, i.e., $f = \phi \otimes 1$.

20: LEMMA Let $K/k$ be a Galois extension and let $G = \text{Gal}(K/k)$ -- then for any $k$-algebra $A$ and for any $k$-scheme $X$,

$$X(A \otimes_K K)^G = X(A).$$

[Note: This generalizes §2, #28 to which it reduces if $A = k$.]

21: DEFINITION Let $G$ be a finite group -- then a $G$-module is an abelian group $M$ supplied with a homomorphism $G \to \text{Aut}(M)$.

22: N.B. A $G$-module is the same thing as a $\mathbb{Z}[G]$-module (in the usual sense when $\mathbb{Z}[G]$ is viewed as a ring).

23: DEFINITION Let $G$ be a finite group -- then a $G$-lattice is a $\mathbb{Z}$-free $G$-module $M$ of finite rank.

24: LEMMA If $T$ is a $k$-torus split by a finite Galois extension $K/k$, then

$$X(T_K) = \text{Mor}_K(T_K, \mathbb{G}_m, K)$$

is a $\text{Gal}(K/k)$-lattice.

25: THEOREM Fix a finite Galois extension $K/k$ -- then the functor

$$T \mapsto X(T_K)$$

from the category of $k$-tori split by $K/k$ to the category of $\text{Gal}(K/k)$-lattices is a contravariant equivalence of categories.

26: N.B. Suppose that $T$ is a $k$-torus split by a finite Galois extension
K/k. Form K[X(T_K)], thus operationally, \( \forall \sigma \in \text{Gal}(K/k) \),

\[
\sigma(\Sigma a_i x_i) = \Sigma \sigma(a_i) \sigma(x_i) \quad (a_i \in K, x_i \in X(T_K)).
\]

Pass now to the invariants

\[
K[X(T_K)] \quad (G = \text{Gal}(K/k)).
\]

Then

\[
T \cong \text{Spec}(K[X(T_K)]^G).
\]

And

\[
T(A \otimes_k K)^G = T(A)
\]

\[
\approx \text{Mor}(\text{Spec}(A), T)
\]

\[
\approx \text{Mor}(\text{Spec}(A), \text{Spec}(K[X(T_K)]^G))
\]

\[
\approx \text{Mor}_K(K[X(T_K)]^G, A)
\]

\[
\approx \text{Mor}_K(K[X(T_K)], A \otimes_k K)^G
\]

\[
\approx \text{Mor}_Z(X(T_K), (A \otimes_k K)^X)^G
\]

\[
\approx \text{Mor}_Z[G](X(T_K), (A \otimes_k K)^X).
\]

[Note: Let \( T = \text{Res}_{K/k}(G_{m,K}) \) -- then on the one hand,

\[
\text{Mor}_Z[G](Z[G], (A \otimes_k K)^X) \approx (A \otimes_k K)^X,
\]

while on the other,

\[
\text{Res}_{K/k}(G_{m,K})(A) = (A \otimes_k K)^X
\]

\[
\approx \text{Mor}_Z[G](X(T_K), (A \otimes_k K)^X).
\]

Therefore

\[
X(T_K) \cong Z[G].
\]
Take \( k = R, K = C \), and let \( \sigma \) be the nontrivial element of \( \text{Gal}(C/R) \) — then every \( R \)-torus \( T \) gives rise to a \( Z \)-free module of finite rank supplied with an involution corresponding to \( \sigma \). And conversely...

There are three "basic" \( R \)-tori.

1. \( T = \mathbb{G}_{m,R} \). In this case,

\[
X(T_C) = X(\mathbb{G}_{m,C}) \simeq \mathbb{Z}
\]

and the Galois action is trivial.

2. \( T = \text{Res}_{C/R}(\mathbb{G}_{m,C}) \). In this case,

\[
X(T_C) \simeq X(\mathbb{G}_{m,C} \times \mathbb{G}_{m,C}) \quad (\text{cf. } \S 3, \#26)
\]

\[
\simeq \mathbb{Z} \times \mathbb{Z}
\]

and the Galois action swaps coordinates.

3. \( T = \mathbb{S}_0^2 \). In this case,

\[
X((\mathbb{S}_0^2)_C) \simeq X(\mathbb{G}_{m,C})
\]

\[
\simeq \mathbb{Z}
\]

and the Galois action is multiplication by \(-1\).

[Note:

\[
\mathbb{S}_0^2 : \text{ALG/R} \rightarrow \text{GRP}
\]

is the functor defined by the rule

\[
\mathbb{S}_0^2(A) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in A \text{ and } a^2 + b^2 = 1 \right\}.
\]

Then \( \mathbb{S}_0^2 \) is an algebraic \( R \)-group such that

\[
(\mathbb{S}_0^2)_C \simeq \mathbb{G}_{m,C}.
\]
so $S_0^2$ is an $R$-torus and $S_0^2(R)$ can be identified with $S (= \{ z \in \mathbb{C} : zz = 1 \})$.

27: THEOREM Every $R$-torus is isomorphic to a finite product of copies of the three basic tori described above.

Here is the procedure. Fix a $\mathbb{Z}$-free module $M$ of finite rank and an involution $\iota : M \to M$ -- then $M$ can be decomposed as a direct sum

$$M_+ \oplus M_{sw} \oplus M_-,$$

where $\iota = 1$ on $M_+$, $\iota$ is a sum of 2-dimensional swaps on $M_{sw}$ (or still, $M_{sw} = \oplus \mathbb{Z}[\text{Gal}(\mathbb{C}/\mathbb{R})]$), and $\iota = -1$ on $M_-$. 

28: SCHOLIUM If $T$ is an $R$-torus, then there exist unique nonnegative integers $a, b, c$ such that

$$T(R) \cong (R^\times)^a \times (\mathbb{C}^\times)^b \times S^c.$$

29: REMARK The classification of $\mathbb{C}$-tori is trivial: Any such is a finite product of the $\mathbb{G}_{m, \mathbb{C}}$.

30: RAPPEL Let $K/k$ be a finite Galois extension and let $A$ be a $k$-algebra -- then there is a norm map

$$A \otimes_k K^\times \to A^\times \cong (A \otimes_k k)^\times.$$ 

31: CONSTRUCTION Let $K/k$ be a finite Galois extension -- then there is a norm map

$$N_{K/k} : \text{Res}_{K/k}(\mathbb{G}_{m, K}) \to \mathbb{G}_{m, k}.$$
[For any $k$-algebra $A$,

$$\text{Res}_{K/k}(G_{m,K})(A)$$

$$= G_{m,K}(A \otimes_k K)$$

$$= (A \otimes_k K)^{\times} \to A^{\times} = G_{m,k}(A).$$

[Note: $N_{K/k}$ is not to be confused with the arrow of adjunction

$$G_{m,k} \to \text{Res}_{K/k}(G_{m,K}).$$.]

32: N.B.

$$N_{K/k} \in X(\text{Res}_{K/k}(G_{m,K})).$$

33: NOTATION Let $\text{Res}_{K/k}^{(1)}(G_{m,K})$ be the kernel of $N_{K/k}$.

34: LEMMA  $\text{Res}_{K/k}^{(1)}(G_{m,K})$ is a $k$-torus and there is a short exact sequence

$$1 \to \text{Res}_{K/k}^{(1)}(G_{m,K}) \to \text{Res}_{K/k}(G_{m,K}) \to \text{Res}_{K/k}(G_{m,K}) \to G_{m,k} \to 1.$$

35: EXAMPLE Take $k = \mathbb{R}$, $K = \mathbb{C}$ -- then

$$\text{Res}_{C/R}^{(1)}(G_{m,C}) \cong S_02$$

and there is a short exact sequence

$$1 \to S_02 \to \text{Res}_{C/R}(G_{m,C}) \to G_{m,R} \to 1.$$

[Note: On $R$-points, this becomes

$$1 \to S \to C^{\times} \to R^{\times} \to 1.$$.]
36: **DEFINITION** Let $T$ be a $k$-torus -- then $T$ is $k$-anisotropic if $X(T) = \{0\}$.

37: **EXAMPLE** $SO_2$ is $R$-anisotropic.

38: **THEOREM** Every $k$-torus $T$ has a unique maximal $k$-split subtorus $T_s$ and a unique maximal $k$-anisotropic subtorus $T_a$. The intersection $T_s \cap T_a$ is finite and $T_s \cdot T_a = T$.

39: **LEMMA** $\text{Res}^{(1)}_{K/k}(\mathbb{G}_m, K)$ is $k$-anisotropic.

**PROOF** Setting $G = \text{Gal}(K/k)$, under the functoriality of #25, the norm map

$$N_{K/k}: \text{Res}_{K/k}(\mathbb{G}_m, K) \to \mathbb{G}_m, K$$

corresponds to the homomorphism $Z \to Z[G]$ of $G$-modules that sends $n$ to $n(\Sigma \sigma)$, the quotient $Z[G]/Z(\Sigma \sigma)$ being $X(T_K)$, where

$$T = \text{Res}^{(1)}_{K/k}(\mathbb{G}_m, K).$$

And

$$Z[G]^G = Z(\Sigma \sigma).$$

40: **N.B.** $\text{Res}^{(1)}_{K/k}(\mathbb{G}_m, K)$ is the maximal $k$-anisotropic subtorus of $\text{Res}_{K/k}(\mathbb{G}_m, K)$.

41: **DEFINITION** Let $G, H$ be algebraic $k$-groups -- then a homomorphism $\phi: G \to H$ is an isogency if it is surjective with a finite kernel.
42: **DEFINITION** Let $G, H$ be algebraic $k$-groups --- then $G, H$ are said to be **isogeneous** if there is an isogeny between them.

43: **THEOREM** Two $k$-tori $T', T''$ per #25 are isogeneous iff the $\mathbb{Q}[[\text{Gal}(k/k)]]$-modules

\[
\begin{bmatrix}
X(T'_K) \otimes_{\mathbb{Z}} \mathbb{Q} \\
X(T''_K) \otimes_{\mathbb{Z}} \mathbb{Q}
\end{bmatrix}
\]

are isomorphic.
1. N.B. The term "LLC" means "local Langlands correspondence" (cf. #26).

Let $K$ be a non-archimedean local field -- then the image of $\text{rec}_K: K^\times \to \mathbb{C}_K^{ab}$ is $\mathbb{W}_K^{ab}$ and the induced map $K^\times \to \mathbb{W}_K^{ab}$ is an isomorphism of topological groups.

2. SCHOLIUM There is a bijective correspondence between the characters of $\mathbb{W}_K$ and the characters of $K^\times$:

$$\text{Mor}(\mathbb{W}_K, \mathbb{C}_K^\times) \cong \text{Mor}(K^\times, \mathbb{C}_K^\times).$$

[Note: "Character" means continuous homomorphism. So, if $\chi: \mathbb{W}_K \to \mathbb{C}_K^\times$ is a character, then $\chi$ must be trivial on $\mathbb{W}_K^\times$ ($\mathbb{C}_K^\times$ being abelian), hence by continuity, trivial on $\mathbb{W}_K^\times$, thus $\chi$ factors through $\mathbb{W}_K^\times / \mathbb{W}_K^{ab} = \mathbb{W}_K^{ab}$.]

Let $T$ be a $K$-torus -- then $T$ is isomorphic to a closed subgroup of some $\text{GL}_{n,K}$ ($\exists \ n$). But $\text{GL}_{n,K}(K)$ is a locally compact topological group, thus $T(K)$ is a locally compact topological group (which, moreover, is abelian).

3. N.B. For the record,

$$\text{GL}_{n,K}(K) = K^\times = \text{GL}_1, K(K).$$

4. EXAMPLE Let $L/K$ be a finite extension and consider $T = \text{Res}_{L/K}(\text{G}_m, L)$ -- then $T(K) = L^\times$.

Roughly speaking, the objective now is to describe $\text{Mor}(T(K), \mathbb{C}_K^\times)$ in terms of data attached to $\mathbb{W}_K$ but to even state the result requires some preparation.
5: N.B. The case when \( T = G_{m,K} \) is local class field theory...

6: EXAMPLE Suppose that \( T \) is \( K \)-split:

\[
T \simeq G_{m,K} \times \cdots \times G_{m,K} \quad \text{(d factors)}.
\]

Then

\[
\prod_{i=1}^{d} \text{Mor}(W_{K}) \simeq \prod_{i=1}^{d} \text{Mor}(K_{x}, C_{})
\]

\[
\simeq \text{Mor}(\bigoplus_{i=1}^{d} K_{x}, C_{})
\]

\[
\simeq \text{Mor}(T(K), C_{}).
\]

Given a \( K \)-torus \( T \), put

\[
\begin{align*}
X^{*}(T) &= \text{Mor}_{K_{\text{sep}}}^{\text{sep}}(T_{\text{sep}}, G_{m,K_{\text{sep}}}) \\
X^{*}(T) &= \text{Mor}_{K_{\text{sep}}}^{\text{sep}}(G_{m,K_{\text{sep}}}, T_{\text{sep}}).
\end{align*}
\]

7: LEMMA Canonically,

\[
X^{*}(T) \otimes_{Z} C_{x} \simeq \text{Mor}(X^{*}(T), C^{x}).
\]

PROOF Bearing in mind that

\[
\text{Mor}_{K_{\text{sep}}}^{\text{sep}}(G_{m,K_{\text{sep}}}, G_{m,K_{\text{sep}}}) \simeq Z,
\]

define a pairing

\[
X^{*}(T) \times X^{*}(T) \overset{<, >}{\longrightarrow} Z
\]
by sending \((\chi^*, x_*)\) to \(\chi^* \circ x_* \in \mathbb{Z}\). This done, given \(\chi_* \otimes z\), assign to it the homomorphism

\[
\chi^* \to z
\]

\[<\chi^*, x_*> \]

3: NOTATION Given a K-torus \(T\), put

\[\hat{T} = \text{Spec}(\mathbb{C}[X_*(T)])\].

9: LEMMA \(\hat{T}\) is a split \(\mathbb{C}\)-torus such that

\[
\begin{align*}
X^*(\hat{T}) &\equiv \text{Mor}_C(\hat{T}, \mathbb{G}_m, \mathbb{C}) = X_*(T) \\
X_*(\hat{T}) &\equiv \text{Mor}_C(\mathbb{G}_m, \hat{T}) = X^*(T).
\end{align*}
\]

Therefore

\[
\text{Mor}(X_*(T), \mathbb{C}^\times) \approx \text{Mor}(X^*(\hat{T}), \mathbb{C}^\times)
\]

\[
\approx X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times
\]

\[
\approx X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times.
\]

10: LEMMA

\[
\hat{T}(\mathbb{C}) \approx X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times.
\]

PROOF In fact,

\[
\hat{T}(\mathbb{C}) \approx \text{Mor}(X^*(\hat{T}), \mathbb{C}^\times) \quad (\text{cf. §4, #12})
\]

\[
\approx \text{Mor}(X_*(T), \mathbb{C}^\times)
\]

\[
\approx X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times.
\]
11: **DEFINITION** \( \hat{T} \) is the complex dual torus of \( T \).

12: **EXAMPLE** Under the assumptions of #6,

\[
\hat{T}(C) \approx X^*(T) \otimes \mathbb{Z} \mathbb{C}^* \\
\approx \mathbb{Z}^d \otimes \mathbb{C}^* \approx (\mathbb{C}^*)^d.
\]

Therefore

\[
\text{Mor}(\mathbb{W}_K, \hat{T}(C)) \approx \text{Mor}(\mathbb{W}_K, (\mathbb{C}^*)^d) \\
\approx \prod_{i=1}^{d} \text{Mor}(\mathbb{W}_K, \mathbb{C}^*) \\
\approx \text{Mor}(T(K), \mathbb{C}^*).
\]

13: **RAPPEL** If \( G \) is a group and if \( A \) is a \( G \)-module, then

\[
H^1(G,A) = \frac{Z^1(G,A)}{B^1(G,A)}.
\]

- \( Z^1(G,A) \) (the **1-cocycles**) consists of those maps \( f:G \to A \) such that \( \forall \sigma, \tau \in G \),

\[
f(\sigma \tau) = f(\sigma) + \sigma(f(\tau)).
\]

- \( B^1(G,A) \) (the **1-coboundaries**) consists of those maps \( f:G \to A \) for which \( \exists \) an \( a \in A \) such that \( \forall \sigma \in G \),

\[
f(\sigma) = \sigma a - a.
\]

[**Note:**

\[
H^1(G,A) = \text{Mor}(G,A)
\]

if the action is trivial.]
14: NOTATION If $G$ is a topological group and if $A$ is a topological $G$-module, then

$$\text{Mor}_C(G,A)$$

is the group of continuous group homomorphisms from $G$ to $A$. Analogously,

$$Z^1_C(G,A) = \text{"continuous 1-cocycles"}$$

$$B^1_C(G,A) = \text{"continuous 1-coboundaries"}$$

and

$$H^1_C(G,A) = \frac{Z^1_C(G,A)}{B^1_C(G,A)}.$$

Let $T$ be a $K$-torus -- then $G_K (= \text{Gal}(K^{sep}/K))$ operates on $X^*(G)$, thus

$W_K \subset G_K$ operates on $X^*(G)$ by restriction. Therefore $\hat{T}(C)$ is a $W_K$-module, so it makes sense to form

$$H^1_C(W_K, \hat{T}(C)).$$

15: NOTATION $\text{TOR}_K$ is the category of $K$-tori.

16: LEMMA The assignment

$$T \mapsto H^1_C(W_K, \hat{T}(C))$$

defines a functor

$$\text{TOR}_K^{\text{op}} \rightarrow \text{AB}.$$
[Note: Suppose that $T_1 \rightarrow T_2$ -- then]

\[(T_1)_K \rightarrow (T_2)_K\]

\[\Rightarrow \]

\[X^*(T_2) \rightarrow X^*(T_1)\]

\[\Rightarrow \]

\[\hat{T}_2(C) \rightarrow \hat{T}_1(C)\]

\[\Rightarrow \]

\[H^1_C(W_K, \hat{T}_2(C)) \rightarrow H^1_C(W_K, \hat{T}_1(C)).\]

17: LEMMA The assignment

\[T \rightarrow \text{Mor}_C(T(K), C^\vee)\]

defines a functor

\[\text{TOR}^{\text{OP}}_K \rightarrow \text{AB}.\]

18: THEOREM The functors

\[T \rightarrow H^1_C(W_K, \hat{T}(C))\]

and

\[T \rightarrow \text{Mor}_C(T(K), C^\vee)\]

are naturally isomorphic.

19: SCHOLIUM There exist isomorphisms

\[\iota_T: H^1_C(W_K, \hat{T}(C)) \rightarrow \text{Mor}_C(T(K), C^\vee)\]
such that if $T_1 \to T_2$, then the diagram

$$
\begin{array}{ccc}
H_c^1(W_K, T_1(C)) & \xrightarrow{T_1} & \text{Mor}_C(T_1(K), C^\times) \\
\uparrow & & \uparrow \\
H_c^1(W_K, T_2(C)) & \xrightarrow{T_2} & \text{Mor}_C(T_2(K), C^\times)
\end{array}
$$

commutes.

20: EXAMPLE Under the assumptions of #12, the action of $G_K$ is trivial, hence the action of $W_K$ is trivial. Therefore

$$H_c^1(W_K, \hat{T}(C)) = \text{Mor}_C(W_K, \hat{T}(C))$$

$$= \text{Mor}_C(T(K), C^\times).$$

[Note: The earlier use of the symbol Mor tacitly incorporated "continuity".]

There is a special case that can be dealt with directly, viz. when $L/K$ is a finite Galois extension and

$$T = \text{Res}_{L/K}(G_m, L).$$

The discussion requires some elementary cohomological generalities which have been collected in the Appendix below.

21: RAPPEL $W_L$ is a normal subgroup of $W_K$ of finite index:

$$W_K/W_L \cong G_K/G_L \cong \text{Gal}(L/K).$$

Proceeding,

$$T_{\text{sep}} \cong \prod_{\sigma \in \text{Gal}(L/K)} \sigma G_{m, L} \quad (\text{cf. #6}),$$
so
\[ X^*(T) \approx Z[\mathfrak{W}_K/\mathfrak{W}_L], \]

where
\[ Z[\mathfrak{W}_K/\mathfrak{W}_L] \approx \text{Ind}_{\mathfrak{W}_L}^{\mathfrak{W}_K} Z \]
\[ \approx Z[\mathfrak{W}_K] \otimes Z[\mathfrak{W}_L] Z. \]

It therefore follows that
\[ \hat{T}(C) \approx X^*(T) \otimes Z C^\times \]
\[ \approx Z[\mathfrak{W}_K] \otimes Z[\mathfrak{W}_L] Z \otimes Z C^\times \]
\[ \approx Z[\mathfrak{W}_K] \otimes Z[\mathfrak{W}_L] C^\times \]
\[ \approx \text{Ind}_{\mathfrak{W}_L}^{\mathfrak{W}_K} C^\times \]

Consequently
\[ H^1(\mathfrak{W}_K, \hat{T}(C)) \approx H^1(\mathfrak{W}_K, \text{Ind}_{\mathfrak{W}_L}^{\mathfrak{W}_K} C^\times) \]
\[ \approx H^1(\mathfrak{W}_L, C^\times) \quad \text{(Shapiro's lemma)} \]
\[ \approx \text{Mor}(\mathfrak{W}_L, C^\times) \]
\[ \approx \text{Mor}(L^\times, C^\times) \]
\[ \approx \text{Mor}(T(K), C^\times), \]

which completes the proof modulo "continuity details" that we shall not stop to sort out.
22: **DEFINITION** The $L$-group of $T$ is the semi-direct product

$$L_T = \hat{T}(G) \rtimes W_K.$$ 

Because of this, it will be best to first recall "semi-direct product theory".

23: **RAPPEL** If $G$ is a group and if $A$ is a $G$-module, then there is a canonical extension of $G$ by $A$, namely

$$0 \rightarrow A \rightarrow A \rtimes G \rightarrow G \rightarrow 1,$$

where $A \rtimes G$ is the semi-direct product.

24: **DEFINITION** A splitting of the extension

$$i$$

$$0 \rightarrow A \rightarrow A \rtimes G \rightarrow G \rightarrow 1$$

is a homomorphism $s: G \rightarrow A \rtimes G$ such that $\pi \circ s = \text{id}_G$.

25: **FACT** The splittings of the extension

$$i$$

$$0 \rightarrow A \rightarrow A \rtimes G \rightarrow G \rightarrow 1$$

determine and are determined by the elements of $Z^1(G,A)$.

Two splittings $s_1, s_2$ are said to be equivalent if there is an element $a \in A$ such that

$$s_1(\sigma) = i(a)s_2(\sigma)i(a)^{-1} \quad (\sigma \in G).$$

If

$$f_1 \leftrightarrow s_1 \quad f_2 \leftrightarrow s_2$$
are the 1-cocycles corresponding to 

\[ \begin{array}{c}
- & \, s_1 \\
- & \, s_2 \\
\end{array} \]

then their difference \( f_2 - f_1 \) is a 1-coboundary.

26: SCHOLIUM The equivalence classes of splittings of the extension

\[ \begin{array}{c}
i & \quad 0 \to A \xrightarrow{i} A \times G \xrightarrow{\pi} G \to 1 \\
\end{array} \]

are in a bijective correspondence with the elements of \( H^1(G,A) \).

Return now to the extension

\[ \begin{array}{c}
0 \to \hat{T}(C) \to \hat{T}(C) \times W_K \to W_K \to 1 \\
\end{array} \]

but to reflect the underlying topologies, work with continuous splittings and call them admissible homomorphisms. Introducing the obvious notion of equivalence, denote by \( \phi_K(T) \) the set of equivalence classes of admissible homomorphisms, hence

\[ \phi_K(T) \simeq H^1(C, \hat{T}(C)). \]

On the other hand, denote by \( A_K(T) \) the group of characters of \( T(K) \), i.e.,

\[ A_K(T) \simeq \text{Mor}_C(T(K), C^\times). \]

27: THEOREM There is a canonical isomorphism

\[ \phi_K(T) \to A_K(T). \]

[This statement is just a rephrasing of #18 and is the LLC for tori.]

28: HEURISTICSE To each admissible homomorphism of \( W_K \) into \( L_T \), it is possible to associate an irreducible automorphic representation of \( T(K) \) (a.k.a. a character of \( T(K) \)) and all such arise in this fashion.
It remains to consider the archimedean case: $\mathbb{C}$ or $\mathbb{R}$.

- If $T$ is a $\mathbb{C}$-torus, then $T$ is isomorphic to a finite product

\[ G_{m,\mathbb{C}} \times \cdots \times G_{m,\mathbb{C}} \]

and

\[ T(\mathbb{C}) = \text{Mor}(X^*(T),\mathbb{C}^x) \]
\[ = X^*_\mathbb{Z}(T) \otimes \mathbb{Z} \mathbb{C}^x. \]

Furthermore, $W_{\mathbb{C}} = \mathbb{C}^x$ and the claim is that

\[ H^1_{\mathbb{C}}(W_{\mathbb{C}}, \hat{T}(\mathbb{C})) \cong \text{Mor}_{\mathbb{C}}(\mathbb{C}^x, \hat{T}(\mathbb{C})) \]

is isomorphic to

\[ \text{Mor}_{\mathbb{C}}(T(\mathbb{C}), \mathbb{C}^x). \]

But

\[ \text{Mor}_{\mathbb{C}}(\mathbb{C}^x, \hat{T}(\mathbb{C})) \]
\[ = \text{Mor}_{\mathbb{C}}(\mathbb{C}^x, X^*(T) \otimes \mathbb{Z} \mathbb{C}^x) \]
\[ = \text{Mor}_{\mathbb{C}}(\mathbb{C}^x, \text{Mor}(X^*(T), \mathbb{C}^x)) \]
\[ = \text{Mor}_{\mathbb{C}}(X^*_\mathbb{Z}(T) \otimes \mathbb{Z} \mathbb{C}^x, \mathbb{C}^x) \]
\[ = \text{Mor}_{\mathbb{C}}(T(\mathbb{C}), \mathbb{C}^x). \]

- If $T$ is an $\mathbb{R}$-torus, then $T$ is isomorphic to a finite product

\[ (G_{m,\mathbb{R}})^a \times (\text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}}))^b \times (S_{02})^c \]

and it is enough to look at the three irreducible possibilities.
1. $T = G_m, R$. The point here is that $\mathcal{W}_{R}^{ab} \cong R^\times \cong T(R)$.

2. $T = \text{Res}_{C/R}(G_m, C)$. One can imitate the argument used above for its non-archimedean analog.

3. $T = S_0$. The initial observation is that $X(T) = Z$ with action $n \to -n$,

so $\hat{T}(C) = C^\times$ with action $z \to \frac{1}{z}$. And ... .

APPENDIX

Let $G$ be a group (written multiplicatively).

1. DEFINITION A left (right) $G$-module is an abelian group $A$ equipped with a left (right) action of $G$, i.e., with a homomorphism $G \to \text{Aut}(A)$.

2. N.B. Spelled out, to say that $A$ is a left $G$-module means that there is a map

$$
\begin{array}{ccc}
G \times A & \to & A \\
(\sigma, a) & \mapsto & \sigma a
\end{array}
$$

such that

$$
\tau(\sigma a) = (\tau \sigma)a, \quad la = a,
$$

thus $A$ is first of all a left $G$-set. To say that $A$ is a left $G$-module then means in addition that

$$
\sigma(a + b) = \sigma a + \sigma b.
$$

[Note: For the most part, the formalities are worked out from the left, the agreement being that

"left $G$-module" = "$G$-module".]
3: NOTATION The group ring $\mathbb{Z}[G]$ is the ring whose additive group is the free abelian group with basis $G$ and whose multiplication is determined by the multiplication in $G$ and the distributive law.

A typical element of $\mathbb{Z}[G]$ is

$$\sum_{\sigma \in G} m_{\sigma} \sigma,$$

where $m_{\sigma} \in \mathbb{Z}$ and $m_{\sigma} = 0$ for all but finitely many $\sigma$.

4: N.B. A $G$-module is the same thing as a $\mathbb{Z}[G]$-module.

5: LEMMA Given a ring $R$, there is a canonical bijection

$$\text{Mor}(\mathbb{Z}[G], R) \cong \text{Mor}(G, R^X).$$

6: CONSTRUCTION Given a $G$-set $X$, form the free abelian group $\mathbb{Z}[X]$ generated by $X$ and extend the action of $G$ on $X$ to a $\mathbb{Z}$-linear action of $G$ on $\mathbb{Z}[X]$ -- then the resulting $G$-module is called a permutation module.

7: EXAMPLE Let $H$ be a subgroup of $G$ and take $X = G/H$ (here $G$ operates on $G/H$ by left translation), from which $\mathbb{Z}[G/H]$.

8: DEFINITION A $G$-module homomorphism is a $\mathbb{Z}[G]$-module homomorphism.

9: NOTATION $\text{MOD}_G$ is the category of $G$-modules.

10: NOTATION Given $A, B$ in $\text{MOD}_G$, write $\text{Hom}_G(A, B)$ in place of $\text{Mor}(A, B)$.

11: LEMMA Let $A, B \in \text{MOD}_G$ -- then $A \otimes \mathbb{Z} B$ carries the $G$-module structure
14. defined by \(\sigma(a \otimes a') = ca \otimes ca'\) and \(\text{Hom}_Z(A, B)\) carries the \(G\)-module structure defined by \((\phi)(a) = \sigma^{-1}(\phi(a))\).

12: **Lemma** If \(G'\) is a subgroup of \(G\), then there is a homomorphism \(Z[G'] \rightarrow Z[G]\) of rings and a functor

\[\text{Res}_G^{G'} : \text{MOD}_G \rightarrow \text{MOD}_{G'}\]

of restriction.

13: **Definition** Let \(G'\) be a subgroup of \(G\) — then the functor of induction

\[\text{Ind}_G^{G'} : \text{MOD}_{G'} \rightarrow \text{MOD}_G\]

sends \(A'\) to

\[Z[G] \otimes_{Z[G']} A'.\]

[Note: \(Z[G]\) is a right \(Z[G']\)-module and \(A'\) is a left \(Z[G']\)-module. Therefore the tensor product

\[Z[G] \otimes_{Z[G']} A'\]

is an abelian group. And it becomes a left \(G\)-module under the operation \(\sigma(r \otimes a') = \sigma r \otimes a'.\)]

14: **Example** Let \(H\) be a subgroup of \(G\). Suppose that \(H\) operates trivially on \(Z\) — then

\[Z[G/H] \cong \text{Ind}_H^G Z.\]

15: **Frobenius Reciprocity** \(\forall A\) in \(\text{MOD}_G\), \(\forall A'\) in \(\text{MOD}_{G'}\),

\[\text{Hom}_G(A', \text{Res}_G^{G'} A) \cong \text{Hom}_G(\text{Ind}_G^{G'} A', A).\]
15.

16: REMARK \( \forall A \) in \( \text{MOD}_G \),

\[
\text{Ind}_G^G \cdot \text{Res}_G^G A = Z[G/G'] \otimes_{Z[G]} A.
\]

\([G \text{ operates on the right hand side diagonally: } \sigma(r \otimes a) = \sigma r \otimes \sigma a.]\)

17: LEMMA There is an arrow of inclusion

\[
Z[G] \otimes_{Z[G']} A' \to \text{Hom}_G(Z[G], A')
\]

which is an isomorphism if \([G:G'] < \infty\).

18: NOTATION Given a \( G \)-module \( A \), put

\[
A^G = \{ a \in A : \sigma a = a \ \forall \sigma \in G \}.
\]

[Note: \( A^G \) is a subgroup of \( A \), termed the \text{invariants} in \( A \).]

19: LEMMA \( A^G = \text{Hom}_G(Z,A) \) (trivial \( G \)-action on \( Z \)).

[Note: By comparison,

\[
A = \text{Hom}_G(Z[G], A).
\]

20: LEMMA \( \text{Hom}_Z(A,B)^G = \text{Hom}_G(A,B) \).

\( \text{MOD}_G \) is an abelian category. As such, it has enough injectives (i.e., every \( G \)-module can be embedded in an injective \( G \)-module).

21: DEFINITION The \text{group cohomology} functor \( H^q(G,-): \text{MOD}_G \to \text{AB} \) is the right derived functor of \( (-)^G \).

[Note: Recall the procedure: To compute \( H^q(G,A) \), choose an injective
resolution

\[ 0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \ldots. \]

Then \( H^*(G, A) \) is the cohomology of the complex \((I)^G\). In particular: \( H^0(G, A) = A^G \).

22: LEMMA \( H^q(G, A) \) is independent of the choice of injective resolutions.

23: LEMMA \( H^q(G, A) \) is a covariant functor of \( A \).

24: LEMMA If

\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \]

is a short exact sequence of \( G \)-modules, then there is a functorial long exact sequence

\[ 0 \rightarrow H^0(G, A) \rightarrow H^0(G, B) \rightarrow H^0(G, C) \]

\[ \rightarrow H^1(G, A) \rightarrow H^1(G, B) \rightarrow H^1(G, C) \rightarrow H^2(G, A) \rightarrow \ldots \]

\[ \ldots \rightarrow H^q(G, A) \rightarrow H^q(G, B) \rightarrow H^q(G, C) \rightarrow H^{q+1}(G, A) \rightarrow \ldots \]

in cohomology.

25: N.B. If \( G = \{1\} \) is the trivial group, then

\[ H^0(G, A) = A, \ H^q(G, A) = 0 \quad (q > 0). \]

[Note: Another point is that for any \( G \), every injective \( G \)-module \( A \) is cohomologically acyclic:

\[ \forall \ q > 0, \ H^q(G, A) = 0. \]

26: THEOREM (SHAPIRO'S LEMMA) If \([G:G'] < \infty\), then \( \forall \ q, \)

\[ H^q(G', A') \approx H^q(G, \text{Ind}_G^G A'). \]
27: EXAMPLE Take $A' = Z[G']$ — then

$$H^q(G', Z[G']) \approx H^q(G, Z[G] \otimes_{Z[G']} Z[G'])$$

$$\approx H^q(G, Z[G]).$$

28: EXAMPLE Take $G' = \{1\}$ (so $G$ is finite) — then $Z[G'] = Z$ and

$$H^q(\{1\}, Z) \approx H^q(G, Z[G]).$$

But the LHS vanishes if $q > 0$, thus the same is true of the RHS. However this fails if $G$ is infinite. E.g.: Take for $G$ the infinite cyclic group: $H^1(G, Z[G]) \approx Z$.

[Note: If $G$ is finite, then $H^0(G, Z[G]) \approx Z$ while if $G$ is infinite, then $H^0(G, Z[G]) = 0$.]

29: EXAMPLE Take $A' = Z$ — then

$$H^q(G', Z) \approx H^q(G, \text{Ind}_{G}^G, Z)$$

$$\approx H^q(G, Z[G/G']).$$
1.

§6. TAMAGAWA MEASURES

Suppose given a Q-torus $T$ of dimension $d$ — then one can introduce

$$T(Q) \subset T(R), T(Q) \subset T(Q_p)$$

and

$$T(Z_p)$$

and

$$T(Q) \subset T(A).$$

1: EXAMPLE Take $T = G_{m,Q}$ — then the above data becomes

$$\mathbb{Q}^\times \subset \mathbb{R}^\times, \mathbb{Q}^\times \subset \mathbb{Q}_p^\times$$

and

$$\mathbb{Z}_p^\times$$

and

$$\mathbb{Q}^\times \subset A^\times = 1.$$

2: LEMMA $T(Q)$ is a discrete subgroup of $T(A)$.

3: RAPPEL $I^1 = \ker | \cdot |_A$, where for $x \in I$,

$$|x|_A = \prod_{p \leq \infty} |x_p|_p.$$ 

And the quotient $I^1/Q^\times$ is a compact Hausdorff space.

Each $\chi \in X(T)$ generates continuous homomorphisms

- $\chi_p : T(Q_p) \to \mathbb{Q}_p^\times \xrightarrow{| \cdot |_p} \mathbb{R}_>0$
- $\chi_\infty : T(R) \to \mathbb{R}_>0 \xrightarrow{| \cdot |_\infty} \mathbb{R}_>0$
from which an arrow

\[
\chi_A : T(A) \rightarrow \mathbb{R}^x
\]

\[
x \rightarrow \prod_{p} \chi_p(x_p).
\]

4: NOTATION

\[
T^1(A) = \bigcap_{\chi \in X(T)} \ker \chi_A.
\]

5: N.B. The infinite intersection can be replaced by a finite intersection
since if \( \chi_1, \ldots, \chi_d \) is a basis for \( X(T) \), then

\[
T^1(A) = \bigcap_{i=1}^{d} \ker (\chi_i)_A.
\]

6: THEOREM The quotient \( T^1(A)/T(Q) \) is a compact Hausdorff space.

7: CONSTRUCTION Let \( \Omega_T \) denote the collection of all left invariant \( d \)-forms
on \( T \), thus \( \Omega_T \) is a 1-dimensional vector space over \( Q \). Choose a nonzero element

\( \omega \in \Omega_T \) — then \( \omega \) determines a left invariant differential form of top degree on
the \( T(Q_p) \) and \( T(R) \), which in turn determines a Haar measure \( \mu_{Q_p, \omega} \) on the \( T(Q_p) \)
and a Haar measure \( \mu_{R, \omega} \) on \( T(R) \).

The product

\[
\prod_{p} \mu_{Q_p, \omega(T(Z_p))}
\]

may or may not converge.
8: DEFINITION A sequence \( \Lambda = \{\Lambda_p\} \) of positive real numbers is said to be a system of convergence coefficients if the product

\[
\prod_p \Lambda_p^{\mu_{Q_p,\omega}(T(Z_p))}
\]

is convergent.

9: N.B. Convergence coefficients always exist, e.g.,

\[
\Lambda_p = \frac{1}{\mu_{Q_p,\omega}(T(Z_p))}
\]

10: LEMMA If the sequence \( \Lambda = \{\Lambda_p\} \) is a system of convergence coefficients, then

\[
\mu_{\omega, \Lambda} \equiv \prod_p \Lambda_p^{\mu_{Q_p,\omega} \times \mu_{R,\omega}}
\]

is a Haar measure on \( T(A) \).

11: N.B. Let \( \lambda \) be a nonzero rational number — then

\[
\mu_{Q_p,\lambda \omega} = |\lambda|_p^p \mu_{Q_p,\omega} \mu_{R,\lambda \omega} = |\lambda|_\infty \mu_{R,\omega}.
\]

Therefore

\[
\mu_{\lambda \omega, \Lambda} \equiv \prod_p \Lambda_p^{\mu_{Q_p,\lambda \omega} \times \mu_{R,\lambda \omega}}
\]

\[
= (\prod_p |\lambda|_p)^p \prod_p \Lambda_p^{\mu_{Q_p,\omega} \times |\lambda|_\infty \mu_{R,\omega}}
\]

\[
= \prod_{p \leq \infty} |\lambda|_p \prod_p \Lambda_p^{\mu_{Q_p,\omega} \times \mu_{R,\omega}}
\]

\[
= \mu_{\omega, \Lambda}.
\]
And this means that the Haar measure \( \mu_{\omega, \lambda} \) is independent of the choice of the rational density \( \omega \).

Let \( K \supset Q \) be a Galois extension relative to which \( T \) splits -- then

\[
X(T_K) = \text{Mor}_K(T_K, G_m,K)
\]
is a Gal(K/Q) lattice. Call \( \Pi \) the representation thereby determined and denote its character by \( \chi_\Pi \). Let

\[
L(s, \chi_\Pi, K/Q) = \prod_p L_p(s, \chi_\Pi, K/Q)
\]
be the associated Artin L-function and denote by \( S \) the set of primes that ramify in \( K \) plus the "prime at infinity".

12: **Lemma** \( \forall \ p \not\in S \),

\[
\mu_{\omega, \lambda}(T(Z_p)) = L_p(1, \chi_\Pi, K/Q)^{-1}.
\]

13: **Scholium** The sequence \( \Lambda = \{ \Lambda_p \} \) defined by the prescription

\[
\Lambda_p = L_p(1, \chi_\Pi, K/Q) \text{ if } p \not\in S
\]
and

\[
\Lambda_p = 1 \text{ if } p \in S
\]
is a system of convergence coefficients termed **canonical**.

14: **Lemma** The Haar measure \( \mu_{\omega, \lambda} \) on \( T(A) \) corresponding to a canonical system of convergence coefficients is independent of the choice of \( K \), denote it by \( \mu_T \).

15: **Definition** \( \mu_T \) is the Tamagawa measure on \( T(A) \).
5.

Owing to Brauer theory, there is a decomposition of the character \( \chi_\Pi \) of \( \Pi \) as a finite sum

\[
\chi_\Pi = d\chi_0 + \sum_{j=1}^{M} m_j \chi_j,
\]

where \( \chi_0 \) is the principal character of \( \text{Gal}(K/Q) \) (\( \chi_0(\sigma) = 1 \) for all \( \sigma \in \text{Gal}(K/Q) \)), the \( m_j \) are positive integers, and the \( \chi_j \) are irreducible characters of \( \text{Gal}(K/Q) \).

Standard properties of Artin L-functions then imply that

\[
L(s, \chi_\Pi, K/Q) = \zeta(s)^d \prod_{j=1}^{M} L(s, \chi_j, K/Q)^{m_j}.
\]

16: FACT

\[
L(1, \chi_j, K/Q)^{m_j} \neq 0 \quad (1 \leq j \leq M).
\]

Therefore

\[
\lim_{s \to \infty} (s-1)^d L(s, \chi_\Pi, K/Q) = \prod_{j=1}^{M} L(1, \chi_j, K/Q)^{m_j} \neq 0.
\]

17: LEMMA The limit on the left is positive and independent of the choice of \( K \), denote it by \( \rho_T \).

18: DEFINITION \( \rho_T \) is the residue of \( T \).

Define a map

\[
T : T(A) \to (\mathbb{R}_{>0})^d
\]
by the rule
\[ T(x) = ((x_1)_A(x), \ldots, (x_d)_A(x)). \]

Then the kernel of \( T \) is \( T^1(A) \), hence \( T \) drops to an isomorphism
\[ T^1: T(A)/T^1(A) \to (\mathbb{R}_{>0})^d. \]

19: DEFINITION The standard measure on \( T(A)/T^1(A) \) is the pullback via \( T^1 \) of the product measure
\[ \prod_{i=1}^{d} \frac{dt_i}{t_i} \]
on \( (\mathbb{R}_{>0})^d \).

Consider now the formalism
\[ d(T(A)) = d(T(A)/T^1(A))d(T^1(A)/T(Q))d(T(Q)) \]
in which:
- \( d(T(A)) \) is the Tamagawa measure on \( T(A) \) multiplied by \( \frac{1}{\rho_T} \).
- \( d(T(A)/T^1(A)) \) is the standard measure on \( T(A)/T^1(A) \).
- \( d(T(Q)) \) is the counting measure on \( T(Q) \).

20: DEFINITION The Tamagawa number \( \tau(T) \) is the volume
\[ \tau(T) = \int_{T^1(A)/T(Q)} \frac{1}{\rho_T} \]
of the compact Hausdorff space \( T^1(A)/T(Q) \) per the invariant measure
\[ d(T^1(A)/T(Q)) \]
such that
\[ \frac{\mu_T}{\rho_T} = \frac{d(T(A)T^1(A))d(T^1(A)/T(Q))d(T(Q))}{\rho_T}. \]

21: N.B. To be completely precise, the integral formula
\[ \int_{T(A)} f(T(A)/T^1(A)) \frac{d(T^1(A))}{T^1(A)} \]
fixes the invariant measure on $T^1(A)$ and from there the integral formula
\[ \int_{T^1(A)/T(Q)} f(T(Q)) \]
fixes the invariant measure on $T^1(A)/T(Q)$, its volume then being the Tamagawa number $\tau(T)$.

[Note: If $T$ is $Q$-anisotropic, then $T(A) = T^1(A)$.

22: EXAMPLE Take $T = G_m, Q$ and $\omega = \frac{dx}{x}$ -- then

\[ \frac{\text{vol} \frac{dx}{|x|_p} (Z^x_p)}{\frac{p-1}{p}} = 1 - \frac{1}{p} \]
and the canonical convergence coefficients are the
\[ (1 - \frac{1}{p})^{-1}. \]

Here $d = 1$ and
\[ \lim_{s \to 1} (s-1)\zeta(s) = 1 \Rightarrow \rho_T = 1. \]

Working through the definitions, one concludes that $\tau(T) = 1$ or still,
\[ \text{vol}(T^1/Q^x) = 1. \]
23: REMARK Take \( T = \text{Res}_{\mathbb{K}/\mathbb{Q}}(G_{m, \mathbb{K}}) \) — then it turns out that \( \tau(T) \) is the Tamagawa number of \( G_{m, \mathbb{K}} \) computed relative to \( \mathbb{K} \) (and not relative to \( \mathbb{Q} \)…). From this, it follows that \( \tau(T) = 1 \), matters hinging on the "famous formula"

\[
\lim_{s \to 1} (s-1) \zeta_{\mathbb{K}}(s) = \frac{\prod_{p} \frac{r_{1}}{r_{2}}}{\omega_{\mathbb{K}} |d_{\mathbb{K}}|^{1/2}} \cdot h_{\mathbb{K}} R_{\mathbb{K}}.
\]

24: LEMMA Let \( F \) be an integrable function on \( (\mathbb{R}^+)^d \) — then

\[
\tau(T) = \frac{\frac{1}{\rho_{T}} \int_{\mathbb{R}^d} F(T(x)) d\mu_{\tau}(x)}{\frac{1}{\rho_{T}} \int_{\mathbb{R}^d} F(x) d\mu_{\tau}(x)}
\]

25: EXAMPLE Take \( T = G_{m, \mathbb{Q}} \) — then

\[
\tau(T) = \frac{\int_{\mathbb{Q}^\times} F(|x|_{\mathbb{A}}) d\mu_{\tau}(x)}{\int_{0}^{\infty} \frac{F(t)}{t} dt},
\]

\( \rho_{T} \) being 1 in this case. To see that \( \tau(T) = 1 \), make the calculation by choosing

\[
F(t) = 2te^{-\pi t^2}.
\]

[Note: Recall that

\[
\prod_{p} \mathbb{Z}_{p}^\times \times \mathbb{R}_{>0}^\times
\]

is a fundamental domain for \( \mathbb{Q}^\times \).]

26: NOTATION Put

\[
H^1(\mathbb{Q}, T) = H^1(\mathbb{Gal}(\mathbb{Q}^{\text{sep}}/\mathbb{Q}), T(\mathbb{Q}^{\text{sep}}))
\]
and for \( p \leq \infty \),
\[
H^1(Q_p, T) = H^1(Gal(Q_{sep}/Q_p), T(Q_{sep})).
\]

27: **Lemma** There is a canonical arrow
\[ H^1(Q, T) \to H^1(Q_p, T). \]

**Proof** Put
\[
G = Gal(\overline{Q}/Q) \quad (\overline{Q} = Q_{sep})
\]
and
\[
G_p = Gal(\overline{Q}_p/Q_p) \quad (\overline{Q}_p = Q_{sep}).
\]

Then schematically

1. There is an arrow of restriction
\[
\rho: G_p \to G
\]
and a morphism \( T(Q) \to T(\overline{Q}_p) \) of \( G_p \)-modules, \( T(Q) \) being viewed as a \( G_p \)-module via \( \rho \).

2. The canonical arrow
\[
H^1(Q, T) \to H^1(Q_p, T)
\]
is then the result of composing the map
\[
H^1(G, T(Q)) \to H^1(G_p, T(Q))
\]
with the map

$$H^1(G_p, T(Q)) \to H^1(G_p, T(\bar{Q}_p)).$$

**28: NOTATION** Put

$$\text{III}(T) = \text{Ker}(H^1(Q, T) \to \bigsqcup_{p \leq \infty} H^1(Q_p, T)).$$

**29: DEFINITION** $\text{III}(T)$ is the Tate-Shafarevich group of $T$.

**30: THEOREM** $\text{III}(T)$ is a finite group.

**31: EXAMPLE** If $K$ is a finite extension of $Q$, then

$$H^1(Q, \text{Res}_{K/Q}(G_m, K)) = 1.$$ Therefore in this case

$$#(\text{III}(T)) = 1.$$

**32: REMARK** By comparison,

$$H^1(Q, \text{Res}(1)_{K/Q}(G_m, K)) \approx Q^\times \cap N_{K/Q}(K^\times).$$

[Consider the short exact sequence

$$1 \to \text{Res}_{K/Q}(G_m, K) \to \text{Res}_{K/Q}(G_m, K) \xrightarrow{N_{K/Q}} G_m, Q \to 1.$$

**33: NOTATION** Put

$$\text{IV}(T) = \text{CoKer}(H^1(Q, T) \to \bigsqcup_{p \leq \infty} H^1(Q_p, T)).$$
THEOREM: \(\mathcal{U}(T)\) is a finite group.

MAIN THEOREM: The Tamagawa number \(\tau(T)\) is given by the formula

\[
\tau(T) = \frac{\#(\mathcal{U}(T))}{\#(\mathfrak{I}(T))}.
\]

EXAMPLE: If \(K\) is a finite extension of \(\mathbb{Q}\), then

\[
H^1(\mathbb{Q}, \text{Res}_{K/Q}(\mathbb{G}_m,K)) = 1.
\]

Therefore in this case

\[
\#(\mathcal{U}(T)) = 1.
\]

It follows from the main theorem that \(\tau(T)\) is a positive rational number.

Still, there are examples of finite abelian extensions \(K \supset \mathbb{Q}\) such that

\[
\tau(\text{Res}_{K/Q}(\mathbb{G}_m,K))^{(1)}
\]

is not a positive integer.