Math 591 – Homework 3

Due 1:30pm on Thursday, February 25, 2016

Please indicate any sources you used to find the solution to a given problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together in small groups, but try the problems on your own first and write up your own solutions.

Problem 1. Let $I = \langle x + y - 1\rangle \subset \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$.

(a) For every $w \in \mathcal{T}(I) \cap \mathbb{Q}^2$, give a point $(x, y) \in \mathcal{V}(I)$ with $\text{val}(x, y) = w$.

(b) Choose some $w \in \mathcal{T}(I) \cap \mathbb{Q}^2$ and some $a \in \mathcal{V}(\text{in}_w(I))$, and describe the set

$$\{(x, y) \in \mathcal{V}(I) \ : \ \text{val}(x, y) = w, \ \text{coeff}(x, y) = a\}.$$

Is this set Zariski-dense in $\mathcal{V}(I)$?

Problem 2. Suppose that $I = \langle f_1, \ldots, f_r\rangle \subset \mathbb{C}\{\{t\}\}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ where each $f_i \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is homogeneous with respect to every vector v in a linear space $L \subset \mathbb{R}^n$.

(a) Show that for any $a \in \mathbb{C}\{\{t\}\}$, $v \in L$, and $p \in \mathcal{V}(I)$, the point $a^v \cdot p = (a^{v_1} p_1, \ldots, a^{v_n} p_n)$ also lies in $\mathcal{V}(I)$.

(b) Prove that if $w \in \mathcal{T}(I)$, then $\mathcal{T}(I)$ contains the affine space $w + L$.

Problem 3. The discriminant of a univariate cubic $p(t) = at^3 + bt^2 + ct + d$ is

$$D \;=\; b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

(a) Check that D is homogeneous with respect to $v = (1, 1, 1, 1)$ and $w = (3, 2, 1, 0)$.

(b) Find a Laurent polynomial $f(x, y) \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ for which the Laurent ideals $\langle D\rangle$ and $\langle f(ac/b^2, bd/c^2)\rangle$ are equal in $\mathbb{C}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}]$.

(c) Describe the maximal cones of $\mathcal{T}(\langle D\rangle) \subset \mathbb{R}^4$ and their multiplicities.

Problem 4.

(a) Give an example of two ideals $I, J \subset \mathbb{C}\{\{t\}\}[x_1, \ldots, x_n]$ (for some n) so that $\mathcal{T}(I) \cap \mathcal{T}(J)$ is non-empty but $\mathcal{T}(I + J)$ is empty.

(b) Is the intersection $\mathcal{T}(I) \cap \mathcal{T}(J)$ transverse?

Problem 5. Suppose that P is a lattice N-gon in the plane. Let $r_1, \ldots, r_N \in \mathbb{Z}^2$ be the primitive integer vectors of the rays of the inner normal fan of P. For $i = 1, \ldots, N$, let $m_i \in \mathbb{Z}_+$ be the lattice length of the edge $\text{face}_{r_i}(P)$. Prove that $\sum_{i=1}^{N} m_i r_i = (0, 0)$.