Math 591 – Homework 2

Due 1:30pm on Thursday, February 11, 2016

Please indicate any sources you used to find the solution to a given problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together in small groups, but try the problems on your own first and write up your own solutions.

Problem 1. Let $f, g \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be Laurent polynomials.

(a) Prove that for any $w \in \mathbb{R}^n$, $\mathrm{in}_w(f \cdot g) = \mathrm{in}_w(f) \cdot \mathrm{in}_w(g)$.

(b) Show that the Newton polytope of $f \cdot g$ is the Minkowski sum of their Newton polytopes, *i.e.*

$$\mathrm{Newt}(f \cdot g) \;=\; \mathrm{Newt}(f) + \mathrm{Newt}(g),$$

where the Minkowski sum of two polytopes A, B is $A + B = \{a + b : a \in A, b \in B\}$.

(c) How does $\mathcal{T}(\mathrm{trop}(f \cdot g))$ relate to $\mathcal{T}(\mathrm{trop}(f))$ and $\mathcal{T}(\mathrm{trop}(g))$?

Problem 2. Let $f(x, y) \in \mathbb{Q}[x_1, \ldots, x_n, y]$ and $F = \mathrm{trop}(f)$, taking $\mathbb{Q}$ with the trivial valuation. Define $g = f(x, t) \in \mathbb{Q}(t)[x_1, \ldots, x_n]$ and $G = \mathrm{trop}(g)$. Show that

$$\mathcal{T}(G) \;=\; \{\ w \in \mathbb{R}^n \ \text{ such that } \ (w, 1) \in \mathcal{T}(F)\ \}.$$

Problem 3. Let K be an algebraically closed field and take an ideal $I \subset K[x_1, \ldots, x_n]$. Show that $V(I) \cap (K^*)^n$ is empty if and only if I contains a monomial.
(Hint: Use Hilbert's Nullstellensatz)

Problem 4. Consider the ideal $I = \langle f, g \rangle \subset \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$ where

$$f = t^2x^2 + xy + t^2y^2 + x + y + t^2 \quad \text{and} \quad g = 5 + 6tx + 17ty - 4t^3xy.$$

Let $F = \mathrm{trop}(f)$ and $G = \mathrm{trop}(g)$.

(a) For each $w \in \mathcal{T}(F) \cap \mathcal{T}(G)$, compute $\mathrm{in}_w(I)$.

(b) Is $\{f, g\}$ a tropical basis for I?

(c) There are four points in the variety $\mathcal{V}(\langle f, g \rangle) \subset (\mathbb{C}\{\{t\}\}^*)^2$.
Compute the leading term of each.