Math 591 – Homework 1

Due 1:30pm on Thursday, January 28, 2016

Please indicate any sources you used to find the solution to a given problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together in small groups, but try the problems on your own first and write up your own solutions.

Problem 1. Prove the tropical fundamental theorem of algebra: every tropical polynomial

$$F(x) = x^{\odot d} \oplus (c_{d-1} \odot x^{\odot d-1}) \oplus \ldots \oplus (c_1 \odot x) \oplus c_0$$

has a factorization as a product of linear forms $\odot_{k=1}^{d}(x \oplus r_k)$.

Define the *multiplicity* of a root w of a tropical polynomial $F(x)$ to be the number of times it appears in the factorization from Problem 1, *i.e.* $\mathrm{mult}(w) = \#\{k \in \{1, \ldots, d\} : r_k = w\}$.

Problem 2. Prove that for any $f \in \mathbb{C}\{\{t\}\}[x]$, the multiplicity of a root w of $\mathrm{trop}(f)$ equals the number of roots of $\mathrm{in}_w(f)$ in $\mathbb{C}^*$ (counted with multiplicity).

Problem 3. How many roots in $\mathbb{C}\{\{t\}\}$ does the polynomial $t^3x^5 - x^2 + t^4$ have? Find the first term of each.

Problem 4. Let $K = \mathbb{Q}$ with the p-adic valuation for some prime p.

(a) Show that the residue field of (K, val) is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

(b) The p-adic completion of $\mathbb{Q}$, denoted $\mathbb{Q}_p$, contains elements of the form

$$a = \sum_{i=m}^{\infty} a_i p^i \quad \text{where } a_i \in \{0, \ldots, p-1\} \text{ and } m \in \mathbb{Z}$$

and inherits the valuation $\mathrm{val}(a) = \min\{i : a_i \neq 0\}$. Use the method discussed in class (Newton's method) to find a square root of -1 in $\mathbb{Q}_5$.

Problem 5. Consider the two polynomials $f, g \in \mathbb{C}(t)[x, y]$ defined by

$$f = t^2x^2 + xy + t^2y^2 + x + y + t^2 \quad \text{and} \quad g = 5 + 6tx + 17ty - 4t^3xy.$$

Let $F = \mathrm{trop}(f)$ and $G = \mathrm{trop}(g)$.

(a) Draw $\mathcal{T}(F)$ and label each $w \notin \mathcal{T}(F)$ with $\mathrm{in}_w(f)$.

(b) Draw $\mathcal{T}(G)$ and label each $w \notin \mathcal{T}(G)$ with $\mathrm{in}_w(g)$.

(c) How many points are in $\mathcal{T}(F) \cap \mathcal{T}(G)$?