Math 582G – Homework 4

Due on Wednesday, March 9, 2022

Please indicate any sources you used for a given problem on the solution to that problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together, but you should try the problems on your own first and write up your own solutions.

Problem 1. Suppose that $f \in \mathbb{R}[x_1, \dots, x_n]$ is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$ and let $K = \overline{C(f, \mathbf{e})}$. The *rank* of a point $\mathbf{a} \in \mathbb{R}^n$, denoted $\operatorname{rank}(\mathbf{a})$, is defined to be the number of its nonzero eigenvalues.

(a) Show that for any $\mathbf{a}_1, \dots, \mathbf{a}_m \in K$ whose span intersects the interior of K, the rank of $\mathbf{a}_j$ equals the degree of y_j in $f(\sum_{j=1}^m y_j \mathbf{a}_j)$.

(b) Show that for any $\mathbf{a}, \mathbf{b} \in K$, $\operatorname{rank}(\mathbf{a} + \mathbf{b}) \leq \operatorname{rank}(\mathbf{a}) + \operatorname{rank}(\mathbf{b})$.

(c) Can you come up with an example $f = \det(\sum_{j=1}^m y_j A_j)$ with $A_j \in \mathbb{R}^{d \times d}_{\text{sym}}$ where $\deg_{y_j}(f) \neq \operatorname{rank}(A_j)$ for some j?

Problem 2. Let $G = ([n], E)$ be a graph. For each edge $e = \{i, j\} \in E$ with $i < j$, define the vector $v_e \in \mathbb{R}^{n-1}$ to be $\mathbf{1}_i - \mathbf{1}_j$ if $j < n$ and $\mathbf{1}_i$ if $j = n$, where $\mathbf{1}_i$ is the ith coordinate unit vector in $\mathbb{R}^{n-1}$. Show the following:

(a) The vectors $\{v_e : e \in E\}$ are linearly dependent if and only if E contains a cycle.

(b) If G is connected, then

$$\det\left(\sum_{e \in E} x_e v_e v_e^T\right) = \sum_T \prod_{e \in T} x_e \;\in\; \mathbb{R}[x_e : e \in E],$$

where the sum is taken over all spanning trees T of the graph G.

Problem 3. Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a nonzero polynomial with nonnegative coefficients that is homogeneous of degree d. Show that $\log(f)$ is concave on $\mathbb{R}^n_+$ if and only if $f^{1/d}$ is concave on $\mathbb{R}^n_+$.