Math 582G – Homework 3

Due on Friday, February 18, 2022

Please indicate any sources you used for a given problem on the solution to that problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together, but you should try the problems on your own first and write up your own solutions.

Problem 1. Let $H(x)$ be the $d \times d$ symmetric matrix with entries in $\mathbb{R}[x_1, \ldots, x_n]$. Show that $H(x)$ equals PP^T for some matrix $d \times m$ matrix P with entries in $\mathbb{R}[x_1, \ldots, x_n]$ if and only if the polynomial $y^T H(x) y \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_d]$ is a sum of squares, where $y = (y_1, \ldots, y_d)^T$.

Problem 2. For $A \in \mathbb{C}^{n\times n}$, let $M_A(t, x, y) = tI + x(A + A^*)/2 + y(A - A^*)/(2i)$.

(a) Show that $|z| \leq r$ for all $z \in \mathcal{W}(A)$ if and only if $\{(x, y) \in \mathbb{R}^2 : M_A(1, x, y) \succeq 0\}$ contains the ball defined by $x^2 + y^2 \leq 1/r^2$. (The smallest such r is called the *numerical radius* of A.)

(b) Using your favorite computer algebra software, find the minimal polynomial vanishing on the boundary of $\{(a, b) \in \mathbb{R}^2 : a + ib \in \mathcal{W}(A)\}$ for

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ i & 0 & 0 \end{pmatrix}.$$

(c) What is the numerical radius of A?

Problem 3. Given a univariate polynomial, $p(t)$ of degree d with roots $r_1, \ldots, r_d \in \mathbb{C}$, define the quadratic form H_p on $\mathbb{R}[t]_{\leq d-1}$ given by $H_p(q) = \sum_{j=1}^{d} q(r_j)^2$.

(a) Show that p is real rooted if and only if H_p is positive semidefinite.

(b) For $p = p_3t^3 + p_2t^2 + p_1t + p_0$, write down the symmetric matrix representing H_p with respect to the basis $\{1, t, t^2\}$ of $\mathbb{R}[t]_{\leq 2}$ in terms of the coefficients $p_3, \ldots, p_0$.

(c) Let $H(x_1, x_2)$ be the 3×3 matrix given by H_p for $p(t) = \det(M_A(t, x_1, x_2))$ for the matrix A in 2(b). Can $H(x_1, x_2)$ be written as a matrix sum of squares PP^T for some matrix P with entries in $\mathbb{R}[x_1, x_2]$?
(If so, this gives a sum-of-squares certificate for hyperbolicity!)