Math 582G – Homework 2

Due on Friday, February 4, 2021

Please indicate any sources you used for a given problem on the solution to that problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together, but you should try the problems on your own first and write up your own solutions.

Problem 1. The *Newton polytope* of a polynomial $f = \sum_\alpha c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is the convex hull of the exponent vectors of f:

$$\mathrm{Newt}(f) \;=\; \mathrm{conv}(\{\alpha : c_\alpha \neq 0\}) \;\subset \mathbb{R}^n.$$

For $w \in \mathbb{R}^n$, it can also be useful to consider the w-initial form of f, $\mathrm{in}_w(f) = \sum_{\alpha\in\mathcal{A}} c_\alpha \underline{x}^\alpha$, where $\mathcal{A} = \{\alpha \in \mathrm{Newt}(f) \cap \mathbb{Z}^n : w^T\alpha \geq w^T\beta \text{ for all } \beta \in \mathrm{Newt}(f)\}$.

(a) Prove that if $f = \sum_{i=1}^k h_i^2$, then $\mathrm{Newt}(h_i) \subseteq \frac{1}{2}\mathrm{Newt}(f)$.

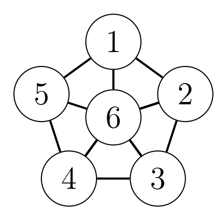
(b) Show that the Motzkin polynomial $1 - 3x^2y^2 + x^4y^2 + x^2y^4$ is not a sum of squares.

Problem 2. For $d \in \mathbb{Z}_{\geq 0}$, let $\mathcal{C}_d = \{(t, t^2, \ldots, t^d) : t \in [-1, 1]\}$.

(a) Write $\mathrm{conv}(\mathcal{C}_d)$ as a spectrahedron.

(b) Show that any point on the boundary of $\mathrm{conv}(\mathcal{C}_d)$ can be written as a convex combination of $\lceil \frac{d+1}{2} \rceil$ points of $\mathcal{C}_d$.
(Hint: consider a supporting hyperplane and corresponding univariate polynomial.)

(c) Show that any point in $\mathrm{conv}(\mathcal{C}_d)$ can be written as a convex combination of $\lceil \frac{d+1}{2} \rceil + 1$ points of $\mathcal{C}_d$. How does this compare to the bound give by Carathéodory?



Problem 3. Let $G = ([6], E)$ be the wheel graph on six vertices:

(a) Write the problem

$$\theta_1(G) = \min\ c \ \text{ such that }\ c - \sum_{i=1}^{6} x_i \in \mathrm{SOS}_{6,2} + I_G$$

explicitly as a semidefinite program

$$\min\langle C, X\rangle \text{ such that } \langle A_i, X\rangle = b_i \text{ for } i = 1, \ldots, m$$

for some m, real symmetric matrices $C, A_1, \ldots, A_m \in \mathbb{R}^{7\times 7}_{\mathrm{sym}}$ and $b \in \mathbb{R}^m$. Here I_G denotes the the ideal generated by $x_i^2 - x_i$ for $i = 1, \ldots, 6$ and x_ix_j for $\{i, j\} \in E$.

(b) Write down the dual semidefinite program.

(c) How does $\theta_1(G)$ compare to the size $\alpha(G)$ of the maximum stable set of G?

See `https://sums-of-squares.github.io/sos/exercises.html` for other exercises and `https://sums-of-squares.github.io/sos/` for SOS computational tools.