Math 582G – Homework 1

Due on Friday, January 21, 2021

Please indicate any sources you used for a given problem on the solution to that problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together, but you should try the problems on your own first and write up your own solutions.

Problem 1. Let K denote the convex cone of quadratic polynomials in $\mathbb{R}[x]$ that are nonnegative on $[-1,1]$, i.e.

$$K \;=\; \mathcal{P}_2([-1,1]) \;\cong\; \{(a,b,c) \in \mathbb{R}^3 : ax^2 + bx + c \geq 0 \text{ for all } x \in [-1,1]\}.$$

(a) Describe the dual cone K^*.

(b) Give a semialgebraic description of K.

(c) Draw the intersections of both K and K^* with the planes "last coordinate"$= 1$.

(d) Does either K or K^* have a non-exposed face?

Problem 2. Let $\mathcal{H}^n$ denote the real vector space of Hermitian matrices,

$$\mathcal{H}^n = \{A \in \mathbb{C}^{n\times n} : A = \overline{A}^T\},$$

where $\overline{A}^T$ is the conjugate transpose of A. A Hermitian matrix A is *positive semidefinite* if $\overline{v}^T A v \geq 0$ for all $v \in \mathbb{C}^n$. Let $\mathcal{H}^n_+$ denote the set of positive semidefinite matrices in $\mathcal{H}^n$.

(a) Show that $\mathcal{H}^n_+$ is a closed, convex cone.

(b) Show that $\mathcal{H}^n_+$ is self-dual under the inner product $\langle A, B\rangle = \text{trace}(AB)$.

(c) What are the extreme points of the convex set $\{A \in \mathcal{H}^n_+ : \text{trace}(A) = 1\}$?

Problem 3. For $A \in \mathbb{R}^{n\times n}_{\text{sym}}$, consider the maximization problem

$$\max_{y\in\mathbb{R}} \; y \quad \text{s.t.} \quad A - yI \in \text{PSD}_n,$$

where I is the $n \times n$ identity matrix.

(a) Write a minimization problem of which this is the dual.

(b) Show that the primal and dual problems attain the same value. What is this optimal value in terms of the eigenvalues of the matrix A?

Here's another good to think about. Do not turn in the solution!

Extra Problem 1. Suppose that C, K are closed convex cones and L is a linear subspace in a finite dimensional real vectorspace V. Show the following:

(a) If $C \subseteq K$ then $C^* \supseteq K^*$.

(b) $(C + K)^* = C^* \cap K^*$

(c) $L^* = L^\perp = \{\ell \in V^* : \ell(v) = 0 \text{ for all } v \in L\}$

(d) $K \subseteq L$, if and only if $K^* + L^\perp = K^*$

Here $C + K = \{u + v : u \in C, v \in K\}$ denotes the Minkowski sum of C and K.