Math 582G

Today: Applications of SOS & moments in combinatorial optimization

Constrained Polynomial Optimization

Given $f, g_1, \dots, g_r \in \mathbb{R}[x_1, \dots, x_n]_{\leq 2d}$, find

$$f^* = \max_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g_1(x) = 0, \dots, g_r(x) = 0$$

$$= \min_{c \in \mathbb{R},\, x \in \mathbb{R}^n} c \quad \text{s.t.} \quad c - f(x) \geq 0 \text{ on } V_{\mathbb{R}}(g_1, \dots, g_r)$$

$$\leq f^*_{sos,d} = \min c \quad \text{s.t.} \quad c - f(x) = \sigma + \sum_{i=1}^{r} h_i g_i$$

for some $\sigma \in SOS_{n,2d}$

Dual version:

$$f^*_{sos,d} = \max_{L \in \mathbb{R}[\underline{x}]^*_{\leq d}} L(f(x)) \quad \text{s.t.} \quad L \in SOS^*_{n,2d}$$

$$L(h_i g_i) = 0 \quad \forall h_i \in \mathbb{R}[\underline{x}]$$

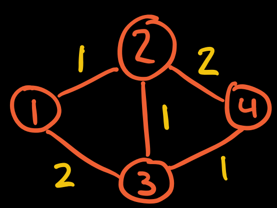
with $\deg(h_i g_i) \leq d$

Interesting applications in comb. opt. when $V_{\mathbb{R}}(g_1, \dots, g_r)$ is finite!

MAXCUT: Let $G=([n],E)$ be a graph on n vertices with edges $E \subseteq \binom{[n]}{2}$ and weights $w: E \to \mathbb{R}_{\geq 0}$.

$$\text{MAXCUT}(G) = \max_{S \subseteq [n]} w(S, [n]\setminus S) = \max_{S \subseteq [n]} \sum_{\substack{i \in S \\ j \in [n]\setminus S}} w(i,j)$$

Ex:



$S = \{1,2\}$ $\quad w(12, 34) = 5$

$S = \{1,4\}$ $\quad w(14, 23) = 6$

Known to be NP Hard in general.

→ hope for good approximations in polynomial time

As a poly. opt. problem:

Claim: $\text{MAXCUT}(G) = \max \sum_{\{i,j\} \in E} w_{ij} \left(\frac{1 - x_i x_j}{2} \right)$ s.t. $x_i^2 = 1$ for $i \in [n]$

(Proof) Take $S = \{ i \in [n] \text{ s.t. } x_i = 1 \}$. Then

$$w_{ij} \frac{(1 - x_i x_j)}{2} = \begin{cases} 0 & \text{if } i,j \in S \text{ or } i,j \notin S \\ w_{ij} & \text{if } i \in S, j \notin S \text{ or } j \notin S, i \in S. \end{cases}$$

SOS approximation:

$$\min c \text{ s.t. } c - \sum_{ij \in E} w_{ij} \frac{(1 - x_i x_j)}{2} = \sigma + \sum_{i=1}^{n} h_i (1 - x_i^2)$$

$\sigma \in SOS_{n,2}$

Dual moment approx

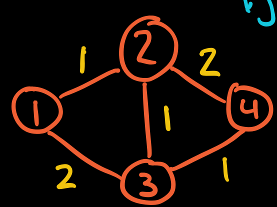
$$\max_{L \in \mathbb{R}[x_1,\dots,x_n]^*_{\le 2}} L\left(\sum_{ij \in E} w_{ij} \frac{(1-x_i x_j)}{2}\right) \quad \text{s.t.} \quad \begin{array}{l} L \ge 0 \text{ on } SOS_{n,2} \\ L(1-x_i^2)=0 \ \forall i \end{array}$$

Write $y_{ij} = L(x_i x_j)$. Then $L(1-x_i^2)=0 \Leftrightarrow y_{ii}=1$

Translation of dual:

(Goemans-Williamson approximation)

$$GW(G) = \max \sum_{ij} w_{ij} \frac{(1-y_{ij})}{2} \quad \text{s.t.} \quad Y = \begin{pmatrix} 1 & y_{12} & \cdots & y_{1n} \\ y_{12} & 1 & & \\ \vdots & & \ddots & \\ y_{1n} & \cdots & & 1 \end{pmatrix} \succeq 0$$

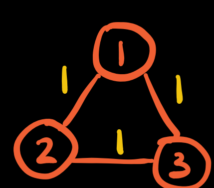
Ex: 

$GW(G) = 6$ acheived by $Y^* = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

Dual SOS certificate:

$$6 - \frac{2}{2}(2 - x_1x_3 - x_2x_4) - \frac{1}{2}(3 - x_1x_2 - x_2x_3 - x_3x_4)$$

$$= \frac{1}{4}\left[x^T \begin{pmatrix} 3 & 1 & 2 & 0 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 0 & 2 & 1 & 3 \end{pmatrix} x + 3(1-x_1^2+1-x_4^2) + 2(1-x_2^2+1-x_3^2) \right]$$

Ex (non-ex) 

$MAXCUT(G) = 2$

$GW(G) = \frac{9}{4}$

acheived by $Y^* = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \rightarrow \sum_{ij} \frac{(1+1/2)}{2} = 3 \cdot \frac{3}{4} = \frac{9}{4}$

$$2 - \frac{1}{2}(3 - x_1x_2 - x_1x_3 - x_2x_3) \geq 0 \text{ on } \{\pm 1\}^3$$

but $\neq \sigma + \sum h_i(1-x_i^2)$ with $\sigma \in SOS_{3,1}$

Thm: $.878\, GW(G) \leq MAXCUT(G) \leq GW(G)$

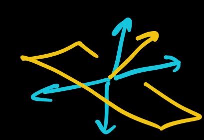
(Proof) Suppose Y^* acheives $GW(G)$.

Write $Y^* = VV^T$ where $V = \begin{pmatrix} - v_1 - \\ \vdots \\ - v_n - \end{pmatrix} \in \mathbb{R}^{n \times r}$.

Note $X_{ij} = \langle v_i, v_j \rangle$

Choose $u \in \mathbb{R}^r$, $\|u\|_2 = 1$ uniformly at random.

Let $S = \{ i \in [n] : \langle v_i, u \rangle > 0 \}$.



$$\mathbb{E}_u(\omega(S, [n] \setminus S)) = \sum_{i,j \in E} w_{ij} \text{Prob}(i,j \text{ separated})$$

$$= \sum_{i,j \in E} w_{ij} \frac{\text{angle}(v_i, v_j)}{\pi}$$

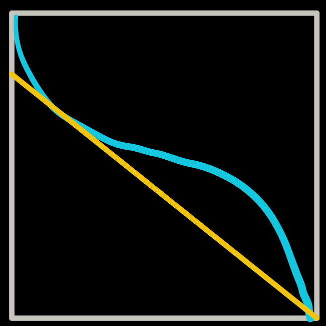
$$= \sum_{i,j \in E} w_{ij} \frac{\arccos(\langle v_i, v_j \rangle)}{\pi}$$

$$= \sum_{i,j \in E} w_{ij} \frac{\arccos(Y_{ij})}{\pi}$$

$$\geq \gamma \cdot \sum_{i,j \in E} w_{ij} \frac{(1 - Y_{ij})}{2} = \gamma\, GW(G)$$

Here $\gamma = .878$ is largest value s.t.

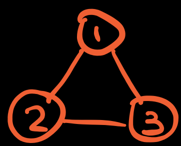
$$\frac{\arccos(t)}{\pi} \geq \gamma \cdot \frac{1-t}{2} \text{ for all } t \in [-1, 1]$$



$$\frac{\arccos(t)}{\pi}$$

$$.878\left(\frac{1-t}{2}\right)$$

Ex:



$$Y^* = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} = VV^T \text{ for } V = \begin{pmatrix} 1 & 0 \\ -1/2 & \sqrt{3}/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix}$$