Math 582G

Today: Moments & pseudomoments
Hwk 1 due on Canvas

Moment Problems

Let $L : \mathbb{R}[x_1, \dots, x_n] \to \mathbb{R}$ be linear, and $S \subseteq \mathbb{R}^n$ closed.

Q: Is there a (nonneg.) measure μ on S s.t.
$$L(f) = \int f \, d\mu \quad \text{for all } f \in \mathbb{R}[x_1, \dots, x_n]?$$

Ex: $n=1$, $S=\mathbb{R}$

$L(x^k) = \frac{1}{k+1} \to$ YES $L(f) = \int_0^1 f \, dx$

$L(x^k) = k+1 \to$ NO $L((x-2)^2) = L(x^2 - 4x + 4) = 3 - 4 \cdot 2 + 4 = -1 < 0$

Haviland's Thm (1930's)

There exists a measure μ on S representing L if and only if $L(f) \geq 0$ for all $f \in P(S)$.

Bounded Degree? $L : \mathbb{R}[x_1, \dots, x_n]_{\leq d} \to \mathbb{R}$

$\mathcal{M}_{\leq d}(S) = \{ L \in (\mathbb{R}[\underline{x}]_{\leq d})^* : L(f) = \int f \, d\mu$ for some nonneg. measure μ on $S \}$

Claim: $\mathcal{M}_{\leq d}(S) = \text{conicalHull}(\{ ev_p : p \in S \})$

Cor: $\mathcal{M}_{\leq d}(S)^* = \mathcal{P}_{\leq d}(S)$

(Proof) Let δ_p = Dirac measure at $\{p\}$ so that

($\supseteq$)

$$\int f \, d\delta_p = f(p) = ev_p(f).$$

Any function $L = \sum \lambda_i ev_{p_i}$ with $p_i \in S$, $\lambda_i \geq 0$ is represented by $\mu = \sum \lambda_i \delta_{p_i}$.

(Sketch of $\subseteq$)

Suppose $L \in \mathcal{M}_{\leq d}(S)$ but $L \notin \text{conicalHull}(ev_p)$

By Separation Thm, $\exists f \in \mathbb{R}[x_1, \ldots, x_n]_{\leq d}$ s.t.

$\ell(f) \geq 0 \;\; \forall \ell \in \text{conicalHull}(ev_p)$ but $L(f) \leq 0$.

$\uparrow \Rightarrow f(p) \geq 0 \;\; \forall p \in S \Rightarrow L(f) = \int f \, d\mu = 0$

$\Rightarrow \text{supp}(\mu) \subseteq \{ x \in S : f(x) = 0 \}$

For $n=1 \Rightarrow \text{supp}(\mu)$ finite

$\Rightarrow \sum_{i=1}^{r} \lambda_i \delta_{p_i}$ for some $p_i \in S$, $\lambda_i \in \mathbb{R}_{\geq 0}$

$n > 1$ induction on dim

Cor: $\mathcal{M}_{\le 2d}(\mathbb{R}^n) \subseteq SOS_{n,2d}^*$

with equality $\iff$ $n=1$, $d=1$, or $(n,d)=(2,2)$

elements $L \in SOS_{n,2d}^*$ known as "pseudo-moments"

or "pseudo-expectations" if $L(f)=1$

$SOS_{n,2d}^*$ as a spectrahedron

$L \in \mathbb{R}[x_1,\dots,x_n]_{\le 2d}^*$ determined by $L(x^\alpha)=y_\alpha$ for $|\alpha|\le 2d$

$L \in SOS_{n,2d}^* \iff L\left(\sum_{i=1}^k h_i^2\right) \ge 0 \quad \forall\, h_i \in \mathbb{R}[x_1,\dots,x_n]_{\le d}$

$\iff L(h^2)\ge 0 \quad \forall\, h \in \mathbb{R}[x_1,\dots,x_n]_{\le d}$

$\iff$ the matrix $\left(L(x^{\alpha+\beta})\right)_{\alpha,\beta}$ representing the quadratic form $h \mapsto L(h^2)$ is PSD

Ex: $n=1$, $d=2$ $\quad L: \mathbb{R}[t]_{\le 4} \to \mathbb{R} \quad L(t^k)=y_k \quad k=0,1,2,3,4$

$h = h_0 + h_1 t + h_2 t^2 = v^T m_2 \qquad$ for $v=\begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix}$ $m_2=\begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$

$L(h^2) = L(v^T m_2 m_2^T v) = v^T L(m_2 m_2^T) v = v^T M_L v$

where $M_L = \left(L(t^{i+j})\right)_{0\le i,j\le 2} = \begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix}$

$SOS_{1,\le 4}^* = \{ L \in \mathbb{R}[t]_{\le 4} : M_L \succeq 0 \} \cong \left\{ y \in \mathbb{R}^5 : \begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \succeq 0 \right\}$

$= \mathcal{M}_{\le 4}(\mathbb{R}) = \overline{\text{conical Hull}\{(1,t,t^2,t^3,t^4) : t \in \mathbb{R}\}}$

Optimization: given $f \in \mathbb{R}[x_1, \dots, x_n]_{\leq 2d}$

$$f^* = \min_{p \in \mathbb{R}^n} f(p) = \min_{p \in \mathbb{R}^n} ev_p(f) = \min_{\substack{L \in \mathcal{M}_{\geq d}(\mathbb{R}^n) \\ L(1)=1}} L(f)$$

$$f^*_{sos} = \min L(f) \quad \text{s.t.} \quad \begin{array}{l} L \in SOS^*_{n,2d} \\ L(1) = 1 \end{array}$$

$\mathcal{M}_{\geq d}(\mathbb{R}^n) \subseteq SOS^*_{n,2d}$

$\Rightarrow f^*_{sos} \leq f^*$

Ex: $\displaystyle\min_{x \in \mathbb{R}} x^4 + 2x^3 + 2 = \min_{y \in \mathbb{R}^4} y_4 + 2y_3 + 2 \quad \text{s.t.} \quad \begin{pmatrix} 1 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \succeq 0$

$f^* = \frac{5}{16} \quad x^* = \frac{-3}{2} \quad (y_1^*, y_2^*, y_3^*, y_4^*) = \left(\frac{-3}{2}, \frac{9}{4}, \frac{-27}{8}, \frac{81}{16}\right)$

an SDP!