Math 582G - Convex Alg. Geometry

Today: Sums of squares and SDPs

Reminder: hwk 1 due Friday

Let $SOS_{n,2d} = \{ \sum_{i=1}^{r} g_i^2 : g_i \in \mathbb{R}[x_1, \ldots, x_n]_{\leq d} \}$.

A representation of f as a sum of squares gives a certificate of nonnegativity of f on $\mathbb{R}^n$.

That is, $SOS_{n,2d} \subseteq P_{2d}(\mathbb{R}^n) \subseteq \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d}$.

Thm (Hilbert) $SOS_{n,2d} = P_{2d}(\mathbb{R}^n)$ if and only if $n=1$ or $d=1$ or $(n,d)=(2,2)$

$(n=1)$ $f \geq 0$ on $\mathbb{R}$ $\Rightarrow$ roots come in conj. pairs

$\Rightarrow f = (g+ih)(g-ih) = g^2 + h^2$ for some $g,h \in \mathbb{R}[x]$.

$2d=2$ $f = x^T A x$ for some $A \in \mathbb{R}^{n \times n}_{sym}$.

$f \geq 0$ on $\mathbb{R}^n \Rightarrow A \succeq 0 \Rightarrow A = \sum_{i=1}^{n} v_i v_i^T$

$\Rightarrow f = \sum_i (v_i^T x)^2$

$(n,d)=(2,2)$ more involved

Motzkin poly: $M(x,y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$

By AM-GM ineq., $\frac{x^4y^2+x^2y^4+1}{3} \geq \sqrt[3]{x^4y^2\cdot x^2y^4\cdot 1} = x^2y^2$

but $M(x,y)$ is not a sum of squares.

Connections with SDPs:

Let m_d = vec. of all mon. x^α in $x_1,\dots,x_n$ with deg $\leq d$

Prop: $f\in SOS_{n,2d} \iff \exists A\in PSD_N$ s.t.

$$f = m_d^T A m_d.$$

(Proof) $f=\sum_{i=1}^r g_i^2$ for some $g_i\in\mathbb{R}[\underline{x}]_{\leq d}$ $\iff f=\sum_{i=1}^r (v_i^T m_d)^2$ for some $v_i\in\mathbb{R}^N$

$$= \sum_{i=1}^r m_d^T v_i v_i^T m_d$$

$$= m_d^T\left(\sum_{i=1}^r v_i v_i^T\right) m_d$$

$\iff f = m_d^T A m_d$ for some $A\in PSD_N$.

Ex $(n=1, d=2)$ $f = x^4+2x^3+2$

$$f = \begin{pmatrix}1\\x\\x^2\end{pmatrix}^T A \begin{pmatrix}1\\x\\x^2\end{pmatrix}$$

$$= \left\langle A, \begin{pmatrix}1 & x & x^2\\ x & x^2 & x^3\\ x^2 & x^3 & x^4\end{pmatrix}\right\rangle = a_{11} + 2a_{12}x + (a_{22}+2a_{13})x^2 + 2a_{23}x^3 + a_{33}x^4$$

$$\Rightarrow a_{11}=2 \quad a_{12}=0 \quad a_{22}+2a_{13}=0 \quad a_{23}=1 \quad a_{33}=1$$

$f\in SOS_{1,2} \iff \exists a\in\mathbb{R}$ s.t. $\begin{pmatrix}2 & 0 & -a\\ 0 & 2a & 0\\ -a & 0 & 1\end{pmatrix} \succeq 0$ det $= 2(1-a)(a^2+a-1)$

PSD $\iff a\in\left[\frac{-1+\sqrt{5}}{2}, 1\right]$

$a=1 \Rightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}(1\ 1\ 0) + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}(-1, 1, 1)$

$\Rightarrow f = (1+x)^2 + (-1+x+x^2)^2$

Global Polynomial Optimization

Given $f \in \mathbb{R}[x_1, \dots, x_n]_{\leq 2d}$,

$$f^* = \min_{x \in \mathbb{R}^n} f(x) = \max_{\lambda \in \mathbb{R}} \lambda \text{ s.t. } f - c \in P_{2d}(\mathbb{R}^n)$$

$$\geq f^*_{sos} = \max_{\lambda \in \mathbb{R}} \lambda \text{ s.t. } f - c \in SOS_{n,2d}$$

Computing f^*_{sos} is a semidefinite program!

Ex: $\min_{x \in \mathbb{R}} 2 + 2x^3 + x^4 = \max_{a, c \in \mathbb{R}} c \text{ s.t. } \begin{pmatrix} 2-c & 0 & -a \\ 0 & 2a & 1 \\ -a & 1 & 1 \end{pmatrix} \succeq 0$

$= 5/16$

Note: This is not always a good lower bound!

for $f = M(x,y)$, $f^* = 0$, $f^*_{sos} = -\infty$

Constrained poly. optimization

Let $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_s(x) \geq 0\}$.

Any polynomial of the form $\sum_{\alpha \in \{0,1\}^s} \sigma_\alpha \prod_{i=1}^{s} g_i^{\alpha_i}$

where $\sigma_\alpha \in SOS_{n,2d}$ is nonnegative on S.

<u>Def</u>: The <u>preorder</u> generated by $g_1, \ldots, g_s$

$$PO(g_1, \ldots, g_s) = \left\{ \sum_{\alpha \in \{0,1\}^s} \sigma_\alpha \prod_{i=1}^{s} g_i^{\alpha_i} : \sigma_\alpha \in SOS_n \right\}.$$

For $f \in \mathbb{R}[x_1, \ldots, x_n]$, let

$$f^* = \min_{x \in S} f(x)$$

$$f^*_{sos} = \max_{c \in \mathbb{R}} c \quad \text{s.t. } f - c \in PO(g_1, \ldots, g_s)$$

$$f^*_{sos,d} = \max_{c \in \mathbb{R}} c \quad \text{s.t. } f - c = \sum_{\alpha \in \{0,1\}^s} \sigma_\alpha \prod_{i=1}^{s} g_i^{\alpha_i}$$

where $\sigma_\alpha \in SOS_n$ and $\deg\left(\sigma_\alpha \prod_{i=1}^{s} g_i^{\alpha_i}\right) \le d$

Then $f^* \ge f^*_{sos} \ge f^*_{sos,d}$ and computing $f^*_{sos,d}$ is a semidefinite prog.

Ex: $S = \{x \in \mathbb{R} : 1 - x^2 \ge 0\} = [-1, 1]$

$PO(1-x^2) = \{\sigma_0 + (1-x^2)\sigma_1 : \sigma_0, \sigma_1 \in SOS_1\}$.

$f = x^4 + 4x^3 \rightarrow$ $f^*_{sos,2} = \min c :$ $f - c = \sigma_0 + \sigma_1(1-x^2)$ $\quad \sigma_0 \in SOS_{1,4}$, $\sigma_1 \in SOS_{1,2}$

$$\min\ c\ :\ f - c = \begin{pmatrix}1\\x\\x^2\end{pmatrix}^T A \begin{pmatrix}1\\x\\x^2\end{pmatrix} + (1-x^2)\begin{pmatrix}1\\x\end{pmatrix}^T B \begin{pmatrix}1\\x\end{pmatrix}$$

$$\text{and } \begin{pmatrix}A & 0\\ 0 & B\end{pmatrix} \in PSD_5$$

linear constraints : coeff of

$x^4 \to 1 = a_{33} - b_{22}$

$x^3 \to 4 = 2a_{23} - 2b_{12}$

$x^2 \to 0 = 2a_{13} + a_{22} + b_{22} - b_{11}$

$x^1 \to 0 = 2a_{12} + 2b_{12}$

$x^0 \to -c = a_{11} + b_{11}$

$f^* = f^*_{sos,1} = -$, $f + 3 = 2(x+1)^2(x^2-x+1) + (1-x^2)(x-1)^2$

For $n=1$, $S=[a,b]$, nonneg. poly. have SOS certificates of lowest possible deg.

e.g. $x^3 + x^2 = (1+x)x^2 + (1-x)\cdot 0$