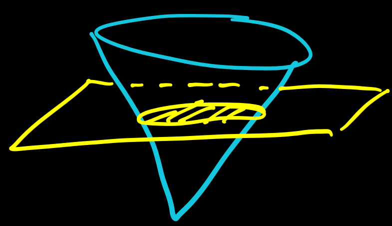
Math 582G - Convex Alg. Geometry

Today: Conic programming

HWk 1 posted on course website, due Fri Jan. 21

V = finite dim'l $\mathbb{R}$-vec. space

$V^* = \{\text{linear functions } \ell : V \to \mathbb{R}\}$



Conic programming

A conic program has the form

(Primal) $\quad \min_{x \in K} c(x) \quad$ s.t. $a_i(x) = b_i \quad \forall i = 1, \dots, m$

where $K \subseteq V$ is a closed convex cone,

$a_1, \dots, a_m \in V^*$, $b_1, \dots, b_m \in \mathbb{R}$

Primal feasible set $= \{x \in K : a_i(x) = b_i \ \forall i\}$

Ex: $K = \mathbb{R}^n_{\geq 0}$ feasible sets = polyhedra

conic program $\rightsquigarrow$ linear program

(many effective methods for solving!)

Ex: $P = [-1,1]^2 = \{z \in \mathbb{R}^2 : -1 \leq z_i \leq 1\}$

Rephrase: $\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and $x \in \mathbb{R}^4_{\geq 0}$

Mult. by $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_3 + x_4 \end{pmatrix}$

P linearly isomorphic to $\left\{ x \in \mathbb{R}^4_{\geq 0} : \begin{array}{l} x_1 + x_2 = 2 \\ x_3 + x_4 = 2 \end{array} \right\}$

Ex: $K = PSD_n$

conic program called a semidefinite program (SDP)

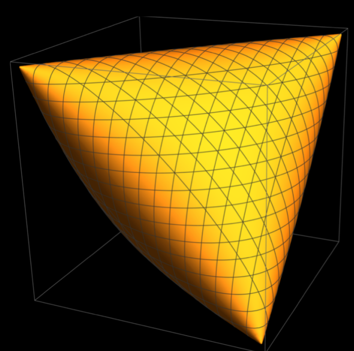
$$\min_{X \succeq 0} \langle C, X \rangle \text{ s.t. } \langle A_i, X \rangle = b_i \quad \forall i = 1, \dots, m$$

where $C, A_1, \dots, A_m \in \mathbb{R}^{n \times n}_{sym}$, $b_1, \dots, b_m \in \mathbb{R}$

feasible sets $\{ X \in PSD_n : \langle A_i, X \rangle = b_i \ \forall i \}$
called <u>spectrahedra</u>

e.g. $\{ X \in PSD_3 : \langle E_{ii}, X \rangle = 1, \ i = 1, 2, 3 \}$

$$= \left\{ \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \in PSD_3 : x, y, z \in \mathbb{R}^3 \right\}$$



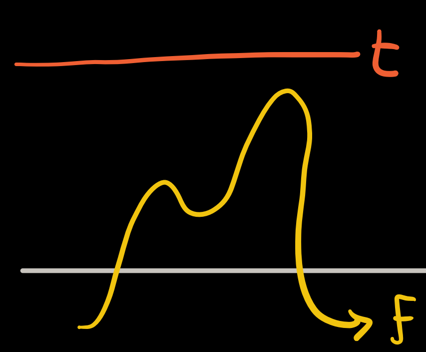
Ex: Polynomial optimization

$$K = P_d(S) = \{ f \in \mathbb{R}[x_1, \dots, x_n]_{\leq d} : f \geq 0 \text{ on } S \}$$

Note that for $f \in \mathbb{R}[x_1, \dots, x_n]_{\leq d}$,

$$\max_{x \in S} f(x) = \min_{t \in \mathbb{R}} t \text{ s.t. } t - f(x) \in P_d(S)$$

(when max is acheived)



All the complexity is in cone K.

To be effective, need to be able to test membership in K!

Duality in conic optimization

(Primal) $p^* = \min_{x \in K} c(x)$ s.t. $a_i(x) = b_i$ $i = 1, \dots, m$

(Dual) $d^* = \max_{y \in \mathbb{R}^m} \sum_{i=1}^m b_i y_i$ s.t. $c - \sum_{i=1}^m y_i a_i \in K^*$

Thm (Weak duality $p^* \geq d^*$)

For any primal feasible x, dual feasible y,

$$c(x) \geq \sum_{i=1}^m b_i y_i.$$

If $c(x) = \sum_{i=1}^m b_i y_i$, then both are optimal and $p^* = d^*$.

(Proof) $c(x) - \sum_{i=1}^{m} b_i y_i = c(x) - \sum_{i=1}^{m} a_i(x) y_i$ $\left(\begin{array}{l}\text{feas. of } x \\ \Rightarrow b_i = a_i(x)\end{array}\right)$

$$= \underbrace{\left(c - \sum_{i=1}^{m} y_i a_i\right)}_{\in K^*} \underbrace{(x)}_{\in K} \geq 0$$

If $c(x^*) = \sum_{i=1}^{m} b_i y_i^*$ (with x^*, y^* feasible)

$c(x^*)$ is a upper bound for $\sum_{i=1}^{m} b_i y_i \Rightarrow y^*$ optimal

$\sum_{i=1}^{m} b_i y_i^*$ is a lower bound for all $c(x) \Rightarrow x^*$ optimal. $\square$

This gives a method for certifying optimal value!

Ex: $K = PSD_3$ $A_i = E_{ii}$, $b_i = 1$ for $i = 1, 2, 3$, $C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(Primal) $\min_{X \in PSD_3} \langle C, X \rangle = 2(X_{12} + X_{13} + X_{23})$ s.t. $X_{11} = X_{22} = X_{33} = 1$.

(Dual) $\max_{y \in \mathbb{R}^3} y_1 + y_2 + y_3$ s.t. $C - \sum_{i=1}^{m} y_i A_i = \begin{pmatrix} -y_1 & 1 & 1 \\ 1 & -y_2 & 1 \\ 1 & 1 & -y_3 \end{pmatrix} \in PSD_3$

For feasible X, y, $\langle C, X \rangle - (y_1 + y_2 + y_3) = \langle X, C - \sum y_i A_i \rangle \geq 0$

For $X^* = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}$ $y^* = (-1, -1, -1)$

$$\langle C, X \rangle = -3 = y_1 + y_2 + y_3 \Rightarrow p^* = -3 = d^*$$

Strong duality ($p^* = d^*$)

- holds for linear programs
- holds if primal and dual are both strictly feasible i.e. $\exists\, x \in \mathrm{int}(K)$ s.t. $a_i(x) = b_i$
  $\exists\, y \in \mathbb{R}^m$ s.t. $c - \sum y_i a_i \in \mathrm{int}(K^*)$

Strong duality can fail for SDPs

Ex (Pataki) Take $K = PSD_3$, $b_1 = 0$, $b_2 = -1$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$p^* = \min_{X \in PSD_3} \langle C, X \rangle = X_{11} + X_{22} \quad \text{s.t.} \quad \langle A_1, X \rangle = X_{11} = 0$$
$$\langle A_2, X \rangle = -X_{22} = -1$$

$\Rightarrow p^* = 1$ (attained by $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$)

$$d^* = \max -y_2 \quad \text{s.t.} \quad C - y_1 A_1 - y_2 A_2 = \underbrace{\begin{pmatrix} 1-y_1 & 0 & y_2 \\ 0 & -1-y_2 & 0 \\ y_2 & 0 & 0 \end{pmatrix}}_{\{1,3\} \text{ minor} \Rightarrow y_2 = 0} \succeq 0$$
$$= 0$$

Note: primal is not strictly feasible

There is no positive definite matrix X with $\langle A_1, X \rangle = X_{11} = 0$