Math 582G - Convex Alg. Geo.

Today: The PSD cone

Eigendecompositions of $A \in \mathbb{R}^{n \times n}_{sym}$

Any real symmetric matrix is diagonalizable with real eigenvalues.

$$A \in \mathbb{R}^{n \times n}_{sym} \implies A = UDU^T$$

where U orthogonal ($U^TU = UU^T = I$)

and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_k = k^{th}$ eig. val. of A.

A matrix $A \in \mathbb{R}^{n \times n}_{sym}$ is

positive semidefinite (PSD) $\iff v^TAv \geq 0$ for all $v \in \mathbb{R}^n$

positive definite (PD) $\iff v^TAv > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$

Characterizations of PSD matrices

For $A \in \mathbb{R}^{n \times n}_{sym}$, TFAE:

1) all eig. val. of $A \geq 0$
2) all principal minors $\det(A_{S,S}) \geq 0$
3) $v^TAv \geq 0$ for all $v \in \mathbb{R}^n$

4) $A = BB^T$ for some $B \in \mathbb{R}^{n\times k}$, $k = \text{rank}(A)$

$$= \sum_{i=1}^{r} c_i c_i^T = \left(\langle r_i, r_j\rangle\right)_{1\le i,j\le n}$$ where $c_1,\dots,c_r$ = col of B, $r_1,\dots,r_k$ = rows of B

(1)⇒(4) $\lambda_k \ge 0\ \forall k \Rightarrow \sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n}\end{pmatrix} \in \mathbb{R}^{n\times n}$

$$A = U\sqrt{D}\sqrt{D}U^T = BB^T \text{ for } B = U\sqrt{D}$$

(4)⇒(3) $v^TAv = v^TBB^Tv = \sum_{i=1}^{r} v^Tc_ic_i^Tv = \sum_{i=1}^{r}(v^Tc_i)^2 \ge 0$

(3)⇒(1) If $\lambda_j < 0$, then $u_j^TAu_j = \lambda_j u_j^Tu_j = \lambda_j < 0$ where u_j = eig. vector corresp. to λ_j.

(1,3)⇒(2) For $S \subseteq [n]$, $v \in \mathbb{R}^S$, $v^TA_{S,S}v = \tilde{v}^TA\tilde{v}$ with $\tilde{v} \in \mathbb{R}^n$, $\tilde{v}_i = 0$ for $i \notin S$ ⇒ $A_{S,S}$ is PSD

$$\Rightarrow \det(A_{S,S}) = \prod \lambda_j(A_{S,S}) \ge 0$$

(2)⇒(1) Consider

$$f(t) = \det(tI + A) = \sum_{k=0}^{n} t^k \sum_{\substack{S\subseteq[n]\\ |S|=n-k}} \det(A_{S,S})$$

roots are $-\lambda_1,\dots,-\lambda_n$.

coefficents of f are ≥ 0 ⇒ f has no roots in $\mathbb{R}_+$

⇒ A has no eig. val. in $\mathbb{R}_-$

Inner product on $\mathbb{R}^{n\times n}_{sym}$: $\langle A,B\rangle = \text{tr}(AB) = \sum_{1\le i,j\le n} A_{ij}B_{ij}$

Claim: The PSD cone is self-dual under $\langle\cdot,\cdot\rangle$.

That is, $A \succeq 0 \iff \langle A,B\rangle \ge 0$ for all $B \succeq 0$

(Proof) ($\Rightarrow$) For $B = vv^T$, $\langle A,B\rangle = tr(Avv^T) = tr(v^TAv) \geq 0$.

($\Leftarrow$) $B \succeq 0 \Rightarrow B = \sum_{i=1}^{k} v_i v_i^T \Rightarrow \langle A,B\rangle = \sum_{i=1}^{k} v_i^T A v_i \geq 0$.

Note for $A, B \in PSD_n$,

$$\langle A,B\rangle = 0 \iff \text{rowspan}(B) \subseteq \ker(A)$$

$$A = \sum_{i=1}^{d} v_i v_i^T, \quad B = \sum_{j=1}^{e} w_j w_j^T \quad \left(\text{rowsp}(B) = \text{span}_{\mathbb{R}}\{w_1, \ldots, w_e\}\right)$$

$$\langle A,B\rangle = \sum_{j=1}^{e} w_j^T A w_j = \sum_j \sum_i w_j^T v_i v_i^T w_j = \sum_{i,j} (v_i^T w_j)^2$$

$$\langle A,B\rangle = 0 \iff \langle v_i, w_j\rangle = 0 \ \forall i,j$$

Thm: The faces of PSD_n are

$$\mathcal{F}_L = \{A \in PSD_n : L \subseteq \ker(A)\}$$

where L runs over all subspaces of $\mathbb{R}^n$.
Moreover $\mathcal{F}_L$ is linearly isomorphic to $PSD_{n-\dim(L)}$.

(Proof)

Let $B = \sum_{j=1}^{d} w_j w_j^T$ where $\{w_j\}_j$ = basis for L.

$\Rightarrow \mathcal{F}_L = \{A \in PSD_n : \langle A, B\rangle = 0\}$ = maximizers of $X \mapsto \langle X, -B\rangle$ over $A \in PSD_n$

$$\mathcal{F}_L = \left\{ \sum_{i=1}^{n-d} v_i v_i^T : v_i \in L^\perp \right\} = \left\{ U \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix} U^T : \tilde{A} \in PSD_{n-d} \right\}$$

where $U = (v_1 \cdots v_r \; w_1 \cdots w_{n-r})$ orthogonal

Claim: If $L = \ker(A)$, A belongs to $\operatorname{relint}(\mathcal{F}_L)$.

"relint(F)" = relative interior of F

= interior of F in its affine span.

(Proof) $\operatorname{rank}(A) = n \Rightarrow A \in \operatorname{int}(PSD_n) = PD_n$

$\Rightarrow$ only face containing A is PSD_n

$\operatorname{rank}(A) = r \Rightarrow A = U \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix} U^T$ where $\tilde{A} \succ 0$

$\Rightarrow A \in \operatorname{relint}(\mathcal{F}_L)$

For any face F of PSD_n, take $A \in \operatorname{relint}(F)$ and $L = \ker(A) \Rightarrow F = \mathcal{F}_L$

Cor: Every face of PSD_n is exposed.

Cor: PSD_n only has faces of dim $\binom{r+1}{2}$ for $r = 0, 1, \ldots, n$.