Math 582G - Convex Algebraic Geometry

Today: Convex Duality

$V =$ finite dim'l $\mathbb{R}$-vec. space, $V^* = \{ \text{linear } \ell : V \to \mathbb{R} \}$

Def: Given a convex set $C \subseteq V$, its <u>dual</u> is

$$C^* = \{ \ell \in V^* : \ell(x) + 1 \geq 0 \ \forall x \in C \}$$

$(= -C^\circ$, where $C^\circ = \{ \ell \in V^* : \ell(x) \leq 1 \ \forall x \in C \}$ "polar of C")

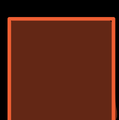
Note: C^* is convex!

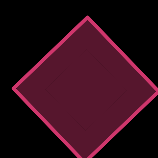
Take $\ell_1, \ell_2 \in C^*$, $\lambda \in [0,1]$, $\ell = \lambda \ell_1 + (1-\lambda) \ell_2$

For $x \in C$,

$$\begin{aligned} \ell(x) + 1 &= \lambda \ell_1(x) + (1-\lambda) \ell_2(x) + 1 \\ &= \underbrace{\lambda}_{\geq 0} \underbrace{(\ell_1(x) + 1)}_{\geq 0} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{(\ell_2(x) + 1)}_{\geq 0} \geq 0 \end{aligned}$$

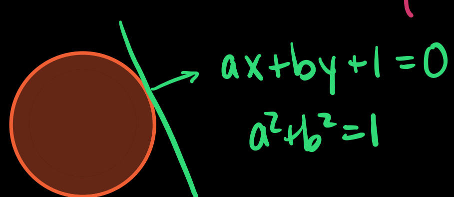
Ex:





$C = [-1,1]^2$ $\quad$ $C^* = \{ (a,b) \in \mathbb{R}^2 : \sigma_1 a + \sigma_2 b + 1 \geq 0 \ \forall \sigma_1, \sigma_2 \in \{\pm 1\} \}$

Ex:



$C = \{ (x,y) \in \mathbb{R} : x^2 + y^2 \leq 1 \}$ $\quad$ $C^* = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \}$

Also C^* is closed in Eucliean topology on $V \cong \mathbb{R}^n$

$$C^* = \bigcap_{x \in C} \{\ell \in V^* : \ell(x) + 1 \geq 0\} = \text{intersection of closed sets} \Rightarrow \text{closed}$$

<u>Biduality Theorem</u>: For any convex set C with $0 \in C$, $(C^*)^* = \overline{C}$ ← Closure in Euclidean topology

Note: we have identified $(V^*)^*$ with V $\left(v \in V \leftrightarrow ev_v \in (V^*)^*\right)$

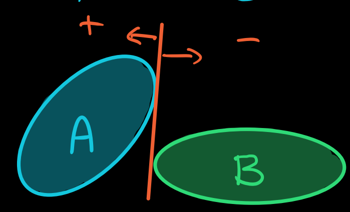
For proof, we need:

<u>Separation Theorems</u>

<u>Thm</u>: If A, B are convex sets and $A \cap B = \emptyset$, there is an affine hyperplane separating A, B.
That is, $\exists\, \ell \in V^*,\ \ell_0 \in \mathbb{R}$ s.t.

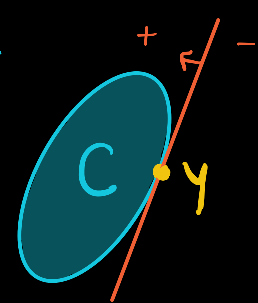
$\ell(x) + \ell_0 \geq 0 \quad \forall\, x \in A$

$\ell(y) + \ell_0 \leq 0 \quad \forall\, y \in B$



<u>Cor 1</u>: If $C \subseteq V$ is a convex set with non-empty interior, for any $y \in \partial C$, there is an affine hyperplane separating y from $\mathrm{int}(C)$.
That is, $\exists\, \ell \in V^*,\ \ell_0 \in \mathbb{R}$ s.t.

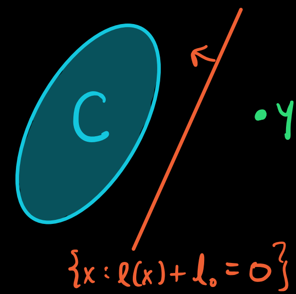
- $\ell(x) + \ell_0 > 0 \quad \forall\, x \in \mathrm{int}(C)$
- $\ell(y) + \ell_0 = 0$.



$H = \{x : \ell(x) + \ell_0 = 0\}$ is a <u>supporting hyperplane of C at y</u>

<u>Cor. 2</u> : If $C \subseteq V$ is a closed convex set and $y \notin C$, then there is an affine hyperplane strictly separating y from C, that is $\exists\, \ell \in V^*,\ \ell_0 \in \mathbb{R}$ s.t.

- $\ell(x) + \ell_0 > 0 \quad \forall x \in C$
- $\ell(y) + \ell_0 < 0$

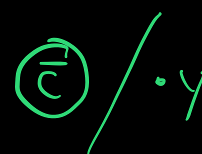


Proof of Biduality Thm: $(C^*)^* = \overline{C}$

($\supseteq$) If $x \in C$, then for every $\ell \in C^*$, $\ell(x)+1 \geq 0$.

$\Rightarrow ev_x(\ell)+1 \geq 0 \quad \Rightarrow \quad ev_x(\cdot) \in (C^*)^* \quad \Rightarrow \quad C \subseteq (C^*)^*$.

Since $(C^*)^*$ is closed, $\overline{C} \subseteq (C^*)^*$.



($\subseteq$) Suppose $y \notin \overline{C}$. By Cor. 2 above, $\exists\, \ell \in V^*,\ \ell_0 \in \mathbb{R}$ s.t. $\ell(x)+\ell_0 > 0 \ \forall x \in C$ and $\ell(y)+\ell_0 < 0$.

Since $0 \in C$, $\ell_0 > 0 \Rightarrow$ for $\tilde{\ell} = \frac{1}{\ell_0}\ell$, $\tilde{\ell}(x)+1 > 0 \ \forall x \in C$

$\Rightarrow \tilde{\ell} \in C^*$

Since $ev_y(\tilde{\ell})+1 = \tilde{\ell}(y)+1 = \frac{1}{\ell_0}(\ell(y)+\ell_0) < 0$, $y \notin (C^*)^*$.

Note: If C is a convex cone, then

$$C^* = \{\ell \in V^* : \ell(x) \geq 0 \quad \forall x \in C\}.$$

C^* is also a convex cone

($\supseteq$) Clear $(\ell(x) \geq 0 \Rightarrow \ell(x)+1 \geq 0)$

($\subseteq$) Suppose $\ell(x) < 0$ for some $x \in C$ (i.e. $\ell \notin$ RHS)

Then for $\lambda \gg 0$, $\lambda \ell(x)+1 = \ell(\lambda x)+1 < 0$.

Since $\lambda x \in C$, $\ell \notin C^*$.

Ex: $(\mathbb{R}^n_{\geq 0})^* = \mathbb{R}^n_{\geq 0}$ (Nonneg. orthant is self-dual)

That is, for $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $x\in\mathbb{R}^n_{\geq 0} \iff \langle a,x\rangle \geq 0$ $\forall a\in\mathbb{R}^n_{\geq 0}$

$$x, a\in\mathbb{R}^n_{\geq 0} \Rightarrow \langle a,x\rangle = \sum_{i=1}^n a_i x_i \geq 0$$

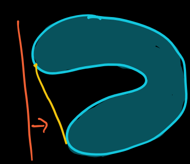
$$x_i < 0 \Rightarrow \langle e_i, x\rangle = x_i < 0$$

Prop: For any subset $S\subseteq V$,

$$\text{Conical Hull}(S) = \left(\{ l\in V^* : l(x)\geq 0 \ \forall x\in S\}\right)^*$$

(Proof) For $l\in V^*$, $l\geq 0$ on S

$\iff l\geq 0$ on conical Hull(S).



Follows from Biduality Theorem with $C=$

Ex: For $n,d\in\mathbb{N}$, let $m_d(x)\in\mathbb{R}[x_1,\dots,x_n]^N$ be the vector of all mon. $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ of degree $\leq \underline{\underline{d}}$

Aside: length of $m_d(x) = \binom{n+d}{d}$

For $p\in\mathbb{R}^n$, $m_d(p)$ represents $ev_p\in(\mathbb{R}[x_1,\dots,x_n]_{\leq d})^*$ in the monomial basis for $\mathbb{R}[x_1,\dots,x_n]_{\leq d}$.

$$f=\sum_\alpha a_\alpha x^\alpha \qquad f(p)=\sum_\alpha a_\alpha p^\alpha = \langle \underline{a}, m_d(p)\rangle$$

For $S \subseteq \mathbb{R}^n$, let

$$P_d(S) = \{ f \in \mathbb{R}[x_1,\dots,x_n]_{\le d} : f(x) \ge 0 \ \forall x \in S \}$$

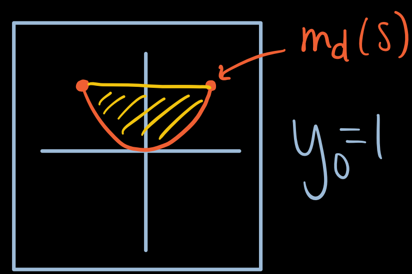
Cor: $P_d(S)^* = \overline{\text{conical Hull}(\underbrace{m_d(S)}_{\{m_d(p)\,:\, p \in S\}})}$

Write $f \in \mathbb{R}[x_1,\dots,x_n]_{\le d}$ as $\ell(m_d(x))$ for some $\ell \in (\mathbb{R}^N)^*$

$$\begin{aligned} f(x) \ge 0 \ \forall x \in S &\iff \ell(m_d(x)) \ge 0 \ \forall x \in S \\ &\iff \ell(y) \ge 0 \ \forall y \in m_d(S) \end{aligned}$$

Ex: $n = 1$, $d = 2$, $S = [-1,1]$

$m_2(t) = (1, t, t^2)$



$$P_2([-1,1]) = \{ f(t) = at^2 + bt + c : f \ge 0 \text{ on } [-1,1] \} = \boxed{?}$$