Math 582G - Convex Algebraic Geometry

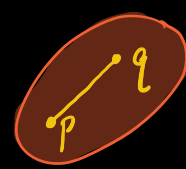
Today: Convexity Basics

Next week: hybrid
Zoom & PDL C-401

V = finite dim'l $\mathbb{R}$-vectorspace $\quad (\cong \mathbb{R}^n$ for $n = \dim(V))$

$V^* = \{$linear functions $\ell : V \to \mathbb{R}\}$ $\quad (\cong V \cong \mathbb{R}^n)$

A set $S \subseteq V$ is <u>convex</u> if for all $p, q \in S$, and $\lambda \in [0,1]$, $\lambda p + (1-\lambda) q \in S$.



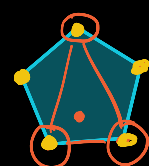
$\Rightarrow$ An aribtrary intersection of convex sets is convex.

The <u>convex hull</u> of $S \subseteq V$ is the intersection of all convex sets containing S:

$$\mathrm{conv}(S) = \bigcap_{\substack{C \text{ convex} \\ S \subseteq C \subseteq V}} C = \left\{ \sum_{i=1}^{r} \lambda_i p_i : \begin{array}{l} r \in \mathbb{N},\ p_1, \ldots, p_r \in S, \\ \lambda_1 \geq 0, \ldots, \lambda_r \geq 0,\ \sum_{i=1}^{r} \lambda_i = 1 \end{array} \right\}$$

check

S finite $\Rightarrow$ $\mathrm{conv}(S)$ is a <u>polytope</u>



<u>Carathéodory's Theorem</u>: Any point in $\mathrm{conv}(S)$ belongs to the convex hull of $\dim(V)+1$ points of S.

<u>Cor</u>: The convex hull of a compact set is compact.

$\dim(V) = n$

(Proof) Let $\Delta_n = \{(\lambda_1,\dots,\lambda_{n+1}) \in \mathbb{R}^{n+1} : \lambda_1 \geq 0,\dots,\lambda_{n+1} \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1\}$.

Suppose $S \subseteq V$ is compact.

$\Longrightarrow S^{n+1} \times \Delta_n$ is compact in $V^{n+1} \times \mathbb{R}^{n+1}$.

By Carathéodory's Thm, $\mathrm{conv}(S) = \varphi(S^{n+1} \times \Delta_n)$

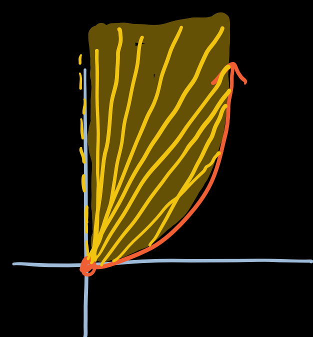
where $\varphi(p_1,\dots,p_{n+1},\underline{\lambda}) = \sum_{i=1}^{n+1} \lambda_i p_i$.

continuous map $\Rightarrow$ $\mathrm{conv}(S)$ is compact.

(Similar argument + Tarski-Seidenberg shows that S semialg. $\Rightarrow$ $\mathrm{conv}(S)$ semialg.)

Need compactness here!

Ex: $S = \{(t, t^2) : t \in \mathbb{R}_{\geq 0}\}$



$\mathrm{conv}(S) = S \cup \{(x,y) \in \mathbb{R}^2_{>0} : y > x^2\}$

Application: quadrature rules

<u>Claim</u>: For any $d \in \mathbb{N}$, $\exists\, r_1,\dots,r_{d+1} \in [0,1]$, $\omega_1,\dots,\omega_{d+1} \in \mathbb{R}_{\geq 0}$

with $\int_0^1 f(t)\,dt = \sum_{i=1}^{d+2} \omega_i f(r_i)$ for all $f \in \mathbb{R}[t]$ with $\deg(f) \leq d$

Ex: $(d=2)$ $f(t) = at^2 + bt + c$

$$\int_0^1 f(t)\,dt = \frac{a}{3} + \frac{b}{2} + c = \frac{1}{4} f(0) + \frac{3}{4} f\left(\frac{2}{3}\right)$$

(Proof): Take $V = \mathbb{R}[t]_{\le d}$, $V^* = \left(\mathbb{R}[t]_{\le d}\right)^*$ (both dim $d+1$)

For $p \in [0,1]$ let $ev_p \in V^*$ with $ev_p(f) = f(p)$.
$\Rightarrow$ $S = \{ ev_p : p \in [0,1] \} \subseteq V^*$ is compact
$\Rightarrow$ $\operatorname{conv}(S)$ is compact

Take $\ell \in V^*$ given by $\ell(f) = \int_0^1 f(t)\,dt$

using compactness of $\operatorname{conv}(S)$

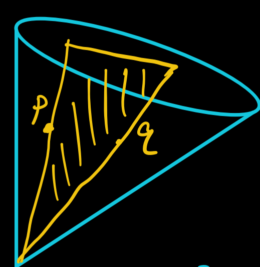
Riemann sums $\quad \ell = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} ev_{\frac{k}{N}} \Rightarrow \ell \in \operatorname{conv}(S)$

Carathéodory $\Rightarrow \ell = \sum_{i=1}^{d+2} \omega_i \, ev_{r_i}$ for some $r_i \in [0,1]$.

Cones and conical hulls

A convex cone is a convex set C with $\mathbb{R}_{\ge 0} C \subseteq C$.
i.e. $p, q \in C \Rightarrow \lambda p + \mu q \in C$ for all $\lambda, \mu \in \mathbb{R}_{\ge 0}$

The conical hull of $S \subseteq V$ is the intersection of all convex cones containg S:



$$\operatorname{ConicalHull}(S) = \left\{ \sum_{i=1}^{r} \lambda_i p_i \;:\; \begin{array}{l} r \in \mathbb{N},\ p_1, \dots, p_r \in S \\ \lambda_1 \ge 0, \dots, \lambda_r \ge 0 \end{array} \right\}$$

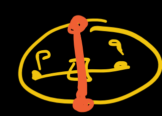
Conic Carathéodory: Any point in $\operatorname{ConicalHull}(S)$ belongs to the conical hull of $\dim(V)$ points of S.

Faces of convex sets

A convex subset F of a convex set C is called a <u>face</u> of C if

$$\lambda p + (1-\lambda) q \in F \quad \text{with } p, q \in C, \ \lambda \in (0,1)$$

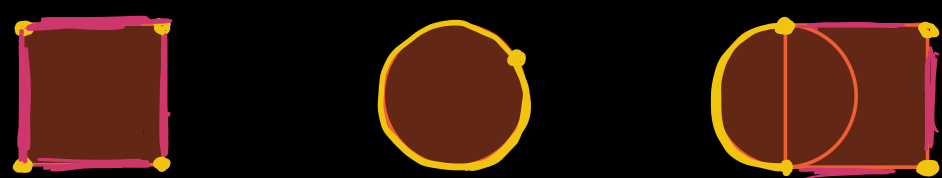
implies that $p, q \in F$.



Trivial faces of C: $\emptyset$, C

Call $p \in C$ an <u>extreme point</u> if $\{p\}$ is a face.

Ex's:



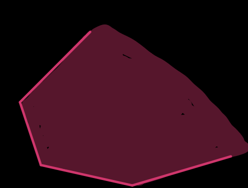
4 extreme pts
4 1-dim'l faces

any bdy pt
is an extr. pt

ext pts
3 1-dim'l faces

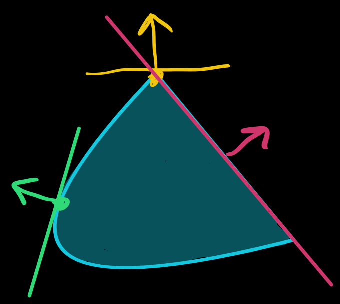
<u>Krein-Milman Theorem</u>: A compact, convex set is the convex hull of its extreme pts.

can fail w/o compactness!



A subset F of a convex set C is called an <u>exposed face</u> of C if for some $\ell \in V^*$

$$F = \{ p \in C : \ell(p) \geq \ell(q) \ \forall q \in C \}.$$



↳ points of C maximized by ℓ

Say that "ℓ exposes F."

Prop: An exposed face of a convex set is a face.

(Proof) Suppose $\ell \in V^*$ exposes $F \subseteq C$ and $\lambda p + (1-\lambda) q \in F$ for some $p, q \in C$, $\lambda \in (0,1)$.

Let m = max value of ℓ on C.

$$m = \ell(\lambda p + (1-\lambda) q) = \lambda \ell(p) + (1-\lambda)\ell(q)$$
$$\leq \lambda m + (1-\lambda) m = m$$

Equality $\Rightarrow$ $\ell(p) = m$ and $\ell(q) = m$

$\Rightarrow$ $p \in F$ and $q \in F$.

Note: Convex sets can have non-exposed faces!

e.g. $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup ([0,2] \times [-1,1])$

point $(0,1)$ is not exposed by any linear function!