Math 582G

Last day: Some open problems

Please fill out course evals.

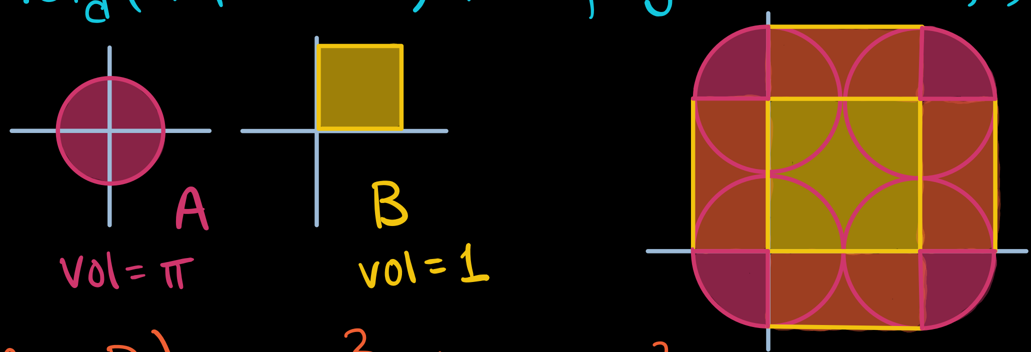
Mixed Volume Polynomials

Let $K_1, \ldots, K_n \subseteq \mathbb{R}^d$ be convex, compact sets.

For $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$, $x_1K_1 + \ldots + x_nK_n = \{ \sum_{i=1}^n x_i k_i : k_i \in K_i \} \subseteq \mathbb{R}^d$ compact

Thm: $\mathrm{Vol}_d(x_1K_1 + \ldots + x_nK_n)$ is a polynomial in $x_1, \ldots, x_n$.

Ex:



A+2B

vol = $\pi + 4 \times 2 + 2^2$

$= \pi + 12$

$$\mathrm{Vol}_2(xA + yB) = \pi x^2 + 4xy + y^2$$

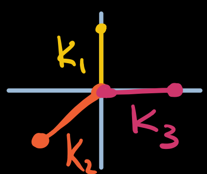
Generalized Brunn-Minkowski Thm:

$\mathrm{Vol}_d(x_1K_1 + \ldots + x_nK_n)$ is completely log-concave.

Ex: $K_i = \mathrm{conv}\{\bar{0}, v_i\}$ for some $v_i \in \mathbb{R}^d$

$$\Rightarrow \mathrm{Vol}_d\left(\sum x_i K_i\right) = \sum_{S \in \binom{[n]}{d}} \left| \det(v_i : i \in S) \right| \prod_{i \in S} x_i$$

e.g.



$\rightarrow x_1x_2 + x_1x_3 + x_2x_3$

Question: What CLC polynomials come this way?

Non-ex: $f = e_2(x_1, x_2, x_3, x_4) = \sum_{\{i,j\} \subseteq [4]} x_i x_j$

If $f = \mathrm{Vol}_2\left(\sum_{i=1}^{4} x_i K_i\right)$ for some convex $K_1, \ldots, K_4 \subseteq \mathbb{R}^2$

then $\dim(K_i) = \deg_{x_i}(f) = 1$ for all i.

Up to translation, $K_i = \mathrm{conv}\{\bar{0}, v_i\}$ for some $v_i \in \mathbb{R}^2$

Check: $\nexists\, v_1, \ldots, v_4 \in \mathbb{R}^2$ s.t. $|\det(v_i v_j)| = 1$ for every $i \neq j$.

Conj (Gurvits) If $f \in \mathbb{R}[x_1, x_2, x_3]_d$ is CLC, then

$\exists$ convex K_1, K_2, K_3 s.t. $f = \mathrm{Vol}_d(x_1 K_1 + x_2 K_2 + x_3 K_3)$.

(Analogue of Helton-Vinnikov Thm for det. reps of stable poly.)

Fractional log-concavity

The Kaufmann-Oppenheim thm on random walks holds more generally. Any bound on eig. val. of walks on links of Δ give bounds on eig. vals walk on $\Delta(d)$.

Corresponding polynomials: "fractionally log-concave"

$f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]_d$ s.t. $\log(f(x_1^\alpha, \ldots, x_n^\alpha))$ is concave on $\mathbb{R}^n_{\geq 0}$.

Ex: Given graph $G = (V, E)$, Alimohammadi, Anari, Shiragur, Vuong show that $f = \sum_{M \text{ matching of } G} \prod_{v \in M} x_v \prod_{v \notin M} y_v \in \mathbb{R}[x_v, y_v : v \in V]$

is $\frac{1}{2}$-log-concave and use this to sample matchings

(monomer-diamer systems in statistical physics)

For $\lambda=1$, $\mathrm{supp}(f)$ must be the bases of a matroid.

Question: What are the possible supports of $\frac{1}{k}$-log-concave polynomials for $k>1$? $k=2$?

Log-concavity w.r.t. other cones

Fix a convex cone $K \subseteq \mathbb{R}^n$. Say f is CLC w.r.t. K if for all $v_1,\ldots,v_r \in K$, $D_{v_1}\cdots D_{v_r} f$ is log-concave on K.

Ex: $f \in \mathbb{R}[x_1,\ldots,x_n]_d$ hyperbolic w.r.t. e, $K = C(f,e)$

Question: Given a cone K, what is $\{f : f \text{ CLC w.r.t } K\}$?

Can we test membership? $K = PSD_n$?

Applications to random walk on infinite sets? $Gr(d,n)$?

Analogue of Borcea-Brändén theory:

Question: What linear operations preserve CLC-ness?

Ex: Multiplication! f, g CLC $\Rightarrow$ $f \cdot g$ CLC

$f \in \mathbb{R}[\underline{x}]$, $g \in \mathbb{R}[\underline{y}]$ $\Rightarrow$ $D_{(v,w)}(f\cdot g) = D_v f(\underline{x})\, g(\underline{y}) + f(x)\cdot D_w g(y)$

$D_{(0,w)}(D_v f(x) g(y)) = D_v f \cdot D_w g = D_{(v,0)}(f(x) D_w g)$

$\Longrightarrow$ sum is CLC (by induction)

Cor: If $T : \mathbb{R}[\underline{x}]_{\leq(1,\ldots,1)} \to \mathbb{R}[\underline{x}]$ is linear and $\mathrm{Symb}(T) = T\left(\prod_{i=1}^{n}(x_i+y_i)\right)$ is CLC then, T preserves CLC

Converse not true!