Math 582G

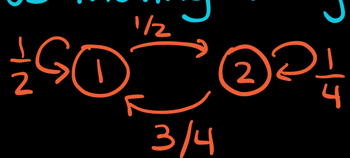
Today: Connections w/random walks

Hwk 4 due today

Please fill out course evaluations!

Markov chains - crash course!

A matrix $P \in \mathbb{R}^{n\times n}_{\geq 0}$ s.t. $P\mathbb{1} = \mathbb{1}$ defines a Markov chain on $[n]$ with prob moving $i \to j$ is P_{ij}.

Ex: $P = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}$ 

If the Markov chain is "reversible and irreducible",

1) there is a unique stationary distribution

$$\pi \in \mathbb{R}^n_{\geq 0} \text{ s.t. } \sum_{i=1}^n \pi_i = 1 \text{ and } \pi P = P$$

2) $e_i P^k \to \pi$ as $k \to \infty$ for any $i \in [n]$

3) $\| e_i P^k - \pi \|_1$ can be bounded as a function of

$$\lambda^*(P) = \max\{|\lambda_2|, |\lambda_n|\}$$

where $\lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = 1$ are eig. val of P.

(Smaller $\lambda^*(P) \to$ smaller $\| e_i P^k - \pi \|$)

In ex: $\pi = (\frac{3}{5}, \frac{2}{5})$, eigval of $P = 1, -\frac{1}{4} \Rightarrow \lambda^*(P) = \frac{1}{4}$

$P^k \to \begin{pmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{pmatrix}$ as $k \to \infty$

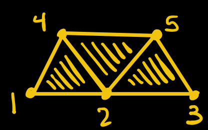
Simplicial Complexes - crash course!

A simplicial complex on $[n]$ is a nonempty collection Δ of subsets of $[n]$ closed under inclusion $\left(\begin{smallmatrix} S\in\Delta \\ T\subseteq S\end{smallmatrix} \Rightarrow T\in\Delta\right)$

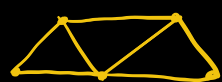
Let $\Delta(k) = \{S\in\Delta : |S| = k\}$ ("faces of dim $k-1$")

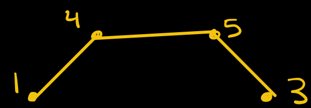
"1-skeleton" of Δ = graph on $\Delta(1)$ with edges $\Delta(2)$

For $S\in\Delta$, $\mathrm{link}_\Delta(S) = \{T\subseteq[n]\backslash S : S\cup T\in\Delta\}$.

Ex: 

$\Delta(3) = \{124, 245, 235\}$

1-skeleton: 

$\mathrm{link}_\Delta(\{2\})$: 

Ex: $\Delta = \{S\subseteq[n] : |S|\le d\}$ $\Delta(k) = \binom{[n]}{k}$ for $k\le d$

Ex: The independent sets of any matroid

e.g. {cycle-free subgraphs of a graph G}

Let $f = \sum_{S\in\binom{[n]}{d}} c_S \prod_{i\in S} x_i \in \mathbb{R}[x_1,\dots,x_n]$ with $c_S \ge 0$

Associate simplicial complex $\Delta = \{T\subseteq[n] : S\subseteq T$ for some S with $c_S\neq 0\}$

with weights $\omega(T) = \sum_{S\supseteq T} c_S = \left(\prod_{i\in T}\partial_i f\right)(\mathbb{1})$

Ex: $f = e_d(x_1,\dots,x_n) = \sum_{S\in\binom{[n]}{d}} \prod_{i\in S} x_i \rightarrow \Delta = \{S\subseteq[n] : |S|\le d\}$

If $T\subseteq[n]$, $|T| = k\le d$, $\omega(T) = \binom{n-k}{d-k}$.

Random walk on $\Delta(k)$:

From $S \in \Delta(k)$, remove $i \in S$ uniformly at random $\to T = S \setminus \{i\} \in \Delta(k-1)$

From $T \in \Delta(k-1)$, add $j \in [n] \setminus T$ with prob. proportional to $w(T \cup \{j\})$

$\to S = T \cup \{j\} \in \Delta(k)$

Let $P_k^{\vee}$ denote the $|\Delta(k)| \times |\Delta(k)|$ trans. matrix for this walk

and $P_{k-1}^{\wedge}$ denote the $|\Delta(k-1)| \times |\Delta(k-1)|$ trans. matrix for walk on $\Delta(k-1)$

given by switching the order

Ex: $f = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \quad \Delta = \{S \subseteq [4] : |S| \leq 2\}$

For $P_1^{\wedge}$, start at $i \in [4]$, add $j \in [4] \setminus \{i\}$ unif. at random

remove $k \in \{i,j\}$ uniformly at random

$P_1^{\wedge}(i,j) = \frac{1}{2}$ for $i=j$ and $\frac{1}{6}$ for $i \neq j$.

$$P_1^{\wedge} = \begin{pmatrix} 1/2 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/2 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/2 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/2 \end{pmatrix} = \frac{1}{2} I + \frac{1}{6} \nabla^2 f = \frac{1}{6} A A^T$$

where $A \in \{0,1\}^{4 \times 6}$ = adj matrix of $\binom{[4]}{1} \times \binom{[4]}{2}$

f log-concave $\Rightarrow \lambda_2(\nabla^2 f) \leq 0 \Rightarrow \lambda_2(P_1^{\wedge}) \leq \frac{1}{2}$.

$P_1^{\wedge} = \frac{1}{6} A A^T \Rightarrow P_1^{\wedge} \succeq 0 \Rightarrow \lambda^*(P_1^{\wedge}) \leq \frac{1}{2}$

For $S, T \in \binom{[4]}{2}$, $P_2^{\wedge}(S,T) = \begin{cases} 0 & \text{if } S \cap T = \emptyset \\ 1/6 & \text{if } |S \cap T| = 1 \\ 1/3 & \text{if } S = T \end{cases}$

$P_2^{\wedge} = \frac{1}{6} A^T A$ has same nonzero eig. val. as $P_1^{\vee}$.

$\Rightarrow \lambda^*(P_2^{\wedge}) \leq 1/2$.

Thm (Kauffman, Oppenheim '18) If

1) the 1-dim'l skeleton of $\text{link}_{\Delta}(S)$ is connected for every $S \in \Delta$ with $|S| \leq d-2$

2) for all $S \in \Delta(d-2)$, the transition matrix for the walk induced on $\mathrm{link}_\Delta(S)$ has $\lambda_2 \leq \frac{1}{2}$

then the transition matrix for $P_d^\vee$ has $\lambda_2 \leq 1 - \frac{1}{d}$.

(Idea of proof): $P_k^\vee$ and $P_{k-1}^\wedge$ have same nonzero eigenvalues. Use connectivity of links to bound eig. val. of $P_k^\wedge$ in terms of eig. val. of $P_k^\vee$.

Translation to polynomials:

$\mathrm{link}_\Delta(S) = \{T \subseteq [n] \setminus S : T \cup S \in \Delta\}$ = simpl. complex assoc. to $\partial^S f$

One dim'l skeleton of $\mathrm{link}_\Delta(S)$ connected $\Leftrightarrow$ $\partial^S f$ indecomposable

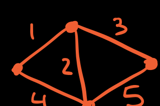
For $S \in \Delta(d-2)$, trans. matrix for walk on $\mathrm{link}_\Delta(S)$ given by

$$\frac{1}{2}\mathrm{Id} + \frac{1}{2} D^{-1} \cdot \nabla^2 \partial^S f \quad \text{where } D = \mathrm{diag}((\nabla^2 \partial^S f)\mathbb{1})$$

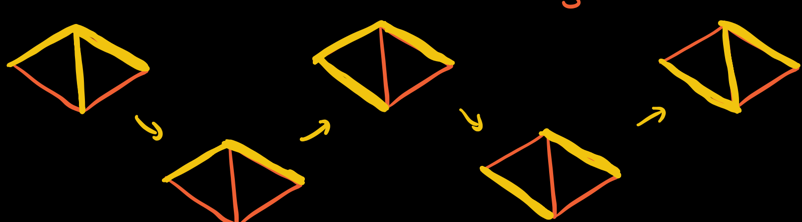
has $\lambda_2 \leq \frac{1}{2} \Leftrightarrow \lambda_2(\nabla^2 \partial^S f) \leq 0 \Leftrightarrow \partial^S f$ log-concave on $\mathbb{R}^n_+$

Thm: (Δ, ω) satisfies (1) & (2) $\Leftrightarrow$ f is complety log-concave

Cor: For any matroid of rank d, $P_d^\vee$ defines a random walk on the bases $\mathcal{B}$ converging to the uniform distribution on $\mathcal{B}$, with $\lambda^*(P) \leq 1 - \frac{1}{d}$.

Ex: Graph matroid of 

8×8 transition matrix $P_3^\vee$ has $\lambda^*(P) \leq 1 - \frac{1}{3} = \frac{2}{3}$.



] random walk on spanning trees mixes quickly