Math 582G

Today: Log-concavity in matroids

Hwk 4 due Wed. 3/9

OH: Today 4-5pm, Th 2-3pm or by appt.

Recall: A homog. polynomial $f \in \mathbb{R}[x_1,\dots,x_n]_d$ with nonneg. coeff. is completely log-concave if for all $v_1,\dots,v_k \in \mathbb{R}^n_{\geq 0}$, $D_{v_1}\cdots D_{v_k} f$ is log-concave on $\mathbb{R}^n_{\geq 0}$.

Thm: Let $f \in \mathbb{R}_{\geq 0}[x_1,\dots,x_n]_d$ with nonneg coeff. be homog. of deg $d \geq 3$. Then f is completely log-concave iff

(i) for all $|\alpha| \leq d-3$, $\partial^\alpha f$ is indecomposable and

(ii) for all $|\alpha| = d-2$ $\partial^\alpha f$ is log-concave.

Cor: $f = \sum_{k=0}^{n} c_k x^k y^{n-k}$ is completely log-concave iff and only if $c_0,\dots,c_n$ has no internal zeros and for every k

$$\frac{c_{k-1}}{\binom{n}{k-1}} \frac{c_{k+1}}{\binom{n}{k+1}} \leq \left(\frac{c_k}{\binom{n}{k}}\right)^2.$$

(Proof) If $c_{j+1} = \dots = c_{k-1} = 0$ for some $j \leq k-2$,

$$\partial_x^j \partial_y^{n-k} = j! \frac{(n-k)!}{(k-j)!} c_j y^{k-j} + (n-k)! \frac{k!}{j!} x^{k-j}$$

has deg $k-j \geq 2$ and is decomposable.

$$\nabla^2 \partial_x^{k-1} \partial_y^{n-k-1} f = n! \begin{pmatrix} c_{k-1}/\binom{n}{k-1} & c_k/\binom{n}{k} \\ c_k/\binom{n}{k} & c_{k+1}/\binom{n}{k+1} \end{pmatrix}$$

has det ≤ 0.

Recall: A <u>matroid</u> $M = ([n], \mathcal{B})$ on ground set $[n]$ is a nonempty collection $\mathcal{B}$ satifying:

$A, B \in \mathcal{B}$, $a \in A \setminus B \Rightarrow \exists b \in B \setminus A$ s.t. $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$.

The independent sets of M are

$\mathcal{I} = \{ I \subseteq [n] : I \subseteq B \text{ for some } B \in \mathcal{B} \}$.

Ex (linear) $v_1, \ldots, v_n \in \mathbb{F}^d$

$\mathcal{B} = \{ B \subseteq [n] : \{v_i\}_{i \in B} \text{ basis for } \operatorname{span}_{\mathbb{F}} \{v_1, \ldots, v_n\} \}$

$\mathcal{I} = \{ I \subseteq [n] : \{v_i\}_{i \in I} \text{ linearly indep. over } \mathbb{F} \}$

Ex (graphic) G connected graph with edges E

$\mathcal{B} = \{ B \subseteq E : B \text{ is a spanning tree of } G \}$

$\mathcal{I} = \{ I \subseteq E : I \text{ has no cycles} \}$

Thm: For any matroid M with bases $\mathcal{B}$, ind. sets $\mathcal{I}$, the polynomials

$$f_{\mathcal{B}} = \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i \quad \text{and} \quad g_{\mathcal{I}} = \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{i \in I} x_i$$

are completely log-concave.

(Sketch of proof for f_B) For any independent set $S \subseteq [n]$,

$M/S = ([n]\setminus S, \{B\setminus S : B \in \mathcal{B}, S \subseteq B\})$ is a matroid (known as the contraction of M by S)

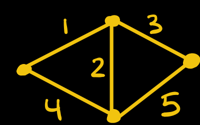
Note: $f_{\mathcal{B}(M/S)} = \prod_{i\in S} \partial_i \cdot f_{\mathcal{B}(M)}$. If $S \notin \mathcal{I}$, $\prod_{i\in S}\partial_i f_{\mathcal{B}} = 0$.

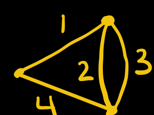
Exchange axiom $\Rightarrow f_{\mathcal{B}(M/S)}$ indecomposable for $|S| \le \text{rank}(M)-2$

For $|S| = \text{rank}(M)-2$, M/S is a matroid of rank 2.

Elements $i,j \in [n]$ are either independent in M/S or "parallel". Parallel elts. form equivalence classes!

$$\Rightarrow \nabla^2 \prod_{i\in S}\partial_i f_{\mathcal{B}} = \begin{pmatrix} 0 & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & 0 & \mathbb{1} \\ \mathbb{1} & \mathbb{1} & 0 \end{pmatrix} = (\mathbb{1}) - \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix}$$ has 1 pos. eig. value!

Ex: $M = ([5], \mathcal{B})$ = graphic matroid of 

$M/5$ = graphic matroid of 

$$\nabla^2\left(\frac{\partial}{\partial x_5} f_{\mathcal{B}}\right) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Cor: For any matroid $M = ([n], \mathcal{B})$ and any $i,j \in [n]$,

$$2\left(1-\frac{1}{d}\right)\text{Prob}(i\in B)\,\text{Prob}(j\in B) \ge \text{Prob}(i,j\in B)$$

Huh, Schröter, Wang show "2" can be removed under mild conditions on i,j

(Proof) Let $f_{\mathcal{B}} = \sum_{B\in\mathcal{B}} \prod_{i\in B} x_i$.

For $S \subseteq [n]$, let $b_S = |\{B \in \mathcal{B} : S \subseteq B\}| = \left(\prod_{i\in S}\partial_i f\right)\Big|_{x=\mathbb{1}}$.

Let $Q = \nabla^2 f(\mathbb{1})$ and consider

$$\begin{bmatrix} \mathbb{1}^T \\ e_i^T \\ e_j^T \end{bmatrix} Q \begin{bmatrix} \mathbb{1} & e_i & e_j \end{bmatrix} = \begin{bmatrix} d(d-1)b & (d-1)b_i & (d-1)b_j \\ (d-1)b_i & 0 & b_{ij} \\ (d-1)b_j & b_{ij} & 0 \end{bmatrix}$$

By log concavity, det $\uparrow$ ≥ 0

$\Rightarrow -d(d-1)b \cdot b_{ij}^2 + 2(d-1)^2 b_i b_j > 0$ divide by $(d-1)b$

$\Rightarrow 2(1-\frac{1}{d})\frac{b_i}{b}\cdot\frac{b_j}{b} > \frac{b_{ij}}{b}$

Cor: For any matroid on n elts with $\mathcal{I}_k = \#\{I \in \mathcal{I} : |I| = k\}$

$$\frac{\mathcal{I}_{k+1}}{\binom{n}{k+1}} \cdot \frac{\mathcal{I}_{k-1}}{\binom{n}{k-1}} \leq \frac{\mathcal{I}_k^2}{\binom{n}{k}^2}.$$

Lemma: If f is CLC, then for any $a_1,\dots,a_m \in \mathbb{R}^n_{\geq 0}$,

$g(y) = f(y_1 a_1 + \dots + y_m a_m)$ is CLC.

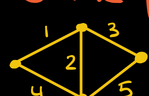
(Idea) For $v = (v_1,\dots,v_m) \in \mathbb{R}^m_{\geq 0}$,

$$D_v g = \sum_{i=1}^m v_i \frac{\partial g}{\partial y_i} = \sum_{i=1}^m v_i D_{a_i} f(\sum_j y_j a_j) = D_{(\sum_i v_i a_i)} f(\sum y_j a_j).$$

(Proof of Cor) By Thm, $g_{\mathcal{I}} = \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{i \in I} x_i$ is CLC.

$\Rightarrow g(x,y) = g_{\mathcal{I}}(x,\dots,x,y) = \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{i \in I} x_i = \sum_{k=0}^n \mathcal{I}_k x^k y^{n-k}$ is CLC

$\Rightarrow$ coeff $\mathcal{I}_k$ satisfy Newton's inequalities

Ex: G = graphic matroid of 

$g_{\mathcal{I}} = y^5 + y^4(x_1 + \dots + x_5) + y^3(x_1x_2 + \dots + x_4x_5) + y^2(x_1x_2x_3 + \dots + x_3x_4x_5)$

$g_{\mathcal{I}}(x,\dots,x,y) = y^5 + 5xy^4 + 10x^2y^3 + 8x^3y^2$

$(\mathcal{I}_k) = (1, 5, 10, 8, 0, 0)$

$\left(\frac{\mathcal{I}_k}{\binom{5}{k}}\right) = (1, 1, 1, \frac{8}{10}, 0, 0)$