Math 582G

Today: Log concave polynomials

Let $f \in \mathbb{R}[x_1,\dots,x_n]_d$ be homog. of deg d with nonneg. coeff

Prop: Let $f \in \mathbb{R}[x_1,\dots,x_n]_d$ be homog. of deg. $d \geq 2$. With $f(a) > 0$. Let $Q = \nabla^2 f(a)$. TFAE

1) f is log-concave at a.
2) $x^T Q x \leq 0$ for all $x \in (Qa)^\perp$
3) $x^T Q x \leq 0$ for all x in some hyperplane H
4) Q has exactly one positive eig. val.

Euler's identity: If g is homog. of degree e, then

$\sum x_i \frac{\partial g}{\partial x_i} = e \cdot g$. (Differentiate $g(\lambda x_1,\dots,\lambda x_n) = \lambda^e g(x)$ w.r.t. λ)

$\Rightarrow Qa = (d-1)\nabla f(a)$ and $a^T Q a = d(d-1) f(a)$

(3)$\Rightarrow$(1) $x^T Q x \leq 0$ for all x in some hyperplane H

Let $b \in \mathbb{R}^n$ and consider

$$A = \begin{bmatrix} - & a^T & - \\ - & b^T & - \end{bmatrix} Q \begin{bmatrix} | & | \\ a & b \\ | & | \end{bmatrix} = \begin{bmatrix} a^T Q a & a^T Q b \\ b^T Q a & b^T Q b \end{bmatrix}$$

$\mathrm{span}_{\mathbb{R}}\{a,b\}$ has dim 2, then it intersects H

$\Rightarrow \exists v \mathbb{R}^2$ s.t. $v^T A v \leq 0 \Rightarrow A$ not pos. def.

$\Rightarrow \det(A) \leq 0$ (Since $A_{11} = a^T Q a > 0$)

$\Rightarrow (a^T Q a)(b^T Q b) - (b^T Q a)(a Q b)$

$= b^T\left((a^T Q a) Q - (Qa)(Qa)^T \right) b \leq 0$ for all b.

$\Rightarrow d(d-1) f(a) \nabla^2 f(a) - (d-1)^2 \nabla f(a) \nabla f(a)^T$

$= d(d-1) f(a)^2 \left[\nabla^2 \log(f(a)) + \frac{1}{d} \nabla f(a) \nabla f(a)^T \right] \preceq 0$

Def: $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ completely log-concave if for every $v_1, \ldots, v_k \in \mathbb{R}^n_{\geq 0}$, $D_{v_1} \cdots D_{v_k} f$ is log-concave.

Ex: f hyperbolic w.r.t. every $v \in \mathbb{R}^n_+$

e.g. $e_k(x_1, \ldots, x_n) = \sum_{S \in \binom{[n]}{k}} x^S$

Prop: $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is completely log-concave
$\Leftrightarrow$ for all $v_1, \ldots, v_{d-2} \in \mathbb{R}^n_{\geq 0}$, $D_{v_1} \cdots D_{v_{d-2}} f$ is log-concave

(Proof) For $a \in \mathbb{R}^n_{\geq 0}$, $D_a^{d-2} f$ log-concave

$\Rightarrow \nabla^2 D_a^{d-2} f = (d-2)! \nabla^2 f(a)$ has only one pos. eig. val.

$\Rightarrow f$ log-concave at $x = a$. (Similar for $D_{v_1} \cdots D_{v_k} f$).

Def: Call f <u>decomposable</u> if $f = g + h$ where g, h are nonzero polynomials in disjoint sets of variables: $g \in \mathbb{R}[x_i : i \in I]$, $h \in \mathbb{R}[x_j : j \notin I]$

Otherwise call f <u>indecomposable</u>.

Ex: $x_1x_2 + x_3x_4$ decomposable

$x_1x_2 + x_2x_3 + x_3x_4$ indecomposable

Remark: If $f \in \mathbb{R}[\underline{x}]_d$ with $d \geq 2$ is decomposable, then it is not log-concave.

$$\nabla^2 f(a) = \left[\begin{array}{c|c} \nabla^2 g(a) & 0 \\ \hline 0 & \nabla^2 h(a) \end{array}\right]$$

$\nabla^2 g(a)$, $\nabla^2 h(a)$ nonneg. entries

$\hookrightarrow$ at least one pos. eig. val each.

Thm: Let $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]_d$ with nonneg coeff. be homog. of deg $d \geq 3$. Then f is completely log-concave iff

(i) for all $|\alpha| \leq d-3$, $\partial^\alpha f$ is indecomposable and

(ii) for all $|\alpha| = d-2$ $\partial^\alpha f$ is log-concave.

Lemma 1: If $g, h \in \mathbb{R}_{\geq 0}[\underline{x}]_d$ are log-concave and $D_b f = D_c g \neq 0$ for some $b, c \in \mathbb{R}^n_{\geq 0}$, then $f + g$ is log-concave.

Idea: For $a \in \mathbb{R}^n_+$, for all $x \in \nabla f(b)^\perp = \nabla g(c)^\perp$

$x^T(\nabla^2 f(a))x$ and $x^T(\nabla^2 g(a))x \leq 0$

$\Rightarrow x^T(\nabla^2(f+g)(a))x \leq 0 \Rightarrow f+g$ log-concave at $x=a$.

Lemma 2: If $f \in \mathbb{R}_{\geq 0}[x_1,\dots,x_n]_d$ with $d \geq 3$ is indecomposable and $\partial_1 f, \dots, \partial_n f$ are log-concave then $D_a f$ is log-concave for all $a \in \mathbb{R}^n_+$.

(Proof) Graph on $[n]$ with edges $\{\{i,j\} : \partial_i \partial_j f \neq 0\}$ is connected $\Rightarrow$ reorder variables s.t. for all $2 \leq j \leq n$, $\exists\, i < j$ s.t. $\partial_i \partial_j f \neq 0$.

Let $b = (a_1, \dots, a_{j-1}, 0, \dots, 0)$ and $c = (0, \dots, 0, a_j, 0, \dots, 0)$.

Then
$$D_b(D_c f) = D_c(D_b f) = \sum_{i=1}^{j} a_i a_j \partial_i \partial_j f \neq 0$$

$\Rightarrow D_b f + D_c f$ log-concave.

Inducting on j gives $D_a f$ is log-concave.

Proof of Thm then follows by induction.