Math 582G

Today: Matroids & polynomials

Matroids: A <u>matroid</u> $M=([n],\mathcal{B})$ on ground set $[n]$ is a nonempty collection $\mathcal{B}$ satisfying:

$$A,B\in\mathcal{B},\ a\in A\setminus B \Rightarrow \exists b\in B\setminus A \text{ s.t. } A\setminus\{a\}\cup\{b\}\in\mathcal{B}.$$

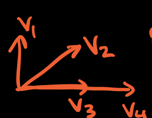
All $B\in\mathcal{B}$ must have the same size, called the rank of M.

Thm (Gelfand, Goresky, MacPherson, Serganova)

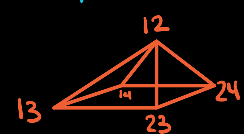
$\mathcal{B}\subseteq 2^{[n]}$ are the bases of a matroid $\iff$ all edges of $\mathrm{Conv}(\{\mathbb{1}_B : B\in\mathcal{B}\})$ are parallel to e_i-e_j for some i,j.

Ex (linear): Given vectors $v_1,\dots,v_n$ in a vectorspace $\mathbb{F}^d$,

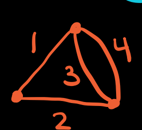
$\mathcal{B}=\{B\subseteq[n] \text{ s.t. } \{v_i : i\in B\} \text{ form a basis for } \mathrm{span}\{v_1,\dots,v_n\}\}$

e.g. 

$\mathbb{R}^2$ $\mathcal{B}=\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$

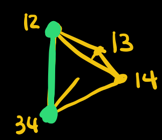


Ex: (Graphic) If $G=([n],E)$ is a connected graph, then $(E,\mathcal{T})$ is a matroid where $\mathcal{T}$ is the set of spanning trees of G.

e.g. 

$\mathcal{B}=\{\{1,2\}\{1,3\}\{1,4\},\{2,3\}\{2,4\}\}$.

Non-ex: $\mathcal{B} = \{\{1,2\},\{1,3\},\{1,4\},\{3,4\}\} \rightsquigarrow$

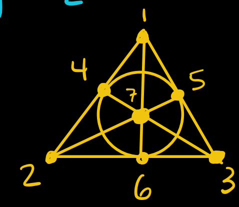


$\{1,2\},\{3,4\} \in \mathcal{B}$, $1 \in \{1,2\}$

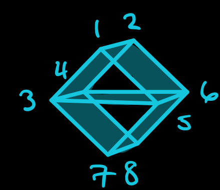
for any $b \in \{3,4\}$ $\quad \{1,2\}\setminus\{1\}\cup\{b\} = \{2,b\} \notin \mathcal{B}$

Ex (Fano Plane) Linear matroid over $\mathbb{F}_2$ rep by $\mathbb{F}_2^3\setminus\{0\}$

$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad \mathcal{B} = \binom{[7]}{3} \setminus \{\{1,2,4\},\ldots,\{4,5,6\}\}$$



Ex (Vámos) $\mathcal{B} = \binom{[8]}{4} \setminus \left\{ \begin{array}{l} \{1,2,3,4\},\{1,2,5,6\},\{3,4,5,6\} \\ \{3,4,7,8\},\{5,6,7,8\} \end{array} \right\}$



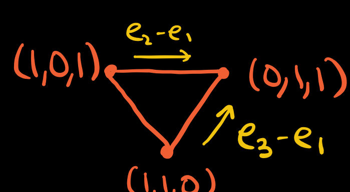
Not a linear matroid over any field!

Matroids and stable polynomials

Thm (COSW) If $f = \sum_{S \in \binom{[n]}{d}} c_S \prod_{i \in S} x_i \in \mathbb{R}[x_1,\ldots,x_n]$ is homog. and multiaffine then its support $\{S : c_S \neq 0\}$ are the bases of a matroid.

Equivalently: every edge of its Newton polytope is parallel to $e_i - e_j$ for some $i,j \in [n]$.

Ex: $x_1x_2 + x_1x_3 + x_2x_3$



($\overset{\Longrightarrow}{\text{Last time}}$ events $i \in S$, $j \in S$ negatively correlated when $\mathrm{Prob}(S) = c_S / f(\mathbb{1})$.)

Thm: (Wagner-Wei, Brändén) The basis generating polynomial of the Vámos matroid,

$$f = \sum_{B \in \mathcal{B}(V_8)} \prod_{i \in B} x_i \in \mathbb{R}[x_1, \dots, x_8]_4$$

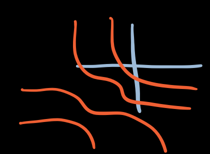
is stable but $f^k \neq \det\left(\sum_{j=1}^{8} x_j A_j\right)$ with $A_j \succeq 0$ for any k.

Thm (Brändén) The bases of the Fano matroid are not the support of any stable polynomial.

Matroids and log-concave polynomials

Def: A polynomial $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ is log-concave on $\mathbb{R}_+^n$ if $f \equiv 0$ or $\log(f) : \mathbb{R}_+^n \to \mathbb{R}$ is concave.

Ex: $f \in \mathbb{R}[x_1, \dots, x_n]$ stable with coeff in $\mathbb{R}_{\geq 0}$



Thm: If $f = \sum_{S \in \binom{[n]}{d}} c_S \prod_{i \in S} x_i \in \mathbb{R}[x_1, \dots, x_n]$ is homog., multiaffine, and log-concave, then its support $\{c_S : S \neq 0\}$ are the bases of a matroid.

Moreover, for any matroid with bases $\mathcal{B}$, the polynomial $f = \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i$ is log-concave.

Remark 1 $\{f \in \mathbb{R}_{\geq 0}[x_1,\ldots,x_n]_{\leq d} : f \text{ is log-concave on } \mathbb{R}_+\}$ is closed is Euclidean topology on $\mathbb{R}[x_1,\ldots,x_n]_{\leq d}$

Why? $\partial_i \partial_j \log(f(x)) = \partial_i \frac{\partial_j f}{f} = \frac{f \cdot \partial_i \partial_j f - \partial_i f \, \partial_j f}{f^2}$

$\Rightarrow \nabla^2 \log(f) = \frac{f \cdot \nabla^2 f - \nabla f \, \nabla f^T}{f^2}$

For any $a \in \mathbb{R}_+^n$,

$\{f \in \mathbb{R}[x_1,\ldots,x_n]_{\leq d} : f(a) \nabla^2 f(a) - \nabla f(a) \nabla f(a)^T \preceq 0\}$ is closed.

$\Rightarrow$ intersection over all $a \in \mathbb{R}_+^n$ closed

Remark 2: f log-concave $\Rightarrow$ $c \cdot f(\lambda_1 x_1, \ldots, \lambda_n x_n)$ log-concave for all $c, \lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}$.

Remark 3: If f is log-concave at $a \in \mathbb{R}_+^n$, then $\nabla^2 f|_{x=a}$ has at most one positive eig. val.

Let $V = \text{span}\{v_1,\ldots,v_k\}$ eigenvectors assoc. to pos. eigenvalues of $\nabla^2 f|_{x=a}$. If $\dim(V) \geq 2$, then $\exists v \in V \setminus \{0\}$ with $\langle v, \nabla f(a) \rangle = 0$.

$$v^T \left(\nabla^2 \log(f)|_{x=a} \right) v = v^T \left(\frac{f(a) \nabla^2 f(a) - \nabla f(a) \nabla f(a)^T}{f(a)^2} \right) v$$

contradicts $v^T(\nabla^2 f(a)) v > 0$ $\rightarrow$

$$= \frac{v^T \nabla^2 f(a) v}{f(a)} \leq 0$$

(Proof of 1st sentence). Let $f = \sum c_S x^S$ be log-concave

Suppose $e = \{\mathbb{1}_A, \mathbb{1}_B\}$ is an edge of $\mathrm{Newt}(f) = \mathrm{conv}\{\mathbb{1}_S : c_S \neq 0\}$

and $\langle \omega, \alpha \rangle \leq \omega_0$ for all $\alpha \in \mathrm{Newt}(f)$ with equality

$\iff v \in e$.

For every $\lambda \in \mathbb{R}_{>0}$,

$$\lambda^{\omega_0} f(\lambda^{-\omega_1} x_1, \ldots, \lambda^{-\omega_n} x_n) = \sum_S c_S \lambda^{\omega_0 - \langle \omega, \mathbb{1}_S \rangle} \prod_{i \in S} x_i \quad \text{is log-concave}$$

Limit as $\lambda \to 0$ is $c_A x^A + c_B x^B$.

Specialize $x_i = 1$ for $i \in A \cap B \Rightarrow$ can assume $A \cap B = \emptyset$.

If $|A| = |B| \geq 2$, $\nabla^2 f|_{x=\mathbb{1}}$ is block diag. $\left(\begin{array}{c|c} \nabla^2 c_A x^A & 0 \\ \hline 0 & \nabla^2 c_B x^B \end{array}\right)$

Both blocks have pos. eig val $\Rightarrow$ $c_A x^A + c_B x^B$ not log concave!