Math 582G

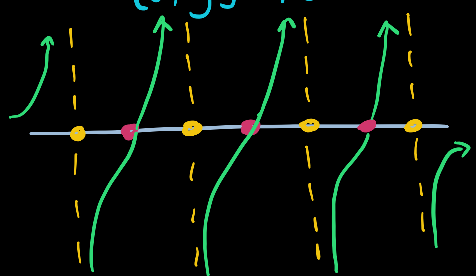
Today: Stable polynomials and negative dependence

Hwk 4 posted, due Wed. March 9

The Wronskian of $f, g \in \mathbb{R}[t]$ is

$$W[f,g] = f' \cdot g - f \cdot g' = g^2 \left(\frac{f}{g}\right)'$$

If f, g are real rooted with roots $\alpha_d \leq \ldots \leq \alpha_1$, $\beta_e \leq \ldots \leq \beta_1$, the roots of f, g are interlaced $(\beta_k \leq \alpha_k \leq \beta_{k-1}$ or $\alpha_k \leq \beta_k \leq \alpha_{k-1})$ iff $W[f,g]$ is lobally nonnegative or nonpositive on $\mathbb{R}$.



$$\frac{f}{g} = \sum_{j=1}^{e} \frac{c_j}{(t-\beta_j)}$$

(roots are interlaced $\Leftrightarrow$ c_j's have same sign)

Thm (Hermite-Biehler) If $f, g \in \mathbb{R}[t]$ are stable, TFAE:

1) $W[f,g] \geq 0$ on $\mathbb{R}$
2) $f + ig \in \mathbb{C}[t]$ is stable
3) $f + sg \in \mathbb{R}[s,t]$ is stable.

$(3 \Rightarrow 1)$ (Sketch) Suppose $f + sg$ stable

Solving for s gives $s = -f/g$.

$\Rightarrow \operatorname{Im}\left(\frac{f(z)}{g(z)}\right) \geq 0$ for $z \in \mathcal{H}$.

Suppose roots $\beta_e \le \ldots \le \beta_1$ of g are simple and $\deg(f) < \deg(g)$. Then for some $c_j \in \mathbb{R}$,

$$\frac{f}{g} = \sum_{j=1}^{e} \frac{c_j}{(t-\beta_j)} \Rightarrow \left(\frac{f}{g}\right)' = \sum_{j=1}^{e} \frac{-c_j}{(t-\beta_j)^2}$$

$$\operatorname{Im}\left(\frac{f(z)}{g(z)}\right) = \sum_{j=1}^{e} \operatorname{Im}\left(\frac{c_j}{z-\beta_j}\right) = \sum_{j=1}^{e} \frac{-c_j \operatorname{Im}(z)}{|z-\beta_j|^2}$$

Plugging in $z = \beta_j + i\varepsilon$ and taking $\varepsilon \to 0$

$\Rightarrow$ coefficients c_j all ≤ 0 $\Rightarrow W[f,g] \ge 0$.

Multivariate version:

Prop: Let $f, g \in \mathbb{R}[x_1, \ldots, x_n]$ be nonzero. TFAE

(1) $f + ig \in \mathbb{C}[\underline{x}]$ is stable

(2) $f + yg \in \mathbb{R}[\underline{x}, y]$ is stable

Cor: If $f + yg \in \mathbb{R}[\underline{x}, y]$ is stable, then for $j \in [n]$,

$$W_j[f,g] = \frac{\partial f}{\partial x_j} \cdot g - f \cdot \frac{\partial g}{\partial x_j} \ge 0 \text{ on } \mathbb{R}^n.$$

Ex: $f = x_1 x_2$ $\quad g = x_1 + x_2$ $\quad f + x_3 g = x_1 x_2 + x_1 x_3 + x_2 x_3$

$W_2[f,g] = x_1(x_1 + x_2) - x_1 x_2 (1) = x_1^2 \ge 0$ on $\mathbb{R}^3$

Thm (Brändén) Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be multiaffine. Then f is stable if and only if for all $i,j \in [n]$

$$\Delta_{ij}(f) = \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} - f \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0 \text{ on } \mathbb{R}^n.$$

Note $\Delta_{ij}(f) = W_j[f, \frac{\partial f}{\partial x_i}]$.

$f + y \frac{\partial f}{\partial x_i} = f|_{x_i=0} + (x_i + y) \frac{\partial f}{\partial x_i} = f|_{x_i \to x_i + y}$ stable

Cor: If $f = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$ is stable and $c_S \in \mathbb{R}_{>0}$ then $F : \{0,1\}^n \to \mathbb{R}$ given by $F(S) = \log(c_S)$ is submodular: $F(S) + F(T) \geq F(S \cap T) F(S \cup T)$

Idea: For $S = A \cup \{i\}$, $T = A \cup \{j\}$ with $A \subseteq [n]$

$$g = \prod_{k \in A} \frac{\partial}{\partial x_k} f = \sum_{S \supseteq A} c_S x^{S \setminus A} \text{ stable}$$

$$\Delta_{ij}(g)|_{x=0} = c_{S \cup i} c_{S \cup j} - c_S c_{S \cup \{i,j\}} \geq 0$$

Applications in negative dependence

Suppose $f = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$ is stable with $c_S \geq 0$.

Associate to f a prob. distribution on $\{S \subseteq [n]\}$

$$\text{Prob}(S) = \frac{c_S}{\sum c_S} = \frac{c_S}{f(\mathbb{1})}.$$

Then for any $i \in S$,

$$\frac{\partial f}{\partial x_i}(\mathbb{1}) = \sum_{S \ni i} c_S = f(\mathbb{1}) \cdot \text{Prob}(i \in S)$$

$$\Delta_{ij}(f)(\mathbb{1}) = \frac{\partial f}{\partial x_i}(\mathbb{1}) \frac{\partial f}{\partial x_j}(\mathbb{1}) - f(\mathbb{1}) \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbb{1})$$

$$= f(\mathbb{1})^2 \big(\text{Prob}(i \in S)\,\text{Prob}(j \in S) - \text{Prob}(i,j \in S)\big) \geq 0)$$

Events $i \in S$, $j \in S$ are negatively correlated!

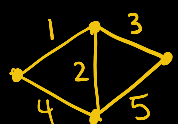
Similarly, $\prod_{j \in T} \frac{\partial}{\partial x_j} f = \sum_{S \supseteq T} c_S x^{S \setminus T}$ is stable $\Rightarrow$ for any $i,j \notin T$,

$$\text{Prob}(T \cup \{i\} \subseteq S)\,\text{Prob}(T \cup \{j\} \subseteq S) \geq \text{Prob}(T \subseteq S)\,\text{Prob}(T \cup \{i,j\} \subseteq S$$

Ex: $G = ([n], E)$ connected graph. By Hwk 4, #2,

$$f = \sum_{\substack{T \text{ spanning} \\ \text{tree of } G}} \prod_{e \in T} x_e \in \mathbb{R}[x_e : e \in E] \text{ is stable}$$

↪ uniform dist. on spanning trees

e.g. 

$$f = e_3(x_1, \ldots, x_5) - x_1x_2x_3 - x_2x_3x_5$$
$$= x_1x_2x_3 + x_1x_2x_5 + \ldots + x_3x_4x_5$$

$$\Delta_{12}(f) = (x_2x_3 + \ldots + x_4x_5)(x_1x_3 + \ldots + x_4x_5) - f(x_3 + x_5)$$
$$= ?$$

$$\Delta_{12}(f)(\mathbb{1}) \geq 0 \Rightarrow \text{Prob}(1 \in T)\,\text{Prob}(j \in T) \geq \text{Prob}(i,j \in T)$$

$\text{Prob}(1 \in T) = \frac{5}{8}$

$\text{Prob}(2 \in T) = \frac{4}{8} = \frac{1}{2}$

$\text{Prob}(1,2 \in T) = \frac{2}{8} = \frac{1}{4} < \frac{1}{2} \cdot \frac{5}{8}$