Math 582G

Today: Stable polynomials II

Hwk 4 posted today, due Wed, March 9

Recall: $f \in \mathbb{C}[x_1,\dots,x_n]$ is stable if $f \equiv 0$ or $f(p) \neq 0$ for $p \in \mathcal{H}^n$
where $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$

Question of the day: What operations preserve stability?

Some examples: If $f, g \in \mathbb{R}[x_1,\dots,x_n]$ are stable, then so are...

1) $D_v f = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}$ for any $v \in \mathbb{R}^n_{\geq 0}$
2) $f \cdot g$
3) $f(x_{\pi(1)},\dots,x_{\pi(n)})$ for any $\pi \in S_n$
4) $f(a, x_2,\dots,x_n)$ for any $a \in \mathbb{R}$
5) $(cx_1+d)^{\deg_{x_1}(f)} f\left(\frac{ax_1+b}{cx_1+d}, x_2,\dots,x_n\right)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$

(1) last class, (2),(3) immediate from def.

(4) limit of $f(a+ib, x_2,\dots,x_n)$ as $b \to 0$

(5) $\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right) = \frac{\mathrm{Im}(z)(ad-bc)}{|cz+d|^2} = \frac{\mathrm{Im}(z)}{|cz+d|^2}$

For $\vec{d} = (d_1,\dots,d_n) \in \mathbb{N}^n$, let $\mathbb{C}[x_1,\dots,x_n]_{\leq \vec{d}} = \{\sum_{\alpha} c_\alpha x^\alpha : \alpha_i \leq d_i \text{ for } i=1,\dots,n\}$

For $\vec{d} = (1,\dots,1)$, these are called multiaffine.

Thm (Borcea, Brändén '09) Let $T: \mathbb{C}[\underline{x}]_{\leq \vec{d}} \to \mathbb{C}[\underline{x}]$ be a linear transformation. Define

$$\mathrm{Symb}(T) = T\left(\prod_{j=1}^{n} (x_j+y_j)^{d_j}\right) = \sum_{\alpha} \prod_{j} \binom{d_j}{\alpha_j} T(\underline{x}^\alpha) \underline{y}^{\vec{d}-\alpha}.$$

If $\mathrm{Symb}(T) \in \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$ is stable then T preserves stability

Ex: $T: \mathbb{C}[x_1,x_2]_{\le(1,1)} \to \mathbb{C}[x_1,x_2]$, $T(f) = x_1x_2 f\left(\frac{-1}{x_1}, \frac{-1}{x_2}\right)$

$T(1) = x_1x_2$ $\quad T(x_1) = -x_2$ $\quad \mathrm{Symb}(T) = T((x_1+y_1)(x_2+y_2))$

$T(x_1x_2) = 1$ $\quad T(x_2) = -x_1$

$$= x_1x_2\left(\frac{-1}{x_1}+y_1\right)\left(\frac{-1}{x_2}+y_2\right) = (x_1y_1-1)(x_2y_2-1)$$

$\mathrm{Symb}(T)$ stable $\Rightarrow T$ preserves stability!

(Proof for $\vec{d}=(1,\dots,1)$) Suppose $T: \mathbb{C}[x_1,\dots,x_n]_{\le(1,\dots,1)} \to \mathbb{C}[x_1,\dots,x_n]$ is a linear transformation and $\mathrm{Symb}(T) = \sum_{S\subseteq[n]} T(\underline{x}^S)\,y^{[n]\setminus S}$ is stable. If $f = \sum_{A\subseteq[n]} c_A \underline{z}^A \in \mathbb{C}[z_1,\dots,z_n]_{\le(1,\dots,1)}$ is stable, then

$$F = f(z)\cdot \mathrm{Symb}(T) = \sum_{A,B\subseteq[n]} c_A z^A \cdot T(x^B)\, y^{[n]\setminus B}$$

in $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_n]$ is stable.

It follows that $\left[\prod_{j=1}^{n}\left(\frac{\partial}{\partial y_j}+\frac{\partial}{\partial z_j}\right)F\right]_{\substack{y=0\\z=0}} \in \mathbb{C}[\underline{x}]$ is stable.

For any $A,B\subseteq[n]$, $\left[\prod_{j=1}^{n}\left(\frac{\partial}{\partial y_j}+\frac{\partial}{\partial z_j}\right) z^A\, T(x^B)\, y^{[n]\setminus B}\right]_{\substack{y=0\\z=0}}$

$$= \sum_{S\subseteq[n]} \underbrace{\left[\left(\prod_{j\notin S}\frac{\partial}{\partial y_j}\prod_{j\in S}\frac{\partial}{\partial z_j}\right) z^A\, T(x^B)\, y^{[n]\setminus B}\right]_{\substack{y=0\\z=0}}}_{=0 \text{ unless } S=A=B} = T(x^A)$$

So $\left[\prod_{j=1}^{n}\left(\frac{\partial}{\partial y_j}+\frac{\partial}{\partial z_j}\right)F\right]_{\substack{y=0\\ z=0}} = \sum_{A\subseteq[n]} c_A\, T(x^A) = T(f)$ is stable. $\square$

Ex: MAP: $\mathbb{C}[x_1,\dots,x_n]_{\le \vec{d}} \to \mathbb{C}[x_1,\dots,x_n]$ defined by

$$MAP\left(\sum_{\alpha} c_\alpha x^\alpha\right) = \sum_{\alpha\in\{0,1\}^n} c_\alpha x^\alpha$$

(Multi Affine Part)

$$Symb(MAP) = MAP\left(\prod_{j=1}^{n}(x_j+y_j)^{d_j}\right) = \prod_{j=1}^{n}\left(d_j x_j y_j^{d_j-1} + y_j^{d_j}\right)$$

$$= \prod_{j=1}^{n}(d_j x_j + y_j)\, y_j^{d_j-1}$$ product of linear stable poly. $\Rightarrow$ stable

Example application: matchings

Def: A (partial) matching of a graph $G=([n],E)$ is subset of disjoint edges $M\subseteq E$.

Let $m_k(G) = \#\{\text{partial matchings of } G \text{ with } k \text{ edges}\}$

Thm (Hielmann-Lieb '72)

$$m(t) = \sum_{k=0}^{\lfloor |E|/2\rfloor} m_k(G)\, t^k \text{ is real rooted}$$

$\Rightarrow m_k(G)$ satisfy Newton's ineq.

$\Rightarrow m_{k-1}(G)\, m_{k+1}(G) \le m_k(G)^2$

Sketch of proof: Let $f_G = \prod_{ij\in E}(1-x_ix_j) \in \mathbb{R}[x_1,\dots,x_n]$.

Then f_G is stable

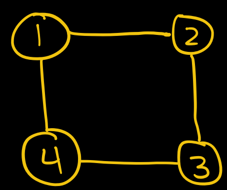
Note $f_G = \sum_{S\subseteq E}\prod_{ij\in S}(-x_ix_j) \Rightarrow \mathrm{MAP}(f_G) = \sum_{\substack{M\subseteq E\\ \text{matching}}}\prod_{ij\in E}(-x_ix_j)$

$\mathrm{MAP}(f_G)(t,\dots,t) = \sum_{M \text{ matching}}(-t^2)^{|M|} = \sum_k m_k(G)(-t^2)^k = m(-t^2)$ is real rooted

$\Rightarrow m(t)$ real rooted (roots $=\{-r^2 : r \text{ is a root of } m(-t^2)\}$) $\square$

Ex: $G = C_4$

$f_G = (1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_1x_4)$



$\mathrm{MAP}(f_G) = 1 - x_1x_2 - x_2x_3 - x_3x_4 - x_1x_4 + 2x_1x_2x_3x_4$

$\mathrm{MAP}(f_G)(t,t,t,t) = 1 - 4t^2 + 2t^4 = m(-t^2)$

where $m(t) = 1 + 4t + 2t^2$.