Math 582G

Today: Stable polynomials

Hwk 3 due today

Stable polynomials

Let $\mathcal{H} = \{a+ib : a \in \mathbb{R}, b \in \mathbb{R}_{>0}\}$ be the upper half plane in $\mathbb{C}$

Def: A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is <u>stable</u> if $f(p) \neq 0$ for all $p \in \mathcal{H}^n$ or $f \equiv 0$.
If, additionally, $f \in \mathbb{R}[x_1, \ldots, x_n]$, f is <u>real stable</u>.

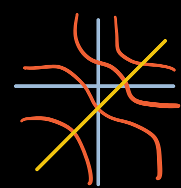
Ex $(n=1)$ $f = x(x+i)(x-2+i) \in \mathbb{C}[x]$, roots $0, -i, 2-i \notin \mathcal{H} \to$ stable

$f = x(x-1)(x+1) \in \mathbb{R}[x]$, roots $0, \pm 1 \to$ real stable

Ex $(n=2)$ $f = xy - 1$. If $(z_1, z_2) \in V_{\mathbb{C}}(f)$, then $z_2 = \frac{1}{z_1} = \frac{\overline{z_1}}{|z_1|^2}$.

If $\mathrm{Im}(z_1) > 0$ then $\mathrm{Im}(z_2) = \frac{-1}{|z_1|^2} \mathrm{Im}(z_1) < 0 \Rightarrow (z_1, z_2) \notin \mathcal{H}^2$.

Thm: Let $f \in \mathbb{R}[x_1, \ldots, x_n]$. TFAE



1) f is stable
2) for all $v \in \mathbb{R}^n_{>0}$, $w \in \mathbb{R}^n$, $f(tv+w) \in \mathbb{R}[t]$ is real rooted
3) $f^{hom} = x_0^{\deg(f)} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$ is hyperbolic w.r.t. every $a \in \{0\} \times \mathbb{R}^n_{>0}$

(Proof) $(2) \Rightarrow (1)$ If f is not stable, then $\exists\, w \in \mathbb{R}^n$, $v \in \mathbb{R}^n_{>0}$ s.t. $f(w+iv) = 0$.

$\Rightarrow t = i$ is a root of $f(tv+w) \Rightarrow$ not real rooted.

$(1) \Rightarrow (2)$ $f(tv+w)$ not real rooted for some $v \in \mathbb{R}^n_{>0}$, $w \in \mathbb{R}^n$

$\Rightarrow$ has root $t = a+ib$ with $b > 0$.

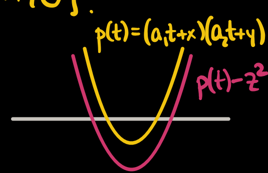
$\Rightarrow 0 = f((a+ib)v + w) = f(\underbrace{(av+w) + ibv}_{\in \mathcal{H}^n}) \Rightarrow f$ not stable

$(2) \Leftrightarrow (3)$ You check!

Ex: $f = xy - 1$ $\quad f^{hom} = xy - z^2$ hyperbolic w.r.t. $(x,y,z) \in \mathbb{R}^2_{>0} \times \{0\}$.

$f^{hom}(a_1 t + x, a_2 t + y, z) = (a_1 t + x)(a_2 t + y) - z^2$

real rooted when $a_1, a_2 \in \mathbb{R}_{>0}$



Cor: If $f \in \mathbb{R}[y_1,\dots,y_m]$ is hyperbolic w.r.t. $e \in \mathbb{R}^m$, then for $a_1,\dots,a_n \in \overline{C(f,e)}$ and $b \in \mathbb{R}^m$, $f(\sum x_i a_i + b) \in \mathbb{R}[x_1,\dots,x_n]$ is stable.

Cor: If $A_1,\dots,A_n, B \in \mathbb{R}^{d\times d}_{sym}$ with $A_1,\dots,A_n \succeq 0$, then

$$f = \det\Big(\sum_{i=1}^n x_i A_i + B\Big) \text{ is stable.}$$

Ex: $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det(xA_1 + yA_2 + B) = \det\begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} = xy - 1.$

Cor: If f is stable and $w \in \mathbb{R}^n_{\geq 0}$, then $D_w f = \sum_{i=1}^n w_i \frac{\partial f}{\partial x_i}$ is stable

Why? f^{hom} hyperbolic w.r.t. $\{0\} \times \mathbb{R}^n_{>0}$

$(0,w)$ belongs to closure of hyp. cone

$$\Rightarrow D_{(0,w)} f^{hom} = \sum_{i=1}^n w_i \frac{\partial f^{hom}}{\partial x_i} = (D_w f)^{hom} \text{ is hyp. w.r.t. } \{0\} \times \mathbb{R}^n_{>0}.$$

Ex: $f = \prod_{i=1}^n x_i$ hyperbolic w.r.t. $\mathbb{R}^n_{>0} \Rightarrow$ stable

$$D_{\mathbb{1}} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} = \sum_{i=1}^n \prod_{j \neq i} x_j = e_{n-1}(x_1,\dots,x_n) \text{ also stable}$$

$$D_{\mathbb{1}}^k f = \sum_{i=1}^k \frac{\partial}{\partial x_i}(D_{\mathbb{1}}^{k-1} f) = k! \sum_{I \in \binom{[n]}{n-k}} \prod_{j \in I} x_j = k!\, e_{n-k}(x_1,\dots,x_n)$$

stable for all k!

Remark: the benefit of coordinate-dependent def. is that the coefficients become meaningful.

Ex (Newton's Inequalities) If $f = \sum_{k=0}^n c_k x^k$ is real rooted and $c_0,\dots,c_n \in \mathbb{R}_{\geq 0}$, then

$$\left(\frac{c_{k-1}}{\binom{n}{k-1}}\right)\cdot\left(\frac{c_{k+1}}{\binom{n}{k+1}}\right) \leq \left(\frac{c_k}{\binom{n}{k}}\right)^2.$$

"log-concavity" of $\frac{c_k}{\binom{n}{k}}$

(Proof) $f^{hom} = \sum_{k=0}^{n} c_k x^k y^{n-k}$ hyp. w.r.t. $(x,y) \in \mathbb{R}_{>0} \times \{0\}$.

Since $c_k \geq 0$, $f^{hom} > 0$ on $\mathbb{R}^2_{>0} \Rightarrow \mathbb{R}^2_{>0} \subseteq$ hyp. cone of f^{hom}.

$\Rightarrow f^{hom}$ stable $\Rightarrow \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial y}\right)^{\beta} f$ stable for any $\alpha, \beta \in \mathbb{N}$.

$$\left(\frac{\partial}{\partial x}\right)^{k-1} \left(\frac{\partial}{\partial y}\right)^{n-k-1} f = \frac{(k+1)!}{2}(n-k-1)!\, c_{k+1} x^2 + k!(n-k)!\, c_k xy + (k-1)!\frac{(n-k+1)!}{2} c_{k-1} y^2$$

$$= \frac{n!}{2}\left(\frac{c_{k+1}}{\binom{n}{k+1}} x^2 + 2\frac{c_k}{\binom{n}{k}} xy + \frac{c_{k-1}}{\binom{n}{k-1}} y^2 \right)$$

Real rooted $\Rightarrow$ discriminant ≥ 0 $\Rightarrow$ desired ineq.