Math 582G

Today: Determinantal Representations

Hwk 3 due today on Canvas

No class on Monday - President's Day

Question: Given $f \in \mathbb{R}[x_1,\ldots,x_n]_d$, hyperbolic w.r.t. $e \in \mathbb{R}^n$ can we write $\overline{C(f,e)} \cong \mathcal{L} \cap PSD_N$ for some N, subspace $\mathcal{L} \subseteq \mathbb{R}^{N\times N}_{sym}$?

Generalized Lax Conj: Yes, always. (OPEN)

Parametrize $\mathcal{L} = \{A(x) : x \in \mathbb{R}^n\}$ $\quad A(x) = \sum_{i=1}^{n} x_i A_i$

$$\Rightarrow \overline{C(f,e)} = \{x : A(x) \succeq 0\}.$$

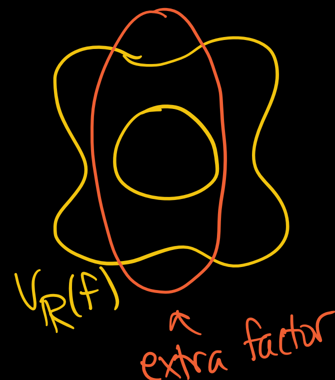
W.l.o.g., we can assume $\mathcal{L} \cap PD_N \neq \emptyset$ (otherwise replace PSD_N with a face $\cong PSD_{N'}$)

Algebraic boundaries?

Suppose $I(\partial \overline{C(f,e)}) = \langle h \rangle$.

Then h divides f and $\det(A(x))$ and

$C(f,e) = C(h,e) = C(\det(A(x)),e) = \{x : A(x) \succ 0\}$



Question: If $f \in \mathbb{R}[x_1,\ldots,x_n]_d$ is hyperbolic w.r.t. $e \in \mathbb{R}^n$

is $f = \det(A(x))$ with $A(x) = \sum x_i A_i$ for some $A_1, \dots, A_n \in \mathbb{R}^{d\times d}_{sym}$ with $A(e) \succ 0$?

$n=2$: $f(x_1, x_2)$, hyperbolic w.r.t. $(1,0) \in \mathbb{R}^2$

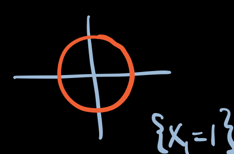
$\Rightarrow f(x_1,x_2) = f(1,0) \cdot \prod_{k=1}^{d} (x_1 - \lambda_k x_2) \Rightarrow f = f(1,0) \cdot \det\left(x_1 I_d - x_2 \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}\right)$

$n=3$ (Lax Conjecture)

Thm (Helton-Vinnikov 2007) If $f \in \mathbb{R}[x_1,x_2,x_3]_d$ is hyperbolic w.r.t. $e \in \mathbb{R}^3$, then $\exists\, A_1, A_2, A_3 \in \mathbb{R}^{d\times d}_{sym}$ s.t. $f = \det(A(x))$ where $A(x) = \sum x_i A_i$ and $A(e) \succ 0$.

Cor: $\overline{C(f,e)} = \{x \in \mathbb{R}^3 : A(x) \succeq 0\}$

Ex: $d=2$ $\quad x_1^2 - x_2^2 - x_3^2 = \det\begin{pmatrix} x_1+x_2 & x_3 \\ x_3 & x_1 - x_2 \end{pmatrix}$



Cor: A compact convex subset $C \subseteq \mathbb{C} \cong \mathbb{R}^2$ is the numerical range of a complex matrix iff K^* is the hyperbolicity cone of some $f \in \mathbb{R}[t,x,y]$ w.r.t. $(1,0,0)$ where $K = \{(\lambda, \lambda a, \lambda b) : a+ib \in C, \lambda \in \mathbb{R}_{\geq 0}\}$.

$(\Rightarrow)$ previously

$(\Leftarrow)$ $K^* =$ hyp. cone of f w.r.t. $(1,0,0)$

$$\underset{H.V}{\Longrightarrow} f = \det(tA_1 + xA_2 + yA_3) \text{ with } A_i \in \mathbb{R}^{d\times d}_{sym}, A_1 \succ 0$$

$$= \det(A_1)\det(tI_d + xB_1 + yB_2) \qquad B_i = U^{-1}A_iU^{-T}$$

$$= (\text{const.})\det\left(tI_d + x\left(\frac{M+M^*}{2}\right) + y\left(\frac{M-M^*}{2i}\right)\right)$$

for $M = B_1 + iB_2 \in \mathbb{C}^{d\times d}_{sym}$

$$\Rightarrow K^* = \{(t,x,y) : tI_d + xB_1 + yB_2 \succeq 0\}$$

$$\Rightarrow K = \{(\langle X, I_d\rangle, \langle X, B_1\rangle, \langle X, B_2\rangle) : X \in PSD_d\}$$

$$= \{(\lambda, \lambda a, \lambda b) : a + ib \in \mathcal{W}(M)\}$$

intersect w/ first coord $= 1$

$$\Rightarrow C = \mathcal{W}(M).$$

fails for $n \geq 4$. That is, there are hyperbolic polynomials $f \in \mathbb{R}[x_1, \dots, x_4]_d$ so that $f \neq \det(\sum x_i A_i)$ for any $A_1, \dots, A_4 \in \mathbb{R}^{d\times d}_{sym}$.

Dimension count

$$\dim(\mathbb{R}[x_1,\dots,x_n]_d) = \binom{n-1+d}{d} \approx n^d \leftarrow \text{grows much faster}$$

$$\dim\left(\{\det(A(x)) : A_1,\dots,A_n \in \mathbb{R}^{d\times d}_{sym}\}\right) \leq \dim\left(\left(\mathbb{R}^{d\times d}_{sym}\right)^n\right)$$

$$= n\binom{d+1}{2} \approx nd^2$$

When $\binom{n-1+d}{d} > n\binom{d+1}{2}$, most polynomials in $\mathbb{R}[x_1,\dots,x_n]_d$ cannot be written as $\det(A(x))$ with $A_i \in \mathbb{R}^{d\times d}_{sym}$

Ex: $f(x)=x_1^2-x_2^2-x_3^2-x_4^2$ $\quad e=(1,0,0,0)$

Suppose $f(x)=\det(A(x))$ for some $A_1,\dots,A_4\in\mathbb{R}^{2\times 2}_{sym}$

$\dim(\mathbb{R}^{2\times 2}_{sym})=3 \Rightarrow$ for some nonzero $v\in\mathbb{R}^4$, $A(v)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

$\Rightarrow f(x+v)=f(x)$ for all $y\in\mathbb{R}^4$

$\Rightarrow \langle\nabla f(x),v\rangle=0$ for all $x\in\mathbb{R}^4 \Rightarrow v=0$.

But $(x_1+x_2)f(x)=\det\begin{pmatrix}x_1-x_2 & x_3 & x_4\\ x_3 & x_1+x_2 & 0\\ x_4 & 0 & x_1+x_2\end{pmatrix}$

Gen. Lax Conj (rephrased): If $f\in\mathbb{R}[x_1,\dots,x_n]_d$ is hyperbolic w.r.t. $e\in\mathbb{R}^n$, then $\exists g\in\mathbb{R}[x_1,\dots,x_n]$ s.t. $f\cdot g=\det(A(x))$ for some $A_i\in\mathbb{R}^{N\times N}_{sym}$ with $A(e)\succ 0$ and $C(g,e)\supseteq C(f,e)$.