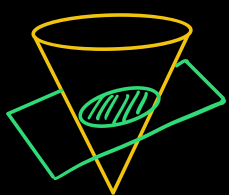
Math 582G

Today: Hyperbolic Programming

A <u>hyperbolic program</u> is an problem of the form

$$\min \langle c,x\rangle \quad \text{s.t.} \quad x\in\overline{C(f,e)} \quad \text{and} \quad \langle a_i, x\rangle = b_i \quad i=1,\dots,m$$

where $f\in\mathbb{R}[x_1,\dots,x_n]_d$ is hyperbolic w.r.t. $e\in\mathbb{R}^n$, $c, a_1,\dots,a_m\in\mathbb{R}^n$, $b_1,\dots,b_m\in\mathbb{R}$.



Ex: $\overline{C(f,e)} = \mathbb{R}^n_{\geq 0} \to$ linear programming (LP)

Ex: $\overline{C(f,e)} = PSD_d \to$ semidefinite programming (SDP)

Güler ('97) and Renegar (early '00's) showed that interior pt methods for LPs and SDPs extend to general hyperbolic programs.

Idea: Use $-\log(f)$ as a "barrier" function for $C(f,e)$

Note: For $a\in C(f,e)$, $f(a)>0 \Rightarrow -\log(f(a))$ defined

$$-\log(f(a))\to\infty \quad \text{as } a\to\partial\overline{C(f,e)} \quad (\text{and } f(a)\to 0)$$

Thm: If $f$ is hyperbolic w.r.t. $e$ and $\overline{C(f,e)}\cap(-\overline{C(f,e)})=\{0\}$ then the Hessian of $-\log(f)$ is positive definite at every point $a\in C(f,e)$.

(Proof) Let $a \in C(f,e)$ and $v \in \mathbb{R}^n$. Then

$$v^T(\nabla^2 \log(f(a))v = \left[\partial_t^2 \log(f(a+tv)\right]_{t=0}.$$

WTS this is $\leq 0$ and $< 0$ for $v \neq 0$.

Since $f$ is hyp. w.r.t. $a$, $f(a+tv) = f(a) \prod_{j=1}^{d} (1+\lambda_k t)$

where $\lambda_k = k^{th}$ eig. val of $v$ w.r.t. $a$.

$$\Rightarrow \frac{d^2}{dt^2} \log(f(a+tv)) = \frac{d^2}{dt^2}\left[\log(f(a)) + \sum_{j=1}^{d} \log(1+\lambda_k t)\right)$$

$$= \sum_{j=1}^{d} \frac{d^2}{dt^2} \log(1+\lambda_k t) = \sum_{j=1}^{d} \frac{d}{dt}\left(\frac{\lambda_j}{(1+\lambda_j t)}\right) = \sum_{j=1}^{d} \frac{-\lambda_j^2}{(1+\lambda_j t)^2}$$

$$\Rightarrow v^T(\nabla^2 \log(f(a)))v = \left[\frac{d^2}{dt^2} \log(f(a+tv))\right]_{t=0} = \sum_{j=1}^{d} -\lambda_j^2$$

Since $\lambda_1, \dots, \lambda_d \in \mathbb{R}$, this is $\leq 0$

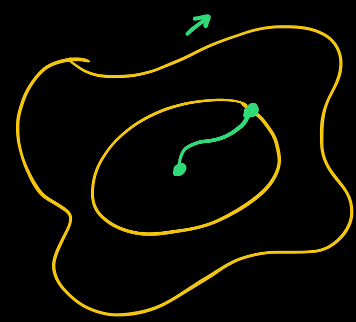
and equals zero $\Leftrightarrow \lambda_1 = \dots = \lambda_d = 0 \Leftrightarrow v \in \overline{C(f,e)} \cap -\overline{C(f,e)}$.

Cor: For any $\lambda \in \mathbb{R}_{>0}$, the Hessian of $\langle c,x\rangle - \lambda \log(f)$ is positive definite at $a \in C(f,e)$.

If $S = \{x \in \overline{C(f,e)} : \langle a_i, x\rangle = b_i, i = 1, \dots, m\}$ is compact then for each $\lambda > 0$, $\langle c,x\rangle - \lambda\log(f(x))$ is minimized by a unique point $x^*(\lambda) \in S$.

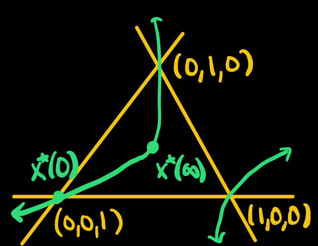
$x^*(\lambda)$ is continuous and $x^*(\lambda) \to$ minimizer of $\langle c, x\rangle$ as $\lambda \to 0$

Def: The central path of a hyperbolic program is $\{x^*(\lambda) : \lambda \in \mathbb{R}_{>0}\}$.



Idea of interior pt. methods: find piecewise linear approximation of central path as $\lambda \to 0$

Ex: $\min x_1 + 2x_2$ s.t. $x \in \mathbb{R}^3_{\geq 0}$, $x_1 + x_2 + x_3 = 1$




$$F_\lambda(x) = x_1 + 2x_2 - \lambda \log(x_1 x_2 x_3)$$

$$\nabla F_\lambda(x) = (1,2,0) - \lambda\left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\right)$$

$$\nabla F_\lambda(x) = \mu(1,1,1) \Rightarrow 0 = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} \end{pmatrix} = \frac{-2}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$$

$$\Rightarrow -2x_2x_3 + x_1x_3 + x_1x_2 = 0.$$

Thm (Renegar) All faces of $\overline{C(f,e)}$ are exposed.

Idea: For $x \in \mathbb{R}^n$, define $\operatorname{rank}(x) = \#$ nonzero eig. val. of $x$.
If $f$ is homog. of degree $d$, this is $d - \left(\text{mult of } 0 \text{ as a root of } f(te - x)\right)$.

- If $\operatorname{rank}(a) = d-1$ and $a \in \overline{C(f,e)}$
  $\mathbb{R}_{\geq 0}\{a\}$ is exposed by $x \mapsto \langle x, \nabla f(a)\rangle$

- If $F$ is a face of $\overline{C(f,e)}$, then $\mathrm{rank}(a)$ is constant on the relative interior of $F$
- If $\mathrm{rank}(a) = r$ for $a \in \mathrm{relint}(F)$ then $F$ is a face of $C(D_e^{d-r-1} f, e)$ with mult. 1 $\Rightarrow$ exposed by $x \mapsto \langle x, \nabla D_e^{d-r-1} f(a) \rangle$.


Open Question: Is hyperbolic programming stronger than semidefinite programming?

Related Open Question: Is every hyperbolicity cone $C(f,e)$ linearly isomorphic to $L \cap PSD_N$ for some $N$ and linear space $L$?