Math 582G

Today: Hyperbolic polynomials II

Hwk 3 due Friday

OH M 4-5pm, Th. 2-3pm PDL C 526

From last time...

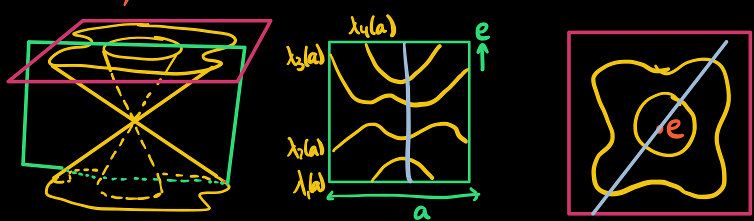
A homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ is hyperbolic w.r.t. $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and for all $a \in \mathbb{R}^n$, $f(te-a) \in \mathbb{R}[t]$ is real rooted.

$\lambda_1(a), \dots, \lambda_n(a)$ = "eigenvalues of a" = roots of $f(te-a)$

$C(f,e)$ = "hyperbolicity cone" = $\{a \in \mathbb{R}^n : \lambda_1(a) > 0\}$

Thm: $C(f,e)$ is a convex cone and f is hyperbolic w.r.t. any $a \in C(f,e)$

Ex: $n=3$, $d=4$ Visualization:



$f(te-a)$ $f(e-ta)$

Prop: If f is hyperbolic w.r.t. e then $\overline{C(f,e)}$ is a basic closed semialgebraic set defined by

$$p_1(x) \geq 0, \ \dots, \ p_d(x) \geq 0$$

where $f(te+x) = f(e) \sum_{k=0}^{d} p_{d-k}(x) t^k$.

Note $p_d(x) = f(0e+x) = f(x)$.

(Proof) Fix $a \in \mathbb{R}^n$ and let

$$p(t) = f(te+a) = f(e)\prod_{k=0}^{d} p_{d-k}(a)t^k = f(e)\prod_{k=1}^{d}(t+\lambda_k(a))$$

If $\lambda_1(a), \dots, \lambda_k(a) \in \mathbb{R}_{\geq 0}$, then $p_{d-k}(a) = \sum_{I \in \binom{[n]}{k}} \prod_{i \in I} \lambda_i(a) \geq 0.$

If $p_1(a), \dots, p_d(a) \geq 0$, then $p(t) > 0$ for $t \in \mathbb{R}_+$

$\Rightarrow$ roots $-\lambda_1(a), \dots, -\lambda_d(a) \in \mathbb{R}_{\leq 0}$.

Ex: $f = \prod_{i=1}^{n} x_i$, $e = (1, \dots, 1)$ $\quad \overline{C(f,e)} = \mathbb{R}^n_{\geq 0}$

$$f(te+a) = \prod_{i=1}^{n}(t+a_i) = \sum_{i=1}^{n} t^k e_{n-k}(a_1, \dots, a_n) \text{ where}$$

$$e_j = \sum_{I \subseteq \binom{[n]}{j}} \prod_{i \in I} x_i.$$

$\Rightarrow \mathbb{R}^n_{\geq 0}$ defined by inequalities

$e_1(x) \geq 0,\ e_2(x) \geq 0,\ \dots,\ e_n(x) \geq 0$

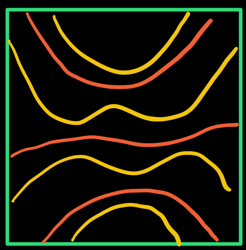
Derivatives

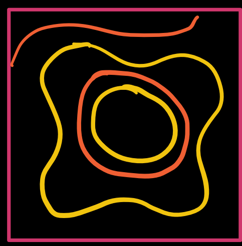
If $q(t) \in \mathbb{R}[t]$ is real rooted with roots $\alpha_1, \dots, \alpha_d$, then $q'(t)$ is also real rooted and its roots $\beta_1, \dots, \beta_{d-1}$ interlace the roots of q: $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \dots \leq \alpha_{d-1} \leq \beta_{d-1} \leq \beta_d$.

For $f \in \mathbb{R}[x_1, \dots, x_n]$, $e \in \mathbb{R}^n$,

let $D_e f = \langle e, \nabla f(x) \rangle = \left[\frac{\partial}{\partial t} f(te+x)\right]\Big|_{t=0} = p_{d-1}(x)$ from ↑

Prop: If f is hyperbolic w.r.t. e and $C(f,e) \subseteq C(D_e f, e)$.





$V_{\mathbb{R}}(f)$ $V_{\mathbb{R}}(D_e f)$

(Proof) For any $a \in \mathbb{R}^n$

$$D_e f(te-a) = \frac{\partial}{\partial t} f(te-a).$$

$\Rightarrow$ roots of $D_e f(te-a)$ are all real and interlace roots of $f(te-a)$.

If $a \in C(f,e)$, all roots of $f(te-a) > 0$

$\Rightarrow$ all roots of $D_e f(te-a) > 0$ $\Rightarrow$ $a \in C(D_e f, e)$.

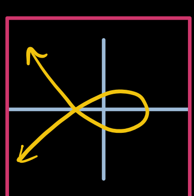
Remark: If $\ell : \mathbb{R}^m \to \mathbb{R}^n$ linear and $\ell(a) \in C(f,e)$, then $g(y) = f(\ell(y_1, \dots, y_m))$ is hyperbolic w.r.t. $\ell(a)$ with

$$C(g, a) = \{ y \in \mathbb{R}^m : \ell(y) \in C(f,e) \}$$

Ex: $f = \det \begin{pmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & \ddots & \vdots \\ x_{1d} & \cdots & x_{dd} \end{pmatrix}$ $e = I_d$ $C(f,e) = PD_d$

If $A(y) = \sum_{i=1}^{m} y_i A_i$ with $A_i \in \mathbb{R}^{d \times d}_{sym}$ and $A(a) \succ 0$, then $g = \det(A(y))$ is hyperbolic w.r.t. a and

$$C(g,a) = \{ y \in \mathbb{R}^m : A(y) \succ 0 \}$$

Ex: 

$\{t=1\}$

$$f = \det \begin{pmatrix} t & 0 & x \\ 0 & t+x & y \\ x & y & t \end{pmatrix} = (t-x)(t+x)^2 - ty^2$$