Math 582G

Today: Real rootedness and hyperbolic polynomials

Hwk 3 posted today, due Fri. Feb. 18.

We say $f \in \mathbb{R}[t]$ is <u>real-rooted</u> if all of its roots (over $\mathbb{C}$) are real, i.e. $f = \lambda \prod_{j=1}^{d} (t - r_j)$ for some $\lambda, r_1, \dots, r_d \in \mathbb{R}$.

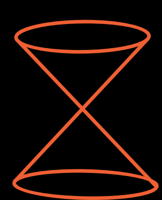
A multivariate polynomial $f = \sum_{\alpha} c_\alpha \underline{x}^{\alpha} \in \mathbb{R}[x_1, \dots, x_n]$ is <u>homogeneous of degree d</u> if $d = |\alpha| = \sum_{i=1}^{n} \alpha_i$ for every α with $c_\alpha \neq 0$.

Equivalently, if for every $\lambda \in \mathbb{R}$, $f(\lambda x) = \lambda^d f(x)$.

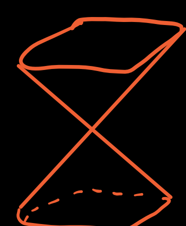
Def: A homog. polynomial $f \in \mathbb{R}[\underline{x}]$ is <u>hyperbolic w.r.t. $e \in \mathbb{R}^n$</u> if $f(e) \neq 0$ and for every $a \in \mathbb{R}^n$, $f(te - a) \in \mathbb{R}[t]$ is real rooted.

Ex: $f = x_1^2 - x_2^2 - x_3^2$ $\quad e = (1, 0, 0)$

$f(te-a) = (t-a_1)^2 - (a_2^2 + a_3^2)$



Non-ex: $f = x_1^4 - x_2^4 - x_3^4$



e = any pt in $\mathbb{R}^4$

Ex: $f = \prod_{i=1}^{n} x_i$ $\quad e = (1, \dots, 1)$

$f(te-a) = \prod_{i=1}^{n} (t - a_i)$

Ex: $f = \det(X)$ $\quad X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1d} \\ x_{12} & x_{22} & & \vdots \\ \vdots & & \ddots & \\ x_{1d} & \cdots & & x_{dd} \end{pmatrix}$ $\quad E = I_d$

$f(tE + A) = \det(tI_d - A) = \prod_{i=1}^{n} (t - \lambda_i)$

where $\lambda_1, \dots, \lambda_n$ = eigval. of A

Note: If f is homogeneous,

$f(te-a)$ is real rooted $\Leftrightarrow$ $f(e-ta)$ is real rooted

Why? Consider $f(se - ta) \in \mathbb{R}[s,t]$, homog. in s and t

$\Rightarrow f(se+ta) = \lambda \cdot s^{d_1} t^{d_2} \prod_{j=1}^{k} (s - r_j t)$ where $r_1, \dots, r_k$ = nonzero roots of $f(se-a)$

$\Rightarrow \quad = \lambda \cdot \prod r_j \, s^{d_1} t^{d_2} \prod_{j=1}^{k} (\frac{1}{r_j} s - t) \Rightarrow \frac{1}{r_1}, \dots, \frac{1}{r_k}$ = nonzero roots of $f(e-ta)$

We can define the "eigenvalues" of $a \in \mathbb{R}^n$ to be the roots $\lambda_1(a) \le \dots \le \lambda_d(a)$ of $f(te-a)$.

Def: The (open) hyperbolicity cone of f is

$$C(f,e) = \{a \in \mathbb{R}^n : \text{roots } \lambda_1(a), \dots, \lambda_d(a) \text{ of } f(te-a) \text{ are positive}\}.$$

Ex: $f = \prod x_i \quad e = (1,..,1)$

For $a \in \mathbb{R}^n$, $\{\text{roots of } f(te-a)\} = a_1, .., a_n$

$$C(f,e) = \mathbb{R}^n_{>0}$$

Ex: $f = \det\begin{pmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & \ddots & \\ x_{1d} & & x_{dd} \end{pmatrix}$, $E = I_d$

For $A \in \mathbb{R}^{d\times d}_{sym}$, roots of $f(tE-A) =$ eig val of A

$$C(f,e) = \{A : A \succ 0\}$$

Thm (Gårding, 1959) If f is hyperbolic w.r.t. e, then

(1) $C(f,e)$ is the connected component of e in $\mathbb{R}^n \setminus V_{\mathbb{R}}(f)$

(2) $C(f,e)$ is convex

(3) f is hyperbolic w.r.t. any $a \in C(f,e)$ and $C(f,a) = C(f,e)$

Lemma 1: $C(f,e)$ is star convex w.r.t. e.

That is, $a \in C(f,e)$ and $\mu \in [0,1]$, $\mu e + (1-\mu)a \in C(f,e)$.

(Proof) Take $0 < \mu < 1$. Then

$$f(te - (\mu e + (1-\mu)a)) = f((t-\mu)e - (1-\mu)a) = \frac{1}{(1-\mu)^d} f\left(\frac{(t-\mu)}{(1-\mu)} - a\right)$$

If λ is an eigval of $\mu e + (1-\mu)a$, then $\frac{\lambda - \mu}{(1-\mu)}$ eig val of a.

$\Rightarrow \frac{\lambda-\mu}{(1-\mu)} > 0 \Rightarrow \lambda > \mu > 0 \Rightarrow \mu e + (1-\mu)a \in C(f,e)$.

(Proof of (1)) $C(f,e)$ is star convex $\Rightarrow$ connected.

Any pt a on the boundary of $\overline{C(f,e)}$ has $\lambda_1(x) = 0 \Rightarrow f(x) = 0$

Lemma 2: For $a \in C(f,e)$, $b \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_{>0}$, and $\mu \in \mathbb{R}_{\geq 0}$, the roots of

$$q_\mu(z) = f(za - i\lambda e - \mu b) \in \mathbb{C}[z]$$

lie in the upper half plane $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

Suppose for some $\mu \geq 0$, q_μ has a root z with $\operatorname{Im}(z) \leq 0$.

Let μ^* be the minimum such μ.

By continuity, q_{μ^*} has a root r with $\operatorname{Im}(r) = 0$ (i.e. $r \in \mathbb{R}$).

$\Rightarrow 0 = q_{\mu^*}(r) = f(ra - i\lambda e - \mu b) = f((ra - \mu b) - i\lambda e)$.

$\Rightarrow i\lambda$ is a root of $f((ra + \mu b) - te)$. $\divideontimes$ $\square$

(Proof of (3)) Taking $\mu = 1$ and $\lambda \to 0$ shows that roots of $f(za - b) \in \mathbb{R}[z]$ have $\operatorname{Im}(z) \geq 0$. By conjugation, roots must also satisfy $\operatorname{Im}(z) \leq 0 \Rightarrow f(ta - b)$ real rooted.

Moreover, a, e belong to same connected component of $\mathbb{R}^n \setminus V_{\mathbb{R}}(f) \Rightarrow C(f,e) = C(f,a)$.

(Proof of (2)) By (3) and Lemma 1, $C(f,e)$ is star convex w.r.t. <u>any</u> $a \in C(f,e) \Rightarrow C(f,e)$ is convex.

Prop: If f is hyperbolic w.r.t. e then $\overline{C(f,e)}$ is a basic closed semialgebraic set defined by

$$p_0(x) \geq 0, \ldots, p_{d-1}(x) \geq 0$$

where $f(te+x) = f(e) \sum_{k=0}^{d} p_k(x) t^k$.

(Proof) Fix $a \in \mathbb{R}^n$ and let

$$p(t) = f(te+a) = f(e) \prod_{k=0}^{d} p_k(a) t^k = f(e) \prod_{k=1}^{d} (t + \lambda_k(a))$$

If $\lambda_1(a), \ldots, \lambda_k(a) \in \mathbb{R}_{\geq 0}$, then $p_k(a) = \sum_{I \in \binom{[n]}{k}} \prod_{i \in I} \lambda_i(a) \geq 0$.

If $p_0(k), \ldots, p_{d-1}(a) \geq 0$, then $p(t) > 0$ for $t \in \mathbb{R}_+$

$\Rightarrow$ roots $-\lambda_1(a), \ldots, -\lambda_d(a) \in \mathbb{R}_{\leq 0}$.