Math 582 G

Today: Algebraic Boundaries

OH: Thurs. 2-3pm PDL-C 526 or Zoom

Algebraic boundaries

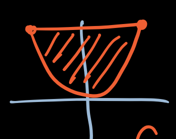
Let $C \subseteq \mathbb{R}^n$ be an n-dim'l convex, semialgebraic set. Then some polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ vanishes on ∂C. (e.g. $f = \prod_i g_i$ over all poly. g_i appearing in description of C)

$\dim(\partial C) = n-1 \Rightarrow \mathcal{I}(\partial C) = \langle f \rangle$ for some f

That is, every polynomial vanishing on ∂C can be written as $h \cdot f$ for some $h \in \mathbb{R}[x_1, \ldots, x_n]$.

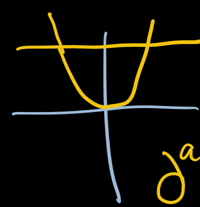
The algebraic boundary of C is

$$\partial^{alg}(C) = V(\mathcal{I}(\partial C)) = V(f)$$

Ex:  C

$C = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 \le y_2 \le 1\}$

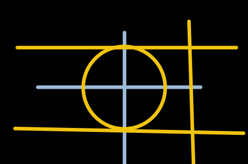
$\partial^{alg}(C) = V((1-y_2)(y_2 - y_1^2))$.

 $\partial^{alg} C$

Ex: 

$C = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\} \cup ([0,2] \times [-1,1])$

$\partial^{alg} C = V((1-x^2-y^2)(x-2)(1-y^2))$



Ex: $C = PSD_d \subseteq \mathbb{R}^{d \times d}_{sym}$ 

$\partial^{alg}(C) = V(\det(X))$

Duality & algebraic boundaries

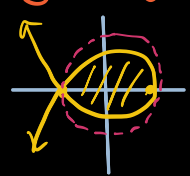
Prop: If $C \subseteq \mathbb{R}^n$ is a compact, convex set and that $f \in \mathbb{R}[x_1, \ldots, x_n]$ vanishes on ∂C. Then
$\partial C^* \subseteq \{ \vec{a} \in \mathbb{R}^n : \exists\, x \in \mathbb{R}^n, \lambda \in \mathbb{R} \text{ with } f(x)=0 \;\; \nabla f(x) = \lambda \vec{a}, \langle \vec{a}, \vec{x} \rangle + 1 = 0 \}$

can calculate polynomials vanishing on this set via elimination (Gröbner bases!)

(Proof) Suppose $\vec{a} \in \partial C^*$. Let $H = \{ \vec{y} \in \mathbb{R}^n : \langle y, x^* \rangle + x_0 = 0 \}$ be a supporting hyperplane of C^* at $\vec{a}$. That is, $\langle y, x^* \rangle + x_0 \geq 0$ for all $y \in C^*$ and $\langle a, x^* \rangle + x_0 = 0$. Since C is compact, $0 \in \operatorname{int}(C^*)$ (check!) so $\langle 0, x^* \rangle + x_0 > 0$ and we can rescale s.t. $x_0 = 1$.

Since $a \in C^*$, $\langle a, x \rangle + 1 \geq 0$ for all $x \in C^*$ so x^* maximizes $x \mapsto \langle a, x \rangle$ over $x \in C$.

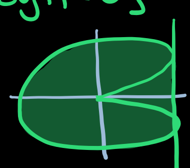
By Lagrange multipliers, $\nabla f(x^*) = \lambda \vec{a}$ for some $\lambda \in \mathbb{R}$

Ex: 

$C = \{ (x,y) : (1-x)(1+x)^2 \geq y^2, \; 1 - x^2 - y^2 \geq 0 \}$

$f = (1-x)(1+x^2) - y^2$

$\{ (a,b) \in \mathbb{R}^2 \text{ s.t. } \exists (x,y) \in \mathbb{R}^2 \text{ with } f(x,y) = 0, \; \nabla f(x,y) = \lambda(a,b), \; ax + by + 1 = 0 \}$
$= V((a-1)(4a^4 + 32b^4 + 13a^2b^2 - 18ab^2 + 4a^3 - 27b^2))$



Algebraic boundaries of spectrahedra

Let $A(y) = A_0 + \sum_{i=1}^{n} y_i A_i$ where $A_i \in \mathbb{R}^{d\times d}_{sym}$, $A(0) = A_0 \succ 0$.

Let $C = \{ y \in \mathbb{R}^n : A(y) \succeq 0 \}$.

Claim: $\partial^{alg}(C) \subseteq V(\det(A(y)))$.

(Proof) If $y \in C$ and $\det(A(y)) \neq 0$ then $A(y) \succ 0$.

$\Rightarrow$ For any $z \in \mathbb{R}^n$ and small enough $\varepsilon > 0$, principal minors of $A(y+\varepsilon z)$ are > 0 $\Rightarrow$ $A(y+\varepsilon z) \succ 0$ $\Rightarrow$ $y + \varepsilon z \in C$.

Therefore, for any $y \in \partial C$, $\det(A(y)) = 0$.

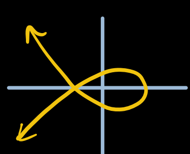
Real rootedness

Prop: If $f = \det(A(y))$ where $A_i \in \mathbb{R}^{d\times d}_{sym}$, $A_0 \succ 0$, then for every $v \in \mathbb{R}^n$, $f(tv) \in \mathbb{R}[t]$ is real rooted.

(Proof) Write $A_0 = UU^T$ for $U \in \mathbb{R}^{d\times d}$. Then

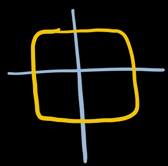
$$f(tv) = \det\left(A_0 + t\sum_i v_i A_i\right) = \det(U)^2 \det\left(I + t\,U^{-1}\left(\sum v_i A_i\right)U^{-T}\right)$$

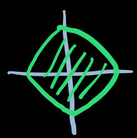
Then λ is a root of $f(tv) \Leftrightarrow -\frac{1}{\lambda}$ is an eig. val. of $\underbrace{U^{-1}\left(\sum v_i A_i\right)U^{-T}}_{\in \mathbb{R}^{d\times d}_{sym}}$

Ex:  $f = \det\begin{pmatrix} 1 & 0 & x \\ 0 & 1+x & y \\ x & y & 1 \end{pmatrix} = (1-x)(1+x)^2 - y^2$

Note that any factor of $f = \det(A(y))$ must also be real rooted on lines through $(0,0)$.

Non-ex: $f = 1 - x^4 - y^4$ $\quad f(t,t) = 1 - 2t^4$ not real rooted!

 $\Rightarrow B = \{(x,y) \in \mathbb{R}^2 : x^4 + y^4 \leq 1\}$ is not a spectrahedron

 Then $C = B^*$ is convex, compact, semialgebraic, but <u>not</u> the numerical range of any matrix!

A converse in the plane $(n=2)$

Thm (Helton-Vinnikov '07)

If $f \in \mathbb{R}[y_1, y_2]_{\leq d}$ with $f(0,0) = 1$ and for all $v \in \mathbb{R}^2$, $f(tv) \in \mathbb{R}[t]$ is real rooted, then there exist $A_1, A_2 \in \mathbb{R}^{d \times d}_{sym}$ s.t.

$$f = \det(I + y_1 A_1 + y_2 A_2).$$

Fails in general!