Math 582G

Today: The numerical range & convex duality

Last week: For any $B, C \in \mathbb{R}^{n\times n}_{sym}$,

$$\{(x^TBx, x^TCx) : x\in\mathbb{R}^n, \|x\|_2=1\} = \{(\langle B,X\rangle, \langle C,X\rangle) : X\in PSD_n, \langle X,I\rangle=1\}$$

and is a convex, compact subset of $\mathbb{R}^2$.

Cor: For any Hermitian matrices $M_1, M_2 \in \mathcal{H}^n$

$$\{(\bar{w}^TM_1w, \bar{w}^TM_2w) : w\in\mathbb{C}^n, \|w\|_2=1\} = \{(\langle M_1,X\rangle, \langle M_2,X\rangle) : X\in\mathcal{H}^n_+, \langle X,I\rangle=1\}$$

(Proof) For $w\in\mathbb{C}^n$, write $w=u+iv$ with $u,v\in\mathbb{R}^n$.

Then $\bar{w}^TM_jw = (u-iv)^TM_j(u+iv)$ is quadratic in (u,v).

If $M_j = A_j+iB_j$ with $A_j=A_j^T$, $B_j=-B_j^T$ real, then

$$\bar{w}^TM_jw = (u-iv)^T(A_j+iB_j)(u+iv) = u^TA_ju+v^TA_jv+u^TB_jv-v^TB_ju = \begin{pmatrix}u\\v\end{pmatrix}^T\begin{pmatrix}A_j & B_j\\ -B_j & A_j\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$$

$\Rightarrow$ Follows from real symmetric version.

Def: The <u>numerical range</u> of a matrix $A\in\mathbb{C}^{n\times n}$ is

$$W(A) = \{\bar{w}^TAw : w\in\mathbb{C}^n, \|w\|_2=1\}$$

Toeplitz-Haussdorf Thm: $W(A)$ is convex in $\mathbb{C}\cong\mathbb{R}^2$.

(Proof) For $A\in\mathbb{C}^{n\times n}$, let $Re(A)=\frac{A+\bar{A}^T}{2}$ and $Im(A)=\frac{A-\bar{A}^T}{2i}$

Then $Re(A)$, $Im(A)$ are Hermitian and $A=Re(A)+i\,Im(A)$

$\Longrightarrow$ $\bar{w}^TAw = \bar{w}^TRe(A)w + i\,\bar{w}^TIm(A)w$.

$\Rightarrow W(A) = \{ \bar{w}^T Re(A) w + i \bar{w}^T Im(A) w : w \in \mathbb{C}^n, \|w\|_2 = 1 \} \subseteq \mathbb{C} \cong \mathbb{R}^2.$

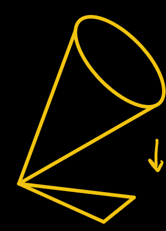
By Cor, this equals $\{ \langle X, Re(A) \rangle + i \langle X, Im(A) \rangle : X \in \mathcal{H}^n_+, \langle X, I \rangle = 1 \}$ and is therefore compact and convex.

Ex: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad Re(A) = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ $\quad Im(A) = \begin{pmatrix} 0 & 1/2i \\ -1/2i & 0 \end{pmatrix}$ $\quad$ What is $W(A)$?

Projections & Convex Duality

Given $A_1, \dots, A_m \in \mathcal{H}^n$, define $\pi_A : \mathcal{H}^n \to \mathbb{R}^m$ as

$$\pi_A(X) = (\langle X, A_1 \rangle, \dots, \langle X, A_m \rangle).$$



Prop: For $K = \pi_A(\mathcal{H}^n_+)$, $K^* = \{ y \in \mathbb{R}^m \text{ s.t. } \sum_{j=1}^m y_j A_j \succeq 0 \}$.

(Proof) Let $\ell : \mathbb{R}^m \to \mathbb{R}$ be given by $\ell(x) = \sum_{j=1}^m x_i y_i$.

For $x = (\langle X, A_j \rangle)$, $\ell(x) = \sum_{j=1}^m \langle X, A_j \rangle y_j = \langle X, \sum_{j=1}^m y_j A_j \rangle \geq 0$ for all $X \in \mathcal{H}^n_+ \Leftrightarrow \sum y_j A_j \succeq 0$.

Prop: For $C = \pi_A(\{ X \in \mathcal{H}^n_+ : \langle X, I \rangle = 1 \})$, $C^* = \{ y \in \mathbb{R}^m : I + \sum_{j=1}^m y_j A_j \succeq 0 \}$.

(Proof) Let $\ell(x) = \sum_{j=1}^m x_j y_j$. For $x = (\langle X, A_j \rangle)_j$ where $\langle X, I \rangle = 1$,

$$1 + \ell(x) = 1 + \sum_j \langle X, A_j \rangle y_j = \langle X, I + \sum y_j A_j \rangle \geq 0 \text{ for all } X \in \mathcal{H}^n_+ \text{ with } \langle X, I \rangle = 1$$

$$\Leftrightarrow I + \sum y_j A_j \succeq 0.$$

Ex: $m_2 = (1, t, t^2)^T$ $\quad SOS_{1,4} = \pi_A(PSD_3)$ where $\langle A_j, X \rangle = \text{coeff}(m_2^T X m_2, t^j)$, $j = 0, \dots, 4$

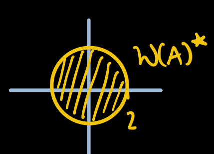
$(SOS_{1,4})^* = \left\{ y \in \mathbb{R}^5 : \begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \succeq 0 \right\}$

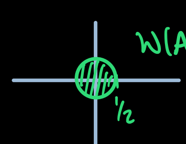
Cor $W(A)^* = \{ (x, y) : I + x Re(A) + y Im(A) \succeq 0 \}$. That is,

$W(A) = \{ a + ib : ax + by + 1 \geq 0 \ \forall x, y \text{ s.t. } I + x Re(A) + y Im(A) \succeq 0 \}$

Ex: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad I + x Re(A) + y Im(A) = \begin{pmatrix} 1 & \frac{x}{2} + \frac{y}{2i} \\ \frac{x}{2} - \frac{y}{2i} & 1 \end{pmatrix} \succeq 0 \Leftrightarrow \det = 1 - \frac{1}{4}(x - iy)(x + iy) \geq 0$

$= 1 - \frac{1}{4}(x^2 + y^2)$





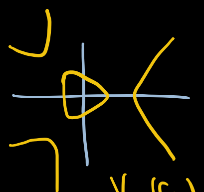
$W(A) = \{ a + ib : a^2 + b^2 \leq \frac{1}{4} \}$

Kippenhahn's Theorem:

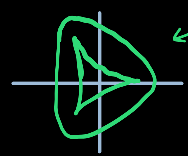
$W_A = \mathrm{conv}\{a+ib : 1+ax+by=0 \text{ is tangent to } V_{\mathbb{R}}(f_A)\}$

where $f_A = \det(I + x\,\mathrm{Re}(A) + y\,\mathrm{Im}(A))$.

Ex: $A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 4 & 0 & 0 \end{pmatrix}$ $\quad f_A = \det\begin{pmatrix} 1 & x-iy & 2x+2iy \\ x+iy & 1 & x-iy \\ 2x-2iy & x+iy & 1 \end{pmatrix} = 4x^3 - 12xy^2 - 6x^2 - 6y^2 + 1$



$V_{\mathbb{R}}(f_A)$



$\leftarrow \{(a,b) : ax+by+1=0 \text{ tangent to } V_{\mathbb{R}}(f_A) \text{ at some point } (x,y)\}$

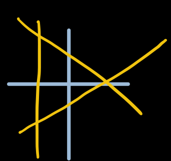
$= V_{\mathbb{R}}(g)$ where $g \in \mathbb{R}[a,b]$ has deg 6

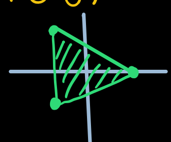
Obtained from eliminating x,y from equations:

$f=0,\quad ax+by+1=0 \quad a\frac{\partial f}{\partial y} - b\frac{\partial f}{\partial x} = 0$ Gröbner bases!

Ex: $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \bar{U}^T\begin{pmatrix} 1 & & \\ & e^{2\pi i/3} & \\ & & e^{4\pi i/3} \end{pmatrix}U$ for some unitary matrix U

$\Rightarrow f_A = (1+x)\left(1 - \frac{x}{2} + \frac{\sqrt{3}}{2}y\right)\left(1 - \frac{x}{2} - \frac{\sqrt{3}}{2}y\right)$





Q: When is a compact, convex set $C \subseteq \mathbb{C} \cong \mathbb{R}^2$ the numerical range of some matrix?

Need C to be semialgebraic

$\Rightarrow C^*$ semialgebraic

$\Rightarrow \partial C^*$ contained in $V_{\mathbb{R}}(g)$ for some $g \in \mathbb{R}[x,y]$

Prop: If $C = W(A)$ for some $A \in \mathbb{C}^{n\times n}$ and $g \in \mathbb{R}[x,y]$ is the minimal polynomial vanishing on ∂C^*, then for every $v \in \mathbb{R}^2$, $g(tv_1, tv_2) \in \mathbb{R}[t]$ is real rooted.

(Proof) If $C = W(A)$, $C^* = \{(x,y) \in \mathbb{R}^2 : I + x\,\mathrm{Re}(A) + y\,\mathrm{Im}(A) \succeq 0\}$

Let $f_A = \det(I + x\,\mathrm{Re}(A) + y\,\mathrm{Im}(A)) \in \mathbb{R}[x,y]$.

Since ∂C^* has codim 1, $\mathcal{I}(\partial C^*) = \langle g \rangle$ for some g.

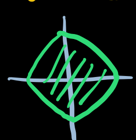
Note that f_A vanishes on ∂C^*, so g divides f_A.

For any $v \in \mathbb{R}^2$, $f_A(tv_1, tv_2) = \det(I + t(\underbrace{v_1 \mathrm{Re}(A) + v_2 \mathrm{Im}(A)}))$.

Roots are $\frac{-1}{\lambda}$ where λ is an eig val of $\uparrow$. $\in \mathcal{H}^n$

$\Rightarrow f_A(tv_1, tv_2)$ real rooted $\Rightarrow g(tv_1, tv_2)$ real rooted.

Non-ex: $f = 1 - x^4 - y^4$, $f(t,t) = 1 - 2t^4$ not real rooted.

Let $B = \{(x,y) : x^4 + y^4 \leq 1\}$ and $C = B^*$.

Then C is convex, compact, semialgebraic, but <u>not</u> the numerical range of any matrix!