Math 582G

Today: Low rank matrix completion & applications

Hwk 2 and midterm course feedback

Last time: $\mathcal{L} \subseteq \mathbb{R}^{n\times n}_{sym}$ affine linear subspace of codim m

$\mathcal{L} \cap PSD_n \neq \emptyset \Rightarrow$ contains a matrix of rank r with $\binom{r+2}{2} < m$.

<u>Barvinok's improvement</u>: If $m = \binom{r+2}{2}$ and $\mathcal{L} \cap PSD_n$ is nonempty and bounded then $\mathcal{L} \cap PSD_n$ contains a matrix of rank $\leq r$.

(Sketch of proof for $r = n-2$, $m = \binom{n}{2}$)

Let $C = \mathcal{L} \cap PSD_n$. Suppose $\mathrm{int}(C)$ nonempty.
(Otherwise $C \subseteq \mathcal{L} \cap F$ for some face $F \cong PSD_{n'}$ of PSD_n)

Then C is n-dim'l convex compact set

$\Rightarrow \partial C$ homeomorphic to S^{n-1}.

For $X \in \partial C$, $X \in \partial(PSD_n) \Rightarrow \mathrm{rank}(X) \leq n-1$.

If $\mathrm{rank}(X) = n-1$ for every $X \in \partial C$, then

$\varphi(X) = \ker(X)$ defines a continuous map

from ∂C to $\mathbb{P}^{n-1}(\mathbb{R}) = \{$lines (through 0) in $\mathbb{R}^n\}$

This map is also injective (since faces of C are char. by kernels). This contradicts the following

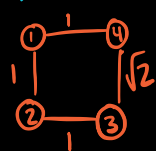
Topological fact: There is no continuous, injective map from $\mathbb{S}^{n-1}$ to $\mathbb{P}^{n-1}(\mathbb{R})$.

Application 1: Graph realizability

Let $G=([n],E)$ be a graph, $d: E \to \mathbb{R}_{\geq 0}$.

Q: Are there pts $p_1,\dots,p_n \in \mathbb{R}^r$ s.t. $\|p_i - p_j\|_2^2 = d_{ij}$ $\forall\, ij \in E$?

If yes, G is "r-realizable"

Ex: 

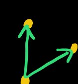
$E=\{12, 23, 34, 14\}$ realizable in $\mathbb{R}^2$, not $\mathbb{R}^1$

$\quad d \quad\;\; 1 \quad 1 \quad 2 \quad 1$

Thm: G is realizable in $\mathbb{R}^r \iff \exists\, d_{ij} \in \mathbb{R}_{\geq 0}$ for $ij \notin E$ s.t.

$$D = \left(d_{in} + d_{jn} - d_{ij}\right)_{1 \leq i,j \leq n-1} \text{ is PSD and rank } \leq r.$$

(Proof) Suppose $p_1,\dots,p_n$ is a realization in $\mathbb{R}^r$.

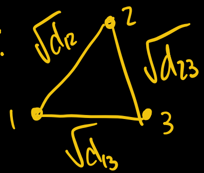


For $i=1,\dots,n-1$, take $v_i = p_i - p_n$. Then

$$d_{in} + d_{jn} - d_{ij} = \langle v_i, v_i\rangle + \langle v_j, v_j\rangle - \langle v_i - v_j, v_i - v_j\rangle$$
$$= 2\langle v_i, v_j\rangle.$$

Then $D = 2\begin{pmatrix} - v_1^T - \\ \vdots \\ - v_{n-1}^T - \end{pmatrix}\begin{pmatrix} | & & | \\ v_1 & \cdots & v_{n-1} \\ | & & | \end{pmatrix}$ is PSD of rank r.

Conversely if D is PSD of rank r, then $D = 2\cdot V^T V$ for some $V \in \mathbb{R}^{r\times(n-1)}$. Take $p_i = i^{\text{th}}$ col of V and $p_n = \bar{0}$.

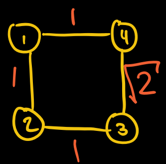
Ex: 

$$D=\begin{pmatrix} 2d_{13} & d_{13}+d_{23}-d_{12} \\ d_{13}+d_{23}-d_{12} & 2d_{23} \end{pmatrix}$$

$\det(D)=\prod\left(\sigma_1\sqrt{d_{23}}+\sigma_2\sqrt{d_{13}}+\sigma_3\sqrt{d_{12}}\right)$

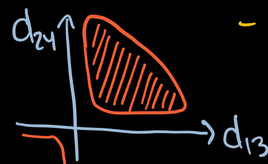
$\sigma\in\{(1,1,1),(1,1,-1),(1,-1,1),(-1,1,1)\}$

$\hookrightarrow$ triangle rule?

Ex: 

$$D=\begin{pmatrix} 2 & d_{24} & 3-d_{13} \\ d_{24} & 2d_{24} & d_{24}+1 \\ 3-d_{13} & d_{24}+1 & 4 \end{pmatrix}$$

$\det(D)=-2\left(d_{13}d_{24}^2+d_{13}^2d_{24}-5d_{13}d_{24}+1\right)$

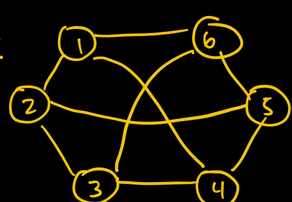


$\mathcal{L}=\left\{(d_{in}+d_{jn}-d_{ij})_{ij}\in\mathbb{R}^{n-1\times n-1}_{sym}:\ d_{ij}\in\mathbb{R}\text{ for } ij\notin E\right\}$

has $\dim=|E^c|$, $\operatorname{codim}=\binom{n}{2}-|E^c|=|E|$

Cor: If G is connected with $\leq\binom{r+2}{2}$ edges, then it is realizable iff it is realizable in $\mathbb{R}^r$.

(Note: G connected $\Rightarrow \mathcal{L}\cap PSD_n$ bounded)

Ex: 

$|E|=9\leq\binom{3+2}{2}$

realizable $\Rightarrow$ realizable in $\mathbb{R}^3$

Application 2: Quadratic Convexity thms

Cor: For any $A,B\in\mathbb{R}^{n\times n}_{sym}$, the image of $\mathbb{S}^{n-1}$ under $\phi(x)=(x^TAx,\ x^TBx)$ is convex.

(Proof) $S=\{(\langle X,A\rangle,\langle X,B\rangle): X\in PSD_n, \langle X,I\rangle=1\}$ is the image of a convex set under a linear map

$\Rightarrow$ S convex. Taking $X = xx^T$ shows S contains $\phi(\mathbb{S}^{n-1})$.

For $(a,b) \in S$, $\{X \in PSD_n : \langle X, I\rangle = 1, \langle X, A\rangle = a, \langle X, B\rangle = b\}$
is nonempty, bounded. Since $3 \le \binom{1+2}{2}$, it
contains a matrix of rank 1 $\Rightarrow$ $X = xx^T$ $x \in \mathbb{S}^{n-1}$