Math 582G

Today: Spectrahedra and matrix completion

Hwk 2 due Friday (OH?) Comb. seminar today?

Let $A_1, \ldots, A_m \in \mathbb{R}^{n\times n}_{sym}$ and $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$.

$$\mathcal{L} = \{X \in \mathbb{R}^{n\times n}_{sym} : \langle A_i, X\rangle = b_i,\ i = 1, \ldots, m\}$$ codim-m affine space

Q: What values of rank(X) appear for $X \in \mathcal{L}$?

$X \in \mathcal{L} \cap PSD_n$? X = extr. pt. of $\mathcal{L} \cap PSD_n$?

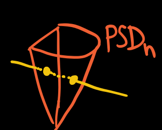
Ex ($n=m=3$) $\mathcal{L} = \{X \in \mathbb{R}^{3\times 3}_{sym} : \langle X, E_{ii}\rangle = 1 \quad i = 1,2,3\}$

Contain matrices of rank 1, 2, 3 (e.g. $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$, $\begin{pmatrix}1&-\frac12&-\frac12\\-\frac12&1&-\frac12\\-\frac12&-\frac12&1\end{pmatrix}$, $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$)

Thm: If X^* is an extreme pt of $\mathcal{L} \cap PSD_n$ and $r = \text{rank}(X^*)$, then $\binom{r+1}{2} \le m$.

(Proof) Let $r = \text{rank}(X^*)$ and $\mathcal{F}$ be the minimal face of PSD_n

$\Rightarrow \quad \mathcal{F} = \{Y \in PSD_n : \ker(Y) \supseteq \ker(X^*)\} \cong \mathbb{R}^{r\times r}_{sym}$.



Since X^* is an extreme pt of $\mathcal{L} \cap PSD_n$,

$\mathcal{F} \cap \mathcal{L} = \{X^*\} \Rightarrow \dim(\mathcal{F}) + \dim(\mathcal{L}) \le \dim(\mathbb{R}^{n\times n}_{sym})$.

Ex: $n = 4$, $m = 5$, $\binom{r+1}{2} \le 5 \Rightarrow r \le 2$.

Most pts on $\partial(\mathcal{L} \cap PSD_4)$ have 3, but can be written as convex comb. of lower rk matrices!

Thm: For a generic choice of $\mathcal{L}$, the rank r of any matrix $X \in \mathcal{L}$ satisfies

$$m + \binom{n-r+1}{2} \leq \binom{n+1}{2}$$

(Sketch of Proof) Let $\mathcal{M}_r = \{X \in \mathbb{R}^{n\times n}_{sym} : \operatorname{rank}(X) \leq r\}$

This is a real alg. set of dimension $\binom{r+1}{2} + r(n-r)$, codim $\binom{n-r+1}{2}$. If $m + \binom{n-r+1}{2} > \binom{n+1}{2}$ then a generic affine space $\mathcal{L} \subseteq \mathbb{R}^{n\times n}_{sym}$ of codim m does not intersect $\mathcal{M}_r$.

Given $n, m \in \mathbb{Z}_{>0}$, the set of $r \in \mathbb{Z}_{\geq 0}$ satisfying

$$\binom{r+1}{2} \leq m \quad \text{and} \quad m + \binom{n-r+1}{2} \leq \binom{n+1}{2}$$

is known as the Pataki interval.

These are the possible ranks of extreme pts in $\mathcal{L} \cap PSD_n$ for generic $\mathcal{L}$ of codim m.

Ex: $n=4, m=5$ $\quad \binom{r+1}{2} \leq m \Rightarrow r \leq 2$

$5 + \binom{5-r}{2} \leq \binom{4+1}{2} \Rightarrow \binom{5-r}{2} \leq 5 \Rightarrow 5-r \leq 3 \Rightarrow r \geq 2$

generically all ext pts of rk 2.

Ex: $n=4$, $m=4$, $\binom{r+1}{2} \leq m \Rightarrow r \leq 2$

$4 + \binom{5-r}{2} \leq \binom{4+1}{2} \Rightarrow \binom{5-r}{2} \leq 6 \Rightarrow 5-r \leq 4 \Rightarrow r \geq 1$

Pataki interval $= \{1, 2\}$ $\quad$ e.g. $\{X \in PSD_4 : \langle X, E_{ii} \rangle = 1\}$

Ex: $\mathcal{L} = \{X \in \mathbb{R}^{4\times 4}_{sym} : \langle X, E_{ii}\rangle = 1,\ i=1,\dots,m\}$

$\min \langle \mathbb{1}, X\rangle$ s.t. $X \in \mathcal{L} \cap PSD_4$

$= -4$, acheived by $X^* = \begin{pmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{pmatrix}$ $\leftarrow$ rk 3 $\Rightarrow$ not extreme $a = -1/3$

$$X^* = \frac{1}{3}\underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}} + \frac{1}{3}\underbrace{\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}} + \frac{1}{3}\underbrace{\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}}$$

rk 2 matrices in $\mathcal{L} \cap PSD_4$ also acheiving -4

Cor: If $m < \binom{r+2}{2}$ and $\mathcal{L} \cap PSD_n$ is nonempty then $\mathcal{L}$ contains a matrix of rank $\le r$.

(Proof) $\mathcal{L} \cap PSD_n$ contains no lines $\Rightarrow$ has an ext. pt. X^*

Ex: $f \in SOS_{n,2d}$ $\mathcal{L} = \{A \in \mathbb{R}^{N\times N}_{sym} : m_d^T A m_d = f\}$

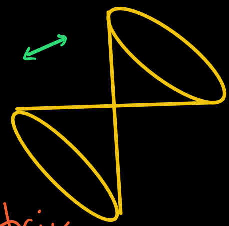
where $N = \binom{n+d}{n}$ and $m = \binom{n+2d}{n}$ coeff(, x^α) match for $|\alpha| \le 2d$

$\Rightarrow$ f is a sum of r squares $\left(f = \sum_{i=1}^{r} h_i^2\right)$ where

$\binom{n+2d}{n} < \binom{r+2}{2} \Longrightarrow r = \frac{1}{2}\left(-3 + \sqrt{9 + 8\binom{n+2d}{n}}\right)$

For fixed n, $r \approx (2d)^{n/2}$ vs. size $N \approx d^n$

Without $\mathcal{L} \cap PSD_n \neq \emptyset$ this can fail!



Ex: $(n=2, m=2)$ $\mathcal{L} = \left\{ \begin{bmatrix} 1 & b \\ b & -1 \end{bmatrix} : b \in \mathbb{R} \right\}$

$m = 2 < \binom{1+2}{2} = 3$ but $\mathcal{L}$ contains no rk-1 matrix

Note: $m \leq \binom{r+2}{2} - 1$ is tight in general.

Given $\binom{r+2}{2}$, we can force rank $\geq r+1$ by prescribing an $(r+1) \times (r+1)$ submatrix.

$$\left(\begin{array}{c|c} I_{r+1} & ? \\ \hline ? & ? \end{array} \right)$$

Next time: applications and

Barvinok's improvement: If $m = \binom{r+2}{2}$ and $\mathcal{L} \cap PSD_n$ is nonempty and bounded then $\mathcal{L} \cap PSD_n$ contains a matrix of rank $\leq r$.