Math 582G

Today: Sums of squares mod an ideal
(after Blekherman, Sinn, Smith, & Velasco)

Set up: $I = \mathcal{I}(V_{\mathbb{R}}(I)) = \langle q_1, \ldots, q_m \rangle$ where $q_i = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^T Q_i \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$

with $X = V_{\mathbb{R}}(I)$ not contained in any hyperplane.

Question: For what varieties X does every $f \in \mathbb{R}[x_0, \ldots, x_n]_2$ that is nonneg on X belong to $SOS_{n,2} + I$?

Ex: $X = \{(1, t, \ldots, t^d) : t \in \mathbb{R}\} = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R} \text{ s.t. } x_0 = 1,\ x_i x_j = x_k x_\ell \text{ whenever } i+j = k+\ell\}$

$f(x_0, \ldots, x_n) \geq 0$ for all $x \in X$

$\Leftrightarrow f(1, t, \ldots, t^d) \geq 0$ for all $t \in \mathbb{R}$

$\Leftrightarrow f(1, t, \ldots, t^d) = \sum_{i=1}^{r} h_i(t)^2$ for some $h_i \in \mathbb{R}[t]_{\leq d}$ "$\langle v_i, (1, t, \ldots, t^d) \rangle$"

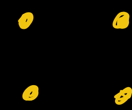
$\Leftrightarrow f(x) = \sum_{i=1}^{r} \langle v_i, x \rangle^2 + I$

Thm (BSSV '16-'20) Every nonneg. quadratic on X is belongs to $SOS_{n,2}$ if and only if X is 2-regular.

A variety is "2-regular" if for any linearspace L for which $L \cap X$ is finite, $L \cap X$ is linearly independent.

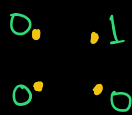
Ex: $I = \langle x(1-x),\ y(1-y) \rangle \subseteq \mathbb{R}[x,y]_{\leq 2}$

$X = V_{\mathbb{R}}(I) = \{0,1\}^2$



For any $\ell \in \mathbb{R}[x,y]_{\leq 1}$, $\ell(1,1) = \ell(1,0) + \ell(0,1) - \ell(0,0)$

Take $q = (x+y)(x+y-1)$



If $q = \sum_i \ell_i^2 + I$, then $\sum \ell_i(p)^2 = 0 \Rightarrow \ell_i(p) = 0$ for $p = \begin{matrix}(0,0)\\(1,0)\\(0,1)\end{matrix}$

$\Rightarrow \ell_i(1,1) = 0 \Rightarrow q(1,1) = 0$ $\cancel{\Rightarrow}$.

Application 1: Toric varieties

A matrix $A = (A_1, \dots, A_n) \in \mathbb{Z}_{\geq 0}^{d \times n}$ defines a monomial map $\mathbb{R}^d \to \mathbb{R}^n$ given by $t = (t_1, \dots, t_d) \mapsto (t^{A_1}, \dots, t^{A_n})$, $t^{A_i} = \prod_{j=1}^d t_j^{A_{ij}}$

Image is a variety $X_A = V(\{x^\alpha - x^\beta : \alpha - \beta \in \ker(A)\})$

Up to lattice isomorphisms of $\mathbb{Z}^n$, X_A is a 2-reg. variety

$\Leftrightarrow$ 1) $A = \{1, \dots, n\} \in \mathbb{Z}^{1 \times n}$

2) $A = \{e_1, \dots, e_n\} \subseteq \mathbb{Z}^{n \times n}$

3) $A = \{\alpha \in \mathbb{Z}_{\geq 0}^2 : 1 \leq |\alpha| \leq 2\} \subseteq \mathbb{Z}^{2 \times 6}$

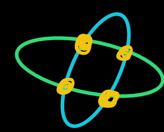
4) $A = \{(e_i, k) : \begin{matrix} i = 1, \dots, m \\ 0 \leq k \leq d_i \end{matrix}\} \subseteq \mathbb{Z}^{(m+1) \times \sum_{i=1}^m (d_i + 1)}$

Ex: 3) $A = \{(1,0), (0,1), (2,0), (1,1), (0,2)\}$

$X_A = \{ (t_1^i t_2^j)_{ij} \in \mathbb{R}^5 : (t_1,t_2) \in \mathbb{R}^2 \}$

For $\ell_1, \ell_2 \in \mathbb{R}[x_1,\dots,x_6]_{\le 1}$, $\ell_i(t^A) = 0$ defines a conic in $\mathbb{R}^2$

$V(\ell_1(t^A)) \cap V(\ell_2(t^A)) = 4$ pts

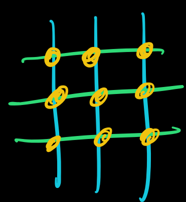


$\Rightarrow X_A \cap V(\ell_1) \cap V(\ell_2) = 4$ pts. in 3-dim'l subspace $\{\ell_1 = \ell_2 = 0\}$

Non-ex: $A = (\alpha \in \mathbb{Z}^2_{\ge 0} : 1 \le \alpha_1 + \alpha_2 \le 3) \subseteq \mathbb{Z}^{2\times 9}_{\ge 0}$

For $\ell_1, \ell_2 \in \mathbb{R}[x_1,\dots,x_9]_{\le 1}$, $\ell_i(t^A)$ defines a cubic curve in $\mathbb{R}^2$

Choose ℓ_1, ℓ_2 s.t. $V_{\mathbb{R}}(\ell_1(t^A)) \cap V_{\mathbb{R}}(\ell_2(t^A)) = 9$ pts



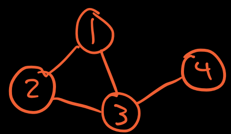
$\Rightarrow X_A \cap V_{\mathbb{R}}(\ell_1) \cap V_{\mathbb{R}}(\ell_2) = 9$ pts in $\underbrace{\{x \in \mathbb{R}^9 : \ell_1(x) = \ell_2(x) = 0\}}_{9-2=7\text{-dim'l hyperplane}}$

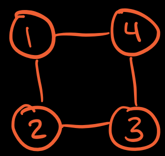
$\Rightarrow X_A \cap L$ finite,
not affinely indep.

Application 2: Unions of subspaces & PSD matrix completion

Given a graph $G = ([n], E)$, let $I_G = \langle x_i x_j : ij \notin E \rangle$.

$$X_G = V_{\mathbb{R}}(I_G) = \bigcup_{\text{clique } K \text{ of } G} \operatorname{span}\{e_i : i \in K\} \subseteq \mathbb{R}^n$$

Ex 1:  $I_G = \langle x_1x_4, x_2x_4 \rangle$ $X = \operatorname{span}\{e_1, e_2, e_3\} \cup \operatorname{span}\{e_3, e_4\}$

Ex 2:  $I_G = \langle x_1x_3, x_2x_4 \rangle$ $X = \bigcup_{i=1}^{4} \operatorname{span}\{e_i, e_{i+1}\}$ (mod 4)

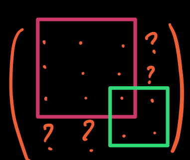
A quadratic $f(x) = x^TAx$ is ≥ 0 on X_G iff for every clique K, the submatrix $A_{K,K}$ is PSD.

It is in $SOS_{n,\leq 2} + I_G$ iff there exists $B \in PSD_n$ s.t. $A_{ij} = B_{ij}$ for all $ij \in E$.

Thm: X_G is 2-regular $\Leftrightarrow$ G is chordal

(i.e. has no induced cycles of length ≥ 4)

(recovers result by Grone, Johnson, Sá, Wolkowicz)

Ex 1:  maximal princ. minors PSD

$\Rightarrow$ has a PSD completion

Ex 2: $A = \begin{pmatrix} 1 & 1 & ? & -1 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ -1 & ? & 1 & 1 \end{pmatrix}$ $x^TAx \geq 0$ for $x \in X_{C_4}$ but

A can't be completed to a PSD matrix