Math 582G

Today: Finishing up theta bodies & software demos

Hwk 2 due Friday, Feb. 4

Theta bodies (cont' from last time)

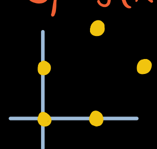
For $S \subseteq \mathbb{R}^n$, let $\mathcal{I}(S) = \{ f \in \mathbb{R}[x_1, \dots, x_n] : f(p) = 0 \ \forall p \in S \}$.

Prop: If $f \in \mathbb{R}[x_1, \dots, x_n]_{\leq d}$ only takes k distinct values on S, and $f \geq 0$ on S, then $f \in SOS_{n, 2(k-1)d} + \mathcal{I}(S)$.

(Proof) Suppose $f(S) \subseteq \{a_1, \dots, a_k\} \subseteq \mathbb{R}_{\geq 0}$. Take polynomials $p_1, \dots, p_k \in \mathbb{R}[t]_{\leq k-1}$ s.t. $p_j(a_i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ eg. $p_j(t) = \dfrac{\prod_{i \neq j} (t - a_i)}{\prod_{i \neq j} (a_j - a_i)}$

Claim: $f - \sum_{j=1}^{k} a_j (p_j(f))^2 \in \mathcal{I}(S)$

If $x \in S$, $f(x) = a_i$ for some $i \Rightarrow f(x) = a_i = \sum_{j=1}^{k} a_j (p_j(f(x)))^2$. $\square$

Ex:  $S = \{(0,0), (1,0), (0,1), (1,2), (2,1)\}$

$f(x,y) = 3 - x - y \quad \Rightarrow f(S) \subseteq \{0, 2, 3\}$

$p_0(t) = (\frac{1}{6})(t-2)(t-3) \quad p_2(t) = (\frac{-1}{2}) t (t-3) \quad p_3 = (\frac{1}{3}) t (t-2)$

$$3 - x - y = 0(p_0(3-x-y))^2 + 2(p_2(3-x-y))^2 + 3(p_3(3-x-y))^2 + h$$

$\in SOS_{2,4} + \mathcal{I}(S)$ for some $h \in \mathcal{I}(S)$

Cor: If $S = \{$vertices of a k-level polytope $P\}$ then $P = TH_{k-1}(\mathcal{I}(S))$.

That is, $\ell \geq 0$ on S for $\ell \in \mathbb{R}[x_1,\dots,x_n]_{\leq 1} \Rightarrow \ell \in SOS_{n,2(k-1)} + \mathcal{I}(S)$

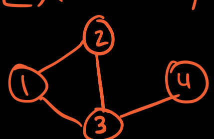
(Proof) For each facet defining ineq. $\ell \geq 0$, ℓ takes on $\leq k$ dist. values on S $\Rightarrow$ $\ell \in SOS_{n,2(k-1)} + \mathcal{I}(S)$

Theta bodies as projections of spectrahedra

$$TH_k(\mathcal{I}) = \{ p \in \mathbb{R}^n : \ell(p) \geq 0 \ \forall \ \ell \in SOS_{n,2k} + \mathcal{I} \}$$

$$= \left\{ (L(x_1),\dots,L(x_n)) : \begin{array}{l} L \in \mathbb{R}[x_1,\dots,x_n]^*_{\leq 2k},\ L(1)=1,\ L \geq 0 \text{ on } SOS_{n,2k} \\ L(h) = 0 \text{ for all } h \in \mathcal{I} \end{array} \right\}$$

Ex: $n=4$, $k=1$ $\quad \mathcal{I} = \mathcal{I}_G = \langle x_1^2 - x_1, \dots, x_4^2 - x_4,\ x_1x_2,\ x_1x_3,\ x_2x_3,\ x_3x_4 \rangle$



G $\quad L \in \mathbb{R}[x_1,\dots,x_4]^*_{\leq 2}$ $\quad L(x_i) = y_i$ $\quad L(x_ix_j) = y_{ij}$

$L(h) = 0$ for $h \in \mathcal{I} \Rightarrow L(x_i^2 - x_i) = L(x_i^2) - L(x_i) = 0 \Rightarrow L(x_i^2) = y_i$

$\hookrightarrow L(x_ix_j) = 0$ for $ij \in E$

$$TH_1(G) = \{ (y_1,y_2,y_3,y_4) \text{ s.t. } M_L \succeq 0 \} \text{ where}$$

$$M_L = L\left(\begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_4 \end{pmatrix} (1\ x_1 \dots x_4) \right) = \begin{pmatrix} L(1) & L(x_1) & L(x_2) & L(x_3) & L(x_4) \\ L(x_1) & L(x_1^2) & L(x_1x_2) & L(x_1x_3) & L(x_1x_4) \\ L(x_2) & L(x_1x_2) & L(x_2^2) & L(x_2x_3) & L(x_2x_4) \\ L(x_3) & L(x_1x_3) & L(x_2x_3) & L(x_3^2) & L(x_3x_4) \\ L(x_4) & L(x_1x_4) & L(x_2x_4) & L(x_3x_4) & L(x_4^2) \end{pmatrix} = \begin{pmatrix} 1 & y_1 & y_2 & y_3 & y_4 \\ y_1 & y_1 & 0 & 0 & y_{14} \\ y_2 & 0 & y_2 & 0 & y_{24} \\ y_3 & 0 & 0 & y_3 & 0 \\ y_4 & y_{14} & y_{24} & 0 & y_4 \end{pmatrix}$$

$\theta_1(G) = \max \sum y_i$ s.t. $M_L \succeq 0$ $\quad$ (an SDP!)

Convex hulls of curve segments

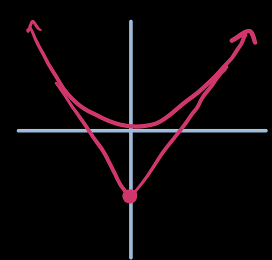
Prop: If $f\in\mathbb{R}[t]_{\le d}$ is nonneg. on $[a,b]$, then

$$f=\begin{cases}\sigma_0(t-a)+\sigma_1(b-t) \text{ for some } \sigma_0,\sigma_1\in SOS_{1,d-1} & \text{if } d \text{ odd}\\ \sigma_0+\sigma_1(t-a)(b-t) \text{ for some } \sigma_0\in SOS_{1,d},\ \sigma_1\in SOS_{1,d-2} & \text{if } d \text{ even}\end{cases}$$

Cor: $\operatorname{conv}\{(t,t^2,\dots,t^d) : t\in[a,b]\}$ equals

$$\left\{L\in\mathbb{R}[t]^*_{\le d} : L(1)=1,\ \begin{matrix}L((t-a)m_k m_k^T)\succeq 0\\ L((b-t)m_k m_k^T)\succeq 0\end{matrix}\right\} \text{ if } d=2k+1 \text{ or}$$

$$\left\{L\in\mathbb{R}[t]^*_{\le d} : L(1)=1,\ \begin{matrix}L(m_k m_k^T)\succeq 0\\ L((t-a)(b-t)m_{k-1} m_{k-1}^T)\succeq 0\end{matrix}\right\} \text{ if } d=2k$$



Ex: $(d=2)$ $P_2([-1,1]) = SOS_{1,2} + (1-t^2)\cdot\mathbb{R}_{\ge 0}$

$$\operatorname{conv}(\{(t,t^2) : t\in[-1,1]\})$$

$$=\left\{(y_1,y_2) : \begin{pmatrix}1 & y_1\\ y_1 & y_2\end{pmatrix}\succeq 0 \text{ and } (1-y_2)\succeq 0\right\}$$

$(d=3)$ $P_3([-1,1]) = (t+1)SOS_{1,2} + (1-t)SOS_{1,2}$

$$\operatorname{conv}(\{(t,t^2,t^3) : t\in[-1,1]\}) = \left\{y\in\mathbb{R}^3 : \begin{pmatrix}y_1+1 & y_2+y_1\\ y_2+y_1 & y_3+y_2\end{pmatrix}\succeq 0\ \begin{pmatrix}1-y_1 & y_1-y_2\\ y_1-y_2 & y_2-y_3\end{pmatrix}\succeq 0\right\}$$