Math 582G

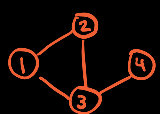
Today: Stable sets and theta bodies

Friday: online only – software demos

Stable sets in graphs

Let $G=([n],E)$ be a graph.

A subset of vertices $S\subseteq[n]$ is a stable set of G (or independent set) if $ij\notin E$ for every $i,j\in S$.

Ex:  has stable sets $\{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,4\}, \{2,4\}\}$

The maximum stable set problem is to find

$$\alpha(G) = \max\{|S| \ \text{ s.t. } S \text{ is a stable set in } G\}$$

This is equivalent to finding the size of the largest clique in the complement graph.

For general graphs this is NP-Hard

Upper bounds via polynomial optimization

Let $STAB_G = \{\mathbb{1}_S : S \text{ stable set of } G\}\subseteq\mathbb{R}^n$,

where $(\mathbb{1}_S)_i = \begin{cases} 1 & \text{if } i\in S \\ 0 & \text{if } i\notin S. \end{cases}$

Claim: $STAB_G = V_{\mathbb{R}}(\{x_i^2-x_i : i=1,\dots,n\}\cup\{x_ix_j : ij\in E\})$.

(Proof) $(\supseteq)$ Take $p\in RHS$. Then $p_i^2-p_i=0 \ \forall i \Rightarrow p\in\{0,1\}^n$.

$\Rightarrow p = \mathbb{1}_S$ for some $S \subseteq [n]$.

For any edge ij, $p_i p_j = 0 \Rightarrow \{i,j\} \not\subset S$.

Let $I_G = \left\{ \sum_{i=1}^n g_i(x_i^2 - x_i) + \sum_{ij \in E} h_{ij} x_i x_j \ : \ g_i, h_{ij} \in \mathbb{R}[x_1, \dots, x_n] \right\}$

Then

$$\alpha(G) = \max \sum_{i=1}^n x_i \quad \text{s.t. } x \in STAB_G$$

$$= \min c \quad \text{s.t. } c - \sum_{i=1}^n x_i \geq 0 \text{ on } STAB_G.$$

For $d \in \mathbb{N}$, define

$$\Theta_d(G) = \min c \quad \text{s.t. } c - \sum_{i=1}^n x_i \in SOS_{n,2d} + I_G$$

Then $\Theta_1(G) \geq \Theta_2(G) \geq \dots \geq \Theta_d(G) \geq \alpha(G)$.

Thm (Lorász) If G is perfect, then $\Theta_1(G) = \alpha(G)$.

(A graph is perfect if neither it nor its complement has induced cycles of length ≥ 5)

By duality,

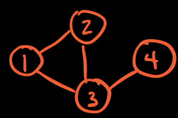
$$\Theta_1(G) = \max_{\substack{L \in \mathbb{R}[\underline{x}]^*_{\leq 2} \\ L(1) = 1}} L\left(\sum x_i\right) \quad \text{s.t. } L \geq 0 \text{ on } SOS_{n,2}, \quad L(x_i^2 - x_i) = 0,\ L(x_i x_j) = 0 \ \forall ij \in E$$

Take $y_i = L(x_i)$, $y_{ij} = L(x_i x_j)$

$$= \max \sum y_i \quad \text{s.t. } M_G(y) \succeq 0$$

where $(M_G(y))_{ij} = \begin{cases} 1 & i = j = 0 \\ y_j & i = 0,\ j \in [n] \\ y_j & i,j \in [n],\ i = j \\ y_{ij} & i,j \in [n],\ ij \notin E \\ 0 & i,j \in [n]\ ij \in E \end{cases}$

Note: For any stable set S,
$\begin{pmatrix} 1 \\ \mathbb{1}_S \end{pmatrix} (1 \ \mathbb{1}_S)$ has this form with $\sum_{i=1}^n y_i = |S|$

Ex: $G=$ 

$$M_G(y)=\begin{pmatrix} 1 & y_1 & y_2 & y_3 & y_4 \\ y_1 & y_1 & 0 & 0 & y_{14} \\ y_2 & 0 & y_2 & 0 & y_{24} \\ y_3 & 0 & 0 & y_3 & 0 \\ y_4 & y_{14} & y_{24} & 0 & y_4 \end{pmatrix}$$

Actually Lovász proved something stronger:

The 1^{st} theta body of G is

$$TH_1(G)=\{(y_1,\dots,y_n) : \exists y_{ij} \text{ with } M_G(y)\succeq 0\}$$
$$=\left\{p\in\mathbb{R}^n : \begin{array}{l} \ell(p)\geq 0 \text{ for every } \ell\in\mathbb{R}[x]_{\leq 1} \\ \text{with } \ell\in SOS_{n,2}+I_G \end{array}\right\}$$

Thm (Lovász) G is perfect if and only if $TH_1(G)=\mathrm{conv}(STAB_G)$

($\Rightarrow$ can maximize any linear function, not just $\sum x_i$)

More general framework:

A subset $I\subseteq\mathbb{R}[x_1,\dots,x_n]$ is an ideal if

(i) $0\in I$ (ii) $f,g\in I\Rightarrow f+g\in I$ (iii) $f\in I, h\in\mathbb{R}[x]\Rightarrow hf\in I$.

Hilbert basis theorem: Every ideal $I\subseteq\mathbb{R}[x]$ has the form $I=\langle f_1,\dots,f_r\rangle=\left\{\sum_{i=1}^{r}h_if_i : h_i\in\mathbb{R}[x]\right\}$.

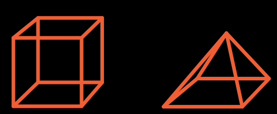
For $d\in\mathbb{Z}_+$, the d^{th} theta body of I is

$$TH_d(I)=\left\{p\in\mathbb{R}^n : \begin{array}{l} \ell(p)\geq 0 \text{ for every } \ell\in\mathbb{R}[x]_{\leq 1} \\ \text{with } \ell\in SOS_{n,2d}+I \end{array}\right\}$$

Then $TH_1(I)\supseteq TH_2(I)\supseteq\dots\supseteq TH_d(I)\supseteq\mathrm{conv}(V_{\mathbb{R}}(I))$.

Thm (Gouveia, Parrilo, Thomas '08)
Let $S \subseteq \mathbb{R}^n$ be finite and $I = \{ g \in \mathbb{R}[x_1,\dots,x_n] \text{ s.t. } g(p)=0 \ \forall p \in S\}$
Then $TH_1(I) = \mathrm{conv}(S)$ if and only if S is the set of vertices of a 2-level polytope.

A polytope P is a <u>2-level</u> if for every facet F of P all vertices of P lie in F or a unique translate $F+v$.

Ex: 

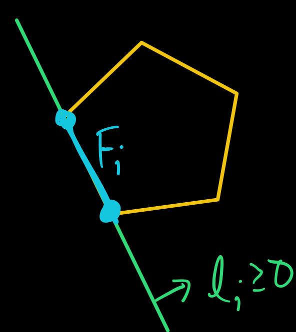
Non-ex: 

Lemma: A graph G is perfect if and only if

$$\mathrm{conv}(STAB_G) = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} x_i \geq 0 & \forall i=1,\dots,n \\ \sum_{i \in K} x_i \leq 1 & \forall \text{ cliques } K \end{array} \right\}$$

Cor: G perfect $\Rightarrow \mathrm{conv}(STAB_G)$ is a 2-level polytope

(Proof of Thm) Let $\ell_1(x) \geq 0, \dots \ell_m(x) \geq 0$ be a minimal list of affine-linear inequalities corresponding to facets $F_1,\dots,F_m$ of $\mathrm{conv}(S)$.



($\Longrightarrow$) If $TH_1(I) = \mathrm{conv}(S)$, then $\ell_i \in SOS_{n,2} + I$

$$\Rightarrow \ell_i = \sum_k h_k^2 + f \quad \text{where} \quad h_k \in \mathbb{R}[x_1,\dots,x_n]_{\leq 1}, f \in I$$

$$\Rightarrow \sum_k h_k(p)^2 = 0 \text{ for all } p \in F_i \cap S$$

$$\Rightarrow h_k(p) = 0 \text{ for all } p \in F_i(S)$$

$$\Rightarrow h_k = \lambda_k \ell_i \text{ for some } \lambda_k \in \mathbb{R}$$

$$\Rightarrow \ell_i = \underbrace{\left(\sum_k \lambda_k^2\right)}_{c_i} \ell_i^2 + f$$

$$\Rightarrow (1 - c_i \ell_i)\ell_i = f = 0 \text{ on } S$$

$$\Rightarrow S \subseteq \{x : \ell_i(x) = 0\} \cup \{x : c_i \ell_i(x) - 1 = 0\}$$

($\Leftarrow$) Reverse proof $\uparrow$ to show $\ell_i \in SOS_{n,2} + I$.

Any other affine linear function $\tilde{\ell} \geq 0$ on S is a conical combination of ℓ_i's $\Rightarrow$ in $SOS_{n,2} + I$