Math 582G - Convex Algebraic Geometry

Today: introduction & overview

TCS seminar Jan. 4, 1:30pm: Shayan Oveis Gharan
"Geometry of Polynomials and Applications to Optimization and Counting"

Main objects of study have algebraic and convex structure / connections to optimization & combinatorics

Algebraic geometry: study of subsets (of $\mathbb{R}^n$, $\mathbb{C}^n$) defined by polynomial equations and inequalities

A polynomial in variables $x_1, \dots, x_n$ over $\mathbb{R}$ has the form $\sum_{\alpha \in A} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $c_\alpha \in \mathbb{R}$ and $A \subseteq \mathbb{N}^n$ is finite

The collection of all polynomials in $x_1, \dots, x_n$ over $\mathbb{R}$ is denoted $\mathbb{R}[x_1, \dots, x_n]$. ($\mathbb{R}$-vec. space, ring, $\mathbb{R}$-alg, ...)

e.g. $1 - x^2 - y^2$, $(\pi+1)x^3y^{17} + \sqrt{3}\,xy^5 \in \mathbb{R}[x,y]$

$f \in \mathbb{R}[x_1, \dots, x_n] \longrightarrow \{p \in \mathbb{R}^n : f(p) \geq 0\}$

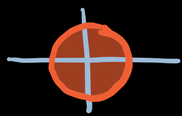
algebraic obj.   geometric object

A semialgebraic set in $\mathbb{R}^n$ is any finite Boolean combination of sets of the form $\{p \in \mathbb{R}^n : f(p) \geq 0\}$

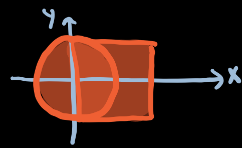
(i.e. obtained using finitely many "and", "or", "not")

intersection   union   complement

Ex's:

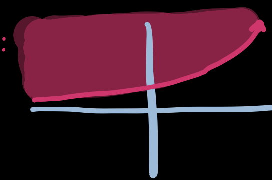


$\{(x,y)\in\mathbb{R}^2 : 1-x^2-y^2\geq 0\}$



$\{(x,y)\in\mathbb{R}^2 : 1-x^2-y^2\geq 0$ or $(0\leq x\leq 2$ and $y^2\leq 1)\}$

Non-ex:



$\{(x,y)\in\mathbb{R}^2 : y\geq e^x\}$

Bigger ex: $\{A\in\mathbb{R}^{n\times n}_{sym} : \underbrace{\text{all eig. val. of } A \geq 0}_{\text{denote by } A\succeq 0}\} = PSD_n$

"\succeq"

$$\| $$

$$\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{1n} & \cdots & & a_{nn}\end{pmatrix}\in\mathbb{R}^{n\times n}_{sym} : \det\begin{pmatrix} a_{i_1i_1} & a_{i_1i_2} & \cdots & a_{i_1i_k} \\ a_{i_2i_1} & a_{i_2i_2} & & \vdots \\ \vdots & & \ddots & \\ a_{i_ki_1} & \cdots & & a_{i_ki_k}\end{pmatrix}\geq 0 \text{ for all } \{i_1,\dots,i_k\}\subseteq[n]\right\}$$

e.g. $PSD_3 = \{A\in\mathbb{R}^{3\times 3}_{sym} : A\succeq 0\}$

$$= \left\{\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c\end{pmatrix} : \begin{array}{l} a\geq 0,\ b\geq 0,\ c\geq 0,\ ab-d^2\geq 0, \\ ac-e^2\geq 0,\ bc-f^2\geq 0, \text{ and} \\ abc-af^2-be^2-cd^2+2def\geq 0\end{array}\right\}$$

$\subseteq \mathbb{R}^{3\times 3}_{sym}\cong\mathbb{R}^6$

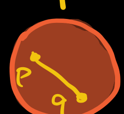
PSD_n is also convex!

<u>Convex geometry</u>: Study of convex subsets of $\mathbb{R}^n$

A set $S\subseteq\mathbb{R}^n$ is <u>convex</u> if it contains the line segment between any two of its points,

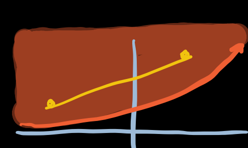
i.e. $p,q\in S \Rightarrow \lambda p+(1-\lambda)q\in S\ \ \forall\lambda\in[0,1]$

Ex:



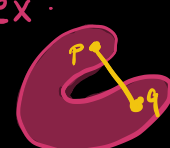
$\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}$

Ex:



$\{(x,y)\in\mathbb{R}^2 : y\geq e^x\}$

Non-ex:



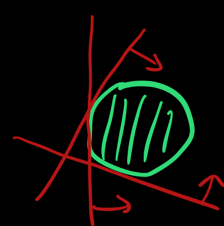
Bigger ex's: Polyhedra, PSD_n, polynomials nonnegative on $S \subseteq \mathbb{R}^n$

both convex and semialgebraic

Positive semidefinite matrices (PSD_n)

For $A \in \mathbb{R}^{n \times n}_{sym}$, TFAE:

- all eigval. of A in $\mathbb{R}_{\geq 0}$ $(A \succeq 0)$
- PSD_n is semialg. — all principal minors of $A \geq 0$
- PSD_n is convex — $v^T A v \geq 0$ for all $v \in \mathbb{R}^n$
- $A = BB^T$ for some $B \in \mathbb{R}^{n \times k}$



$$PSD_n = \{ A \in \mathbb{R}^{n \times n}_{sym} : A \succeq 0 \}$$

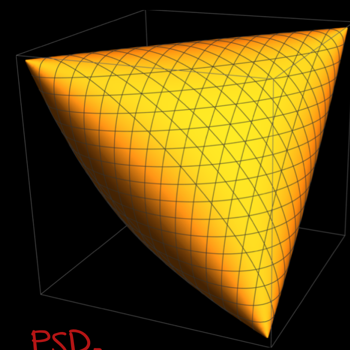
Cor: PSD_n is convex

(Proof) Let $A, B \in PSD_n$, $\lambda \in [0,1]$, $v \in \mathbb{R}^n$

$$v^T(\lambda A + (1-\lambda)B)v = \underbrace{\lambda}_{\geq 0} \cdot \underbrace{v^T A v}_{\geq 0} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{v^T B v}_{\geq 0} \geq 0$$

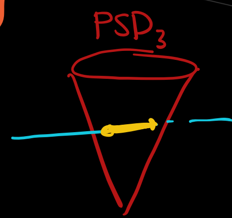
$$\Rightarrow \lambda A + (1-\lambda) B \succeq 0$$

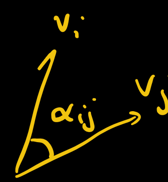
Ex: $S = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}$



$= \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} x^2 \le 1,\ y^2 \le 1,\ z^2 \le 1, \text{ and} \\ 1 - x^2 - y^2 - z^2 + 2xyz \ge 0 \end{array} \right\}$

$\cong \{ A \in PSD_3 : A_{11} = A_{22} = A_{33} = 1 \}$





Claim: $S = \{ (\cos(\alpha_{12}), \cos(\alpha_{13}), \cos(\alpha_{23})) \in \mathbb{R}^3 :$
$\exists\, v_1, v_2, v_3 \in \mathbb{R}^k$ s.t. α_{ij} = angle between $v_i, v_j \}$

(Proof) $A \in PSD_3 \iff A = BB^T$ for some $B \in \mathbb{R}^{3\times k}$

Write $B = \begin{pmatrix} — & v_1 & — \\ — & v_2 & — \\ — & v_3 & — \end{pmatrix}$. Then $A_{ij} = \langle v_i, v_j \rangle$

$\Rightarrow \|v_i\|^2 = \langle v_i, v_i \rangle = A_{ii} = 1$ for $i = 1, 2, 3$

$\Rightarrow A_{ij} = \langle v_i, v_j \rangle = \cos(\alpha_{ij})$