Math 582D – Homework 3

Due Friday, February 21, 2025

You are welcome to talk with other students in the class about problems but should write up solutions on your own. Please indicate any sources you used to find the solution to a given problem. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Friday. You should justify all your answers in order to receive full credit.

Problem 1. Suppose that P is a lattice N-gon in the plane. Let $w_1, \ldots, w_N \in \mathbb{Z}^2$ be the primitive integer vectors of the rays of the inner normal fan of P. For $i = 1, \ldots, N$, let $m_i \in \mathbb{Z}_+$ be the lattice length of the edge $\text{face}_{w_i}(P)$. Prove that $\sum_{i=1}^N m_i w_i = (0, 0)$.

Problem 2. Let $f \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ be a Laurent polynomial. Suppose that Newt(f) belongs to the affine linear space $\alpha + L$ where $\alpha \in \mathbb{Z}^n$ and L is a d-dimensional $\mathbb{Q}$-rational subspace of $\mathbb{R}^n$.

(a) Show that we can write $L^\perp$ as the rowspan of a $(n-d) \times n$ matrix A so that all integer points in $L^\perp$ have the form uA for $u \in \mathbb{Z}^{n-d}$.
(Hint: all submodules of free $\mathbb{Z}$-modules are free.)

(b) Let $\phi : (\mathbb{C}^*)^{n-d} \to (\mathbb{C}^*)^n$ be defined by $\phi(z) = z^A = (z^{A_1}, \ldots, z^{A_n})$ where $A_1, \ldots, A_n$ are the columns of A. Show that this map is one-to-one.

(c) Show that $V(f) \cap (\mathbb{C}^*)^n$ is invariant under the map $x \mapsto z^A \cdot x = (x_1 z^{A_1}, \ldots, x_n z^{A_n})$.

(d) The discriminant of a univariate cubic $p(t) = at^3 + bt^2 + ct + d$ is

$$D \quad = \quad b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

Find a polynomial $g \in \mathbb{Q}[x_1^\pm, x_2^\pm]$ for which $\langle D \rangle$ and $\langle g(ac/b^2, bd/c^2) \rangle$ are equal in $\mathbb{C}[a^\pm, b^\pm, c^\pm, d^\pm]$. Use this to describe the maximal cones of $V(\text{trop}(D)) \subset \mathbb{R}^4$ and their multiplicities.

Problem 3. Let Σ_1 and Σ_2 be the weighted polyhedral complexes supporting $V(\text{trop}(\langle x_2 - x_1^2, \ldots, x_n - x_1^n \rangle))$ and $V(\text{trop}(\langle 1 + x_1 + \ldots + x_n \rangle))$, respectively.

(a) For $n = 3$ compute the stable intersection $\Sigma_1 \cap_{\text{st}} \Sigma_2$ with multiplicities.

(b) What would you expect as the stable intersection $\Sigma_1 \cap_{\text{st}} \Sigma_2$ for general n? Why?