Math 582D – Homework 2

Due Friday, January 31, 2025

You are welcome to talk with other students in the class about problems but should write up solutions on your own. Please indicate any sources you used to find the solution to a given problem. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Friday. You should justify all your answers in order to receive full credit.

Problem 1. Let K be a field with a valuation that splits. Let $f, g \in K[x_1, \ldots, x_n]$.

(a) Show that $\mathrm{in}_w(f \cdot g) = \mathrm{in}_w(f) \cdot \mathrm{in}_w(g)$.

(b) Show that $V(\mathrm{trop}(f \cdot g)) = V(\mathrm{trop}(f)) \cup V(\mathrm{trop}(g))$.

(c) Show that if f is homogeneous, then $\mathrm{in}_w(f) = \mathrm{in}_{w+\lambda(1,\ldots,1)}(f)$ for all $\lambda \in \mathbb{R}$.

(d) For $f = \sum_\alpha c_\alpha x^\alpha \in K[x_1, \ldots, x_n]$, define its homogenezation to be

$$f^{\mathrm{hom}} = x_0^{\deg(f)} f(x_1/x_0, \ldots, x_n/x_0) = \sum_{\alpha \in A} x_0^{\deg(f)-|\alpha|} x^\alpha.$$

Show that $\mathrm{in}_{(0,w)}(f^{\mathrm{hom}}) = x_0^k (\mathrm{in}_w f)^{\mathrm{hom}}$ for some k.

(e) Suppose that the valuation on K is trivial and that $w \in (\mathbb{R}_{<0})^n$. Show that if $\{g_1, \ldots, g_s\} \subset I$ form a Gröbner basis for an ideal $I \subset K[x_1, \ldots, x_n]$ with respect to w then $I = \langle g_1, \ldots, g_s \rangle$.

Hensel's Lemma (Theorem 7.3 [E]). Let R be a ring that is complete with respect to an ideal $\mathfrak{m}$ and let $f \in R[x]$. If $\alpha \in R$ satisfies $f(\alpha) \equiv 0 \mod \mathfrak{m}$ and $f'(\alpha)$ is equivalent to a unit mod $\mathfrak{m}$ then there is a unique $\beta \in R$ with $f(\beta) = 0$ and $\alpha \equiv \beta \bmod \mathfrak{m}$.

See [E, Exercise 7.26] for a multivariate version. Examples of $(R, \mathfrak{m})$ for which R is complete with respect to $\mathfrak{m}$ include $(\Bbbk[[t]], \langle t \rangle)$ and $(\mathbb{Z}_p, \langle p \rangle)$ for and prime p, where $\mathbb{Z}_p$ is the ring of p-adic integers consisting of formal sums $\sum_{i=0}^\infty a_i p^k$ with $a_i \in \{0, \ldots, p-1\}$. Every integer has a unique such representation (with only finitely many terms). Addition and multiplication in $\mathbb{Z}_p$ coincide with the usual operations on $\mathbb{Z}$.

Problem 2.

(a) Let K be a field with a valuation that splits and suppose that the valuation ring R is complete with respect to its maximal ideal $\mathfrak{m}$. Use Hensel's lemma to show that when $w \in \Gamma_{\mathrm{val}}$ and $\mathrm{in}_w(f)$ has a root $a \in \Bbbk^*$ of multiplicity one, then $f(x)$ has a unique root $\alpha \in K$ with $\mathrm{val}(\alpha) = w$ and $\overline{t^{-w}\alpha} = a$.

(b) Show that there is a root $\alpha \in \mathbb{Q}[[t]]$ of $x^5 - x + t$ with $\mathrm{val}(\alpha) = 0$ and compute α mod $\langle t^3 \rangle$.

(c) Show that there is a square root of -1 in $\mathbb{Z}_5$ and compute it mod $\langle 5^4 \rangle$.

(d) Give an example of $f \in \mathbb{C}((t))[x]$ and $w \in \mathbb{Z} \cap V(\mathrm{trop}(f))$ for which f has no root in $\mathbb{C}((t))$ of valuation w.

References

[E] David Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Grad. Texts in Math. 150, Springer-Verlag, New York, 1995.

Problem 3. Consider the ideal $I = \langle f, g \rangle \subset \mathbb{C}((t))[x, y]$ where

$$f = t^2x^2 + xy + t^2y^2 + x + y + t^2 \quad \text{and} \quad g = 5 + 6tx + 17ty - 4t^3xy.$$

(a) Draw $V(\mathrm{trop}(f))$ and label each $w \notin V(\mathrm{trop}(f))$ with $\mathrm{in}_w(f)$.
(b) Draw $V(\mathrm{trop}(g))$ and label each $w \notin V(\mathrm{trop}(g))$ with $\mathrm{in}_w(g)$.
(c) How many points are in $V(\mathrm{trop}(f)) \cap V(\mathrm{trop}(g))$?
(d) There are four points in the variety $V(\langle f, g \rangle) \subset (\mathbb{C}((t))^*)^2$. Compute the leading terms of each, i.e., for each root (x, y), compute $\mathrm{val}(x, y)$ and $\overline{(t^{-\mathrm{val}(x)}x, t^{-\mathrm{val}(y)}y)}$.