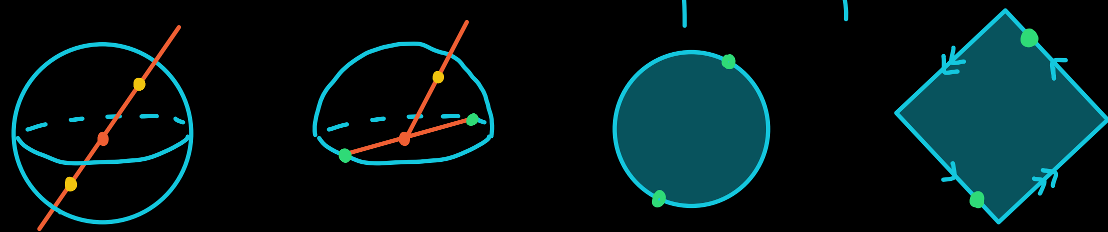
Tropical Geometry: Viro's patchworking of real algebraic curves

↖ One of the precursors to trop. geometry (Viro 1980's)

Motivation: topology of real smooth plane curves.

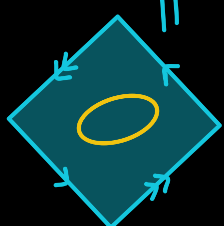
$\mathbb{P}^2(\mathbb{R}) = \{\text{lines through origin in } \mathbb{R}^3\}$
$\cong S^2$ with antipodal pts identified
$\cong$ half of S^2 with antipodal pts on boundary identified



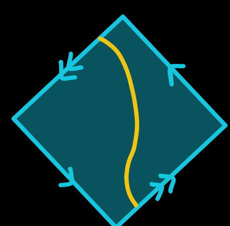
$f = \sum c_{ij} x^i y^j z^{d-i-j} \rightsquigarrow V(f)$ is a curve in $\mathbb{P}^2(\mathbb{R})$.

$V(f)$ smooth $\Rightarrow$ homeomorphic to a union of disjoint circles embedded into $\mathbb{P}^2(\mathbb{R})$

Two types of embedding



"oval"
$\mathbb{P}^2(\mathbb{R}) \setminus C$ disconnected
(disk + mobius strip)



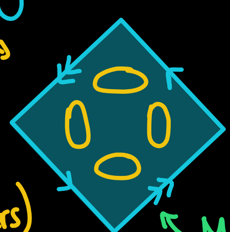
"pseudoline"
$\mathbb{P}^2(\mathbb{R}) \setminus C$ connected
(disk)

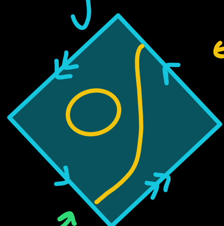
$V(f)$ smooth $\Rightarrow$ union of ovals + a pseudoline if d is odd

Hilbert's 16^{th} problem: What are the possibly topologies of smooth real alg. curves of degree d in $\mathbb{P}^2(\mathbb{R})$?

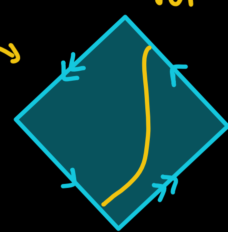
Still open for $d=8$!

← 2 possible → topologies for $d=4$ (among others)





← 2 possible → topologies for $d=3$



↖ M-curves ↗

Thm (Harnack 1876) For $f \in \mathbb{R}[x,y,z]_d$ with $V(f)$ smooth,

$$\#\text{connected components of } V(f) \subseteq \mathbb{P}^2(\mathbb{R}) \leq \binom{d-1}{2} + 1$$

Curves $V(f)$ with "$=$" are called **M-curves**

M for "maximal"

Viro's patchworking (for constructing curves with prescribed topology)

Input: (1) a regular triangulation of $\Delta_d = \mathrm{conv}\{(0,0),(d,0),(0,d)\}$ with integer vertices

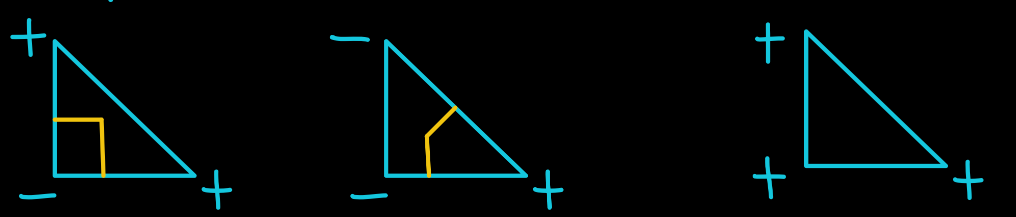
(2) assignment of signs of vertices of the triangulation

Extend to a triangulation of $\Diamond_d = \mathrm{conv}\{(i,j) : i,j \in \mathbb{Z}, |i|+|j| \leq d\}$ by reflecting across coord axes.

Extend signs by :

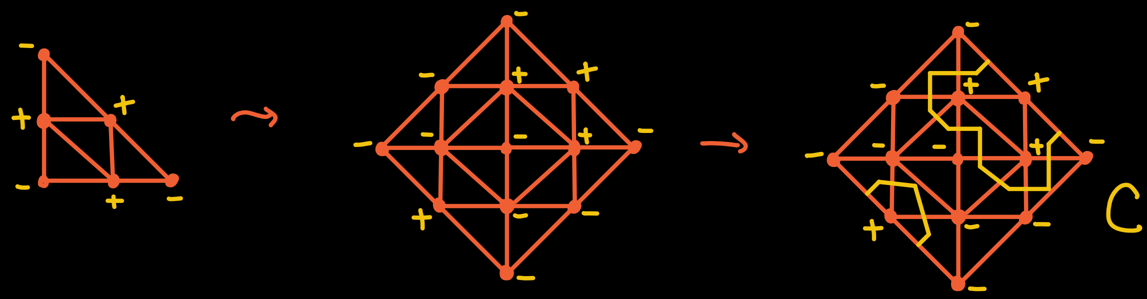
$$\sigma(-i,j) = (-1)^i \sigma(i,j)$$
$$\sigma(i,-j) = (-1)^j \sigma(i,j) \qquad \text{for } i,j \in \mathbb{Z}_{\geq 0}.$$
$$\sigma(-i,-j) = (-1)^{i+j} \sigma(i,j)$$

Let C be the subset of the dual polyhedral complex that separates $+$ from $-$.



. . .

Ex $(d=2)$



Identify opposite sides of $\Diamond_d$ gives $\overline{\Diamond_d} \cong \mathbb{P}^2(\mathbb{R})$

Let $\overline{C}$ be the image of C in this compactification.

Thm: There is a smooth curve $V(f) \subseteq \mathbb{P}^2(\mathbb{R})$ of deg. d s.t. $(\overline{C}, \overline{\Diamond_d})$ is homeomorphic to $(V(f), \mathbb{P}^2(\mathbb{R}))$.

Construction: Let $w : (\Delta_d \cap \mathbb{Z}^2) \to \mathbb{R}$ be the height function inducing our regular triangulation of Δ_d.

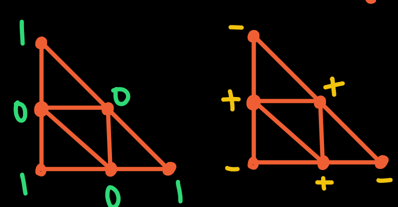
For $t \in \mathbb{R}_+$, define

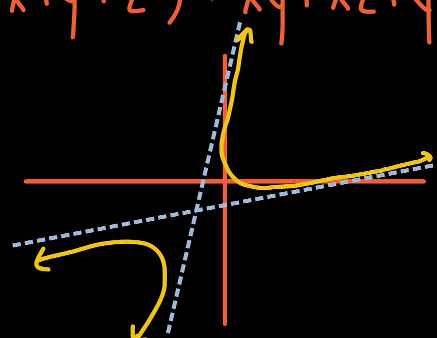
$$f_t = \sum_{(i,j) \in \Delta_d \cap \mathbb{Z}^2} \sigma(i,j)\, t^{w(i,j)} x^i y^j z^{d-i-j} \in \mathbb{R}[x,y,z]_d.$$

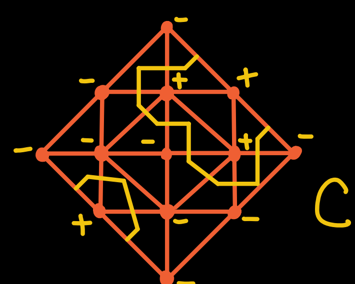
Thm: For sufficiently small $t>0$, $(V(f_t), \mathbb{P}^2(\mathbb{R}))$ is homeomorphic to $(\overline{C}, \overline{\Diamond_d})$.

(Remove coord axes to get topology in $(\mathbb{R}^*)^2$!)

Ex $(d=2)$ $\quad f_t = -t(x^2+y^2+z^2) + xy + xz + yz$







Thm: If the input to Viro's patchworking is

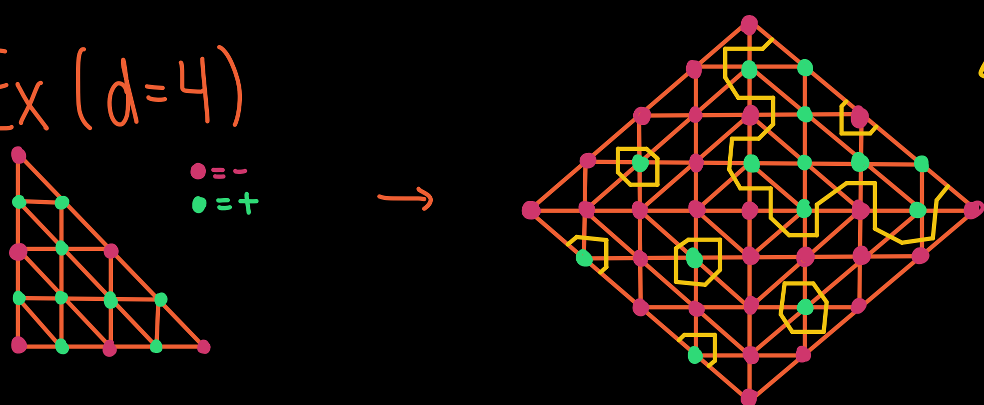
(1) a unimodular triangulation (all triangles have area $\frac{1}{2}$)

(2) signs $\sigma(i,j) = \begin{cases} - & \text{if } i,j \text{ both even} \\ + & \text{o.w.} \end{cases}$

Then the output is an M-curve.

(Idea of proof) For each interior lattice pt of Δ_d, there is a unique orthant in which it will have different signs than all of its adjacent vertices. This gives an oval in $\bar{C}$ around this pt $\Rightarrow \binom{d-1}{2}$ ovals. There is also another conn. component crossing the boundary of the positive orthant $\Rightarrow +1$ other conn. comp.

Ex ($d=4$)



$\swarrow$ $\bar{C}$ has $4 = \binom{4-1}{2} + 1$ ovals!

$\Rightarrow V(f_t)$ is an M-curve of deg 4.