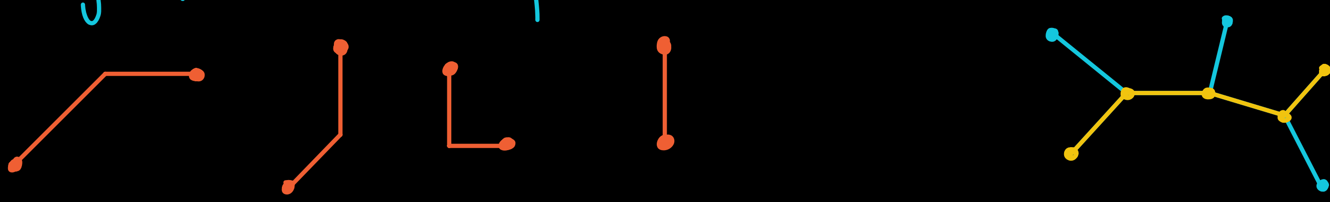
Tropical Geometry: Tropical convexity & working over $\mathbb{R}\{\{t\}\}$

Tropical convexity

Between any two pts in $\mathbb{R}^n$, there is a unique tropical line segment. It will be the concatenation of $\leq n$ ordinary line segments (These are paths between two leaves in a tree!)



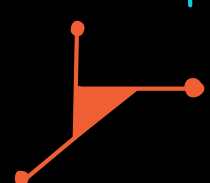
Check: the line segment between p, q is parametrized by $(a \odot p) \oplus (b \odot q)$ where $\min\{a,b\} = 0$.

Def: $S \subseteq \mathbb{R}^n$ is <u>tropically convex</u> if $\forall x, y \in S$, the line segment between x, y is contained in S. The <u>tropical convex hull</u> of $S \subseteq \mathbb{R}^n$, $\mathrm{tconv}(S)$, is the smallest tropically convex set containing S:

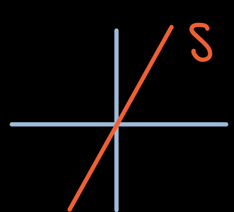
$$\mathrm{tconv}(S) = \left\{ \bigoplus_{i=1}^{k} (a_i \odot p_i) : k \in \mathbb{Z}_+,\ p_i \in S,\ a_i \in \mathbb{R},\ \bigoplus_{i=1}^{k} a_i = 0 \right\}$$

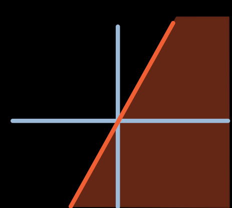
A <u>tropical polytope</u> is the trop. convex hull of finitely many pts.

Ex: $S = \{(0,1), (2,0), (-1,-2)\}$

$\mathrm{tconv}(S) =$  a tropical triangle!

Ex: $S = \{x \in \mathbb{R}^2 : x_2 = x_1\}$





$\mathrm{tconv}(S) = \{x \in \mathbb{R}^2 : x_2 \leq 2x_1\}$

not a trop. polytope!

Basic results for tropically convex sets:

- the intersection of t-convex sets is t-convex.
- t-convex sets are contractible
- tropical linear spaces are t-convex
- the projection of a t-convex set onto a coord. hyperplane is t-convex

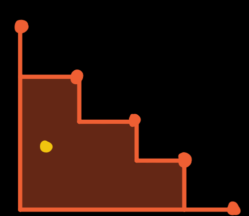
Tropical Carathéodory's Thm: If $x \in \text{tconv}(S)$ for some $S \subseteq \mathbb{R}^n$, then $\exists\, p_1, \ldots, p_{n+1} \in S$ s.t. $x \in \text{tconv}(\{p_1, \ldots, p_{n+1}\})$.

(Proof) If $x \in \text{tconv}(S)$, $x = \bigoplus_{i=1}^{k} (a_i \odot p_i)$ for some $p_i \in S$, $a_i \in \mathbb{R}$ with $\bigoplus_{i=1}^{k} a_i = 0$.
If $k \leq n+1$, done. Otherwise $k > n+1$. Note that, in $\mathbb{R}^{n+1}$,

$$(\underset{x_0}{0}, x) = \bigoplus_{i=1}^{k} (a_i \odot (\underset{p_{i0}}{0}, p_i)).$$

For each $j = 0, \ldots, n$, $\exists\, i \in \{1, \ldots, k\}$ s.t. $x_j = a_i \odot p_{ij}$. Let I be the collection of such j. Then $|I| \leq n+1$ and $x = \bigoplus_{i \in I} (a_i \odot p_i)$ with $\bigoplus_{i \in I} a_i = 0$.

Ex: $S = \{(i, -i) : i = 1, \ldots, 5\}$



$x = (3/2, -7/2)$

$= (\tfrac{1}{2} \odot (1, -1)) \oplus (1 \odot (2, -2)) \oplus (0 \odot (3, -3)) \oplus (\tfrac{1}{2} \odot (4, -4)) \oplus (\tfrac{3}{2} \odot (5, -5))$

$= (\tfrac{1}{2} \odot (1, -1)) \oplus (0 \odot (3, -3)) \oplus (\tfrac{3}{2} \odot (5, -5))$ with $\tfrac{1}{2} \oplus \tfrac{3}{2} \oplus 0 = 0$

Remark: there's no difference between tropical "convex hull" and "affine span" or between "conical hull" and linear span!

$\text{aff span}(S) = \{\sum \lambda_i p_i : p_i \in S, \lambda_i \in K, \sum \lambda_i = 1\}$

$\xrightarrow{\text{trop}} \{\bigoplus_{i=1}^{k} a_i p_i : p_i \in S, a_i \in \mathbb{R}, \bigoplus_i a_i = 0\}$

Tropically, constraints $\lambda_i \geq 0$ disappear.

$\text{span}(S) = \{\sum \lambda_i p_i : p_i \in S, \lambda_i \in K\} \xrightarrow{\text{trop}} \{\bigoplus_{i=1}^{k} a_i p_i : a_i \in \mathbb{R}, p_i \in S\}$

Connection to classical convexity/polytopes

Take $K = \mathbb{R}\{\{t\}\}$ $K_+ = \{a \in \mathbb{R}\{\{t\}\} : a t^{-\text{val}(a)} \in \mathbb{R}_{>0}\}$

Remark: For $a, b \in K_+$, $\text{val}(a+b) = \min\{\text{val}(a), \text{val}(b)\}$ no cancellation of leading terms!

Prop: For any $S \subseteq K_+^n$, $\text{val}(\text{conv}(S)) = t\text{-conv}(\text{val}(S)) \cap (\Gamma_{\text{val}})^n$.

(Proof) ($\subseteq$) $x \in \text{conv}(S) \Rightarrow x = \sum_{i=1}^{k} \lambda_i q_i$ where $q_i \in S$, $\lambda_i \in K_+$, $\sum_i \lambda_i = 1$.

Then $\text{val}(x) = \bigoplus_{i=1}^{k} (\text{val}(\lambda_i) \odot \text{val}(q_i))$ where $\bigoplus_{i=1}^{k} \text{val}(\lambda_i) = 0$.

($\supseteq$) $w \in t\text{-conv}(\text{val}(S)) \cap (\Gamma_{\text{val}})^n \Rightarrow w = \bigoplus_{i=1}^{k} (a_i \odot p_i)$ with $p_i = \text{val}(q_i)$ with $q_i \in S$ and $\bigoplus_{i=1}^{k} a_i = 0$. We can take $a_i \in \Gamma_{\text{val}}$ and $\lambda_i = t^{a_i} / \sum_{i=1}^{k} t^{a_i}$.

Note: $\text{val}(\sum_{i=1}^{k} t^{a_i}) = \bigoplus_{i=1}^{k} a_i = 0$ and $\text{val}(\lambda_i) = a_i$.

Then $\sum_{i=1}^{k} \lambda_i q_i \in \text{conv}(S)$ and $\text{val}(\sum_{i=1}^{k} \lambda_i q_i) = \bigoplus_{i=1}^{k} (a_i \odot p_i) = w$.

Moreover the map $K_+ \to \mathbb{R}$ given by $a \mapsto -\text{val}(a)$ is order preserving: $a < b$ in $K_+ \Rightarrow -\text{val}(a) \leq -\text{val}(b)$.

e.g. $a = a_v t^v + \text{h.o.t.}$ $a \in \mathbb{R}_{>0}$, $b = b_w t^w + \text{h.o.t.}$ $b \in \mathbb{R}_{>0}$,

$a < b \Leftrightarrow b - a \in K_+ \Leftrightarrow w < v$ or $w = v$ and $b_w > a_v$.

Highlights of trop. convexity

- Tropical linear programming

(Classical LP) $\min \sum_{i=1}^{n} c_i x_i$ s.t. $x \in P$, P polytope in K_+^n

(Tropical LP) $\min \bigoplus_{i=1}^{n} c_i \odot x_i$ s.t. $x \in P$, P tropical polytope

Works of Allamigeon, Benchimol, Gaubert, Joswig (+)
build up theory of tropical LP, transfer complexity statements.

Thm (ABGJ 2014) Log-barrier methods for solving linear programs are not strongly polynomia time.

Complexity must depend on bit size of coefficients

Proof involves lifting tropical LPs

- Any tropical polytope is the union of finitely many polytropes (=sets that are both classical and tropical polytopes)

Thm 5.2.19 The combinatorial types of $\mathrm{tconv}(p_1, \ldots, p_r) \subseteq \mathbb{R}^n$ are in natural bijection with regular poly. subdiv. of $\Delta_n \times \Delta_{r-1}$.