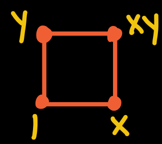
Tropical Geometry: Bernstein's Theorem and hints of tropical convexity.

Bernstein's Thm Given lattice polytopes $P_1, \dots, P_n \subseteq \mathbb{R}^n$ and generic polynomials $f_1, \dots, f_n \in K[x_1, \dots, x_n]$ with $\mathrm{Newt}(f_i) = P_i$,

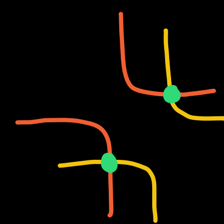
$$|V(f_1, \dots, f_n) \cap (K^*)^n| = MV(P_1, \dots, P_n).$$

Ex: $P_1 = P_2 = [0,1]^2$



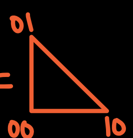
$a_{11}xy + a_{01}y + a_{10}x + a_{00} = 0$

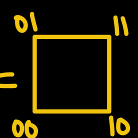
$b_{11}xy + b_{01}y + b_{10}x + b_{00} = 0$



$|V(f_1, f_2) \cap (K^*)| = 2$

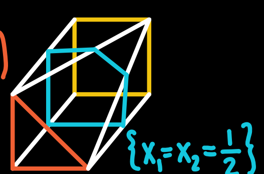
Recall: Lemma 4.6.6. Given any triangulation of $C(P_1, \dots, P_n)$, the mixed volume $MV(P_1, \dots, P_n)$ is the sum of the volumes of the mixed cells in the induced subdiv. of $P_1 + \dots + P_n$.

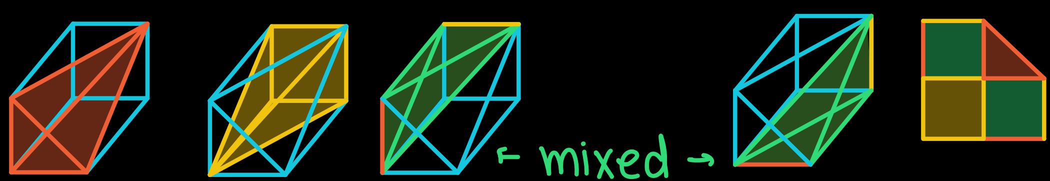
Ex: $P_1 =$ 

$P_2 =$ 

$C(P_1, P_2) = \mathrm{conv}(e_1 \times P_1, e_2 \times P_2) \subseteq \{x_1 + x_2 = 1\} \subseteq \mathbb{R}^4$

$C(P_1, P_2)$





Thm 4.6.8. (Tropical Bernstein)

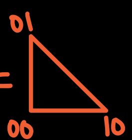
Let $P_1, \dots, P_n$ be lattice polytopes and $\Sigma_1, \dots, \Sigma_n$ the associated tropical hypersurfaces (codim-1 part of inner normal fan). Then the stable intersection $\Sigma_1 \cap_{st} \dots \cap_{st} \Sigma_n$ is the origin, with multiplicity $MV(P_1, \dots, P_n)$.

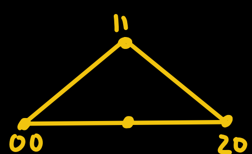
(Idea) In $(\varepsilon v_1+\Sigma_1)\cap\ldots\cap(\varepsilon v_n+\Sigma_n)$, intersections points have the form $(\varepsilon v_1+\sigma_1)\cap\ldots\cap(\varepsilon v_n+\sigma_n)$ where each σ_i is dual to an edge Q_i of P_i. If $q_i\in\mathbb{Z}^n$ is the primitive vector in dir. Q_i and m_i is the lattice length of Q_i then

$$\mathrm{vol}(Q_1+\ldots+Q_n)=m_1\cdots m_n\cdot\left|\det\begin{pmatrix}| & & |\\ q_1 & \cdots & q_n\\ | & & |\end{pmatrix}\right|$$

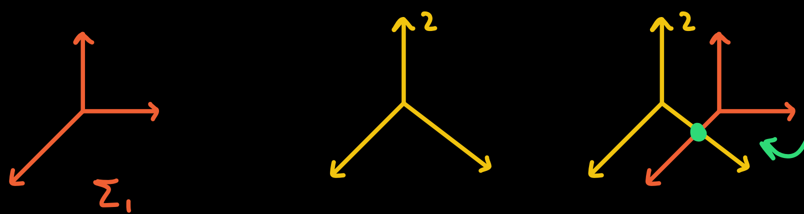
Also m_i = multiplicity of σ_i in Σ_i. So the intersection mult. assigned to this pt is

$$m_1\cdots m_n\cdot[\mathbb{Z}^n : L_{\sigma_1}+\ldots+L_{\sigma_n}] = m_1\cdots m_n[\mathbb{Z}^n:\mathbb{Z}q_1+\ldots+\mathbb{Z}q_n]$$
$$= m_1\cdots m_n\cdot\left|\det\begin{pmatrix}| & & |\\ q_1 & \cdots & q_n\\ | & & |\end{pmatrix}\right|.$$

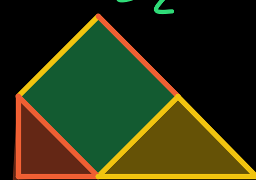
Ex: $P_1=$ 

$P_2=$ 

$MV(P_1,P_2)=2$



int mult $=m_1 m_2\cdot|\det(q_1\ q_2)|$
$=1\cdot 1\cdot\left|\det\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\right|$
$=2$



Thm (Prop 4.6.10, Cor 4.6.11) If K has the trivial valuation, then for generic polynomials $f_1,\ldots,f_r$ with $\mathrm{Newt}(f_i)=P_i$, $\mathrm{trop}(V(f_1,\ldots,f_n))=\mathrm{trop}(V(f_1))\cap_{st}\ldots\cap_{st}\mathrm{trop}(V(f_r))$.

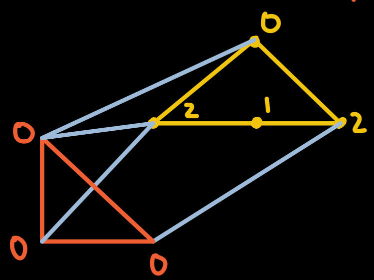
$\Rightarrow$ Bernstein's Thm

Another way to see Bernstein Thm:

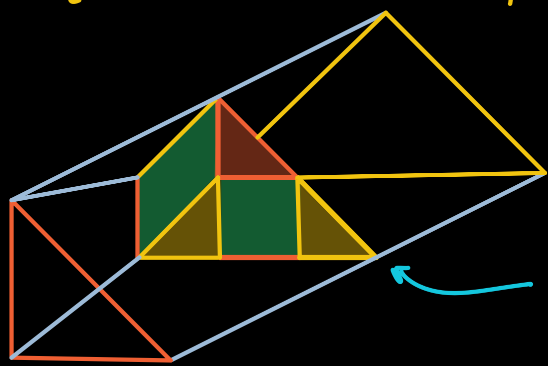
Thm 4.6.18 Let $f_1,\dots,f_r \in K[x_1,\dots,x_n]$ with $\mathrm{Newt}(f_i)=P_i$. If the valuations of the coeff of the f_i's induces a triangulation of $C(P_1,\dots,P_r)$, then $\{f_1,\dots,f_r\}$ forms a tropical basis for $\langle f_1,\dots,f_r\rangle$ i.e.

$$V(\mathrm{trop}(\langle f_1,\dots,f_r\rangle)) = V(\mathrm{trop}(f_1)) \cap \dots \cap V(\mathrm{trop}(f_r)).$$

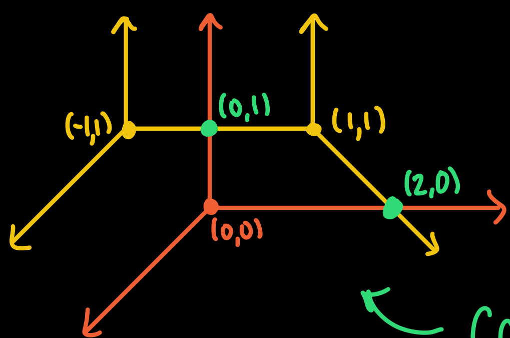
Ex: $f_1 = 1+x+y$ $\quad f_2 = t^2+tx+t^2x^2+2xy$



$C(P_1,P_2)$



induced mixed subdiv. of P_1+P_2



dual to $V(\mathrm{trop}(f_1)) \cup V(\mathrm{trop}(f_2))$!

Can solve for initial terms of pts in $V(f_1,\dots,f_n)$ by solving binomial equations $\mathrm{in}_w(f_1)=\dots=\mathrm{in}_w(f_n)=0$.

$w=(2,0)$ $\quad \mathrm{in}_w(f_1)=1+y,\ \mathrm{in}_w(f_2)=1+2xy \quad \Rightarrow (x,y)=(1/2,-1)$

pt in $V(f_1,f_2) = \left(\frac{1}{2}t^2 + \text{h.o.t},\ -1t^0+\text{h.o.t.}\right)$

Next week: tropical convexity and tropicalizations over $\mathbb{R}\{\{t\}\}$.

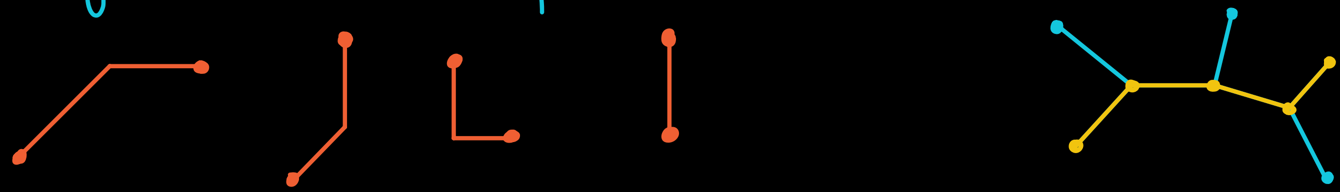
For $K=\mathbb{R}\{\{t\}\}$, we can define $K_+ = \{a \in K^* : \overline{t^{-\mathrm{val}(a)}a} \in \mathbb{R}_{>0}\}$

leading coeff >0

Useful observation: For $a, b \in K_+$, $\mathrm{val}(a+b) = \min\{\mathrm{val}(a), \mathrm{val}(b)\}$.
Leading terms are positive $\Rightarrow$ can't cancel
This will have nice consequences for tropicalizing subsets of K_+^n, like polytopes!

Tropical convexity

Between any two pts in $\mathbb{R}^n$, there is a unique tropical line segment. It will be the concatenation of $\leq n-1$ ordinary line segments (These are paths between two leaves in a tree!)



Check: the line segment between p, q is parametrized by $(a \odot p) \oplus (b \odot q)$ where $\min\{a, b\} = 0$.

Def: $S \subseteq \mathbb{R}^n$ is <u>tropically convex</u> if $\forall x, y \in S$, the line segment between x, y is contained in S. The <u>tropical convex hull</u> of $S \subseteq \mathbb{R}^n$ is the smallest tropically convex set containing S.

This will let us consider tropical versions of polytopes, and many convexity thms (Farkas' Lemma, Carathéodory...)