Tropical Geometry: Complete Intersections and Bernstein's Theorem

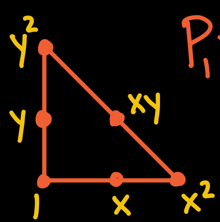
Q: Given polytopes $P_1, \dots, P_n \subseteq \mathbb{R}^n$ and generic polynomials $f_1, \dots, f_n \in K[x_1, \dots, x_n]$ with $\text{Newt}(f_i) = P_i$, how many points are in

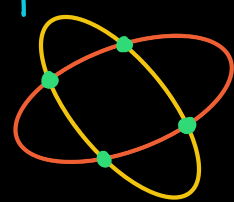
$$V(f_1, \dots, f_n) \cap (K^*)^n \ ?$$

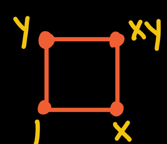
Assumption: K alg. closed. Answer doesn't depend on K!

(Bezout's Thm) For f_i generic of deg d_i,

$$V(f_1, \dots, f_n) \cap (K^*)^n \text{ has } d_1 \cdot d_2 \cdots d_n \text{ points.}$$

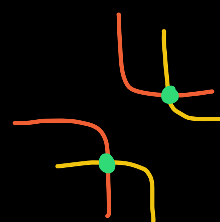
Ex ($d_1 = d_2 = 2$) 

$P_1 = P_2$ $\quad |V(f_1, f_2)| = 4$ 

Ex: $P_1 = P_2 = [0,1]^2$ 

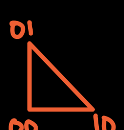
$|V(f_1, f_2) \cap (K^*)| = 2$

$$a_{11}xy + a_{01}y + a_{10}x + a_{00} = 0$$
$$b_{11}xy + b_{01}y + b_{10}x + b_{00} = 0$$



In general, the answer will be the mixed volume of $P_1, \dots, P_n$. (Bernstein's Thm)

The normalized volume of a lattice polytope $P \subset \mathbb{R}^n$, $\text{nvol}(P)$ is the usual Euclidean multiplied by $n!$

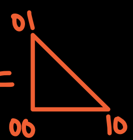
Ex: nvol(

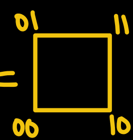
$) = 1$ $\quad$ nvol( $) = 2$ $\quad$ nvol( $) = 6$

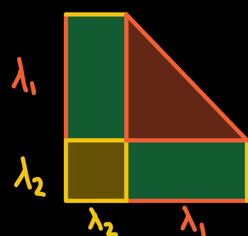
$\subseteq \mathbb{R}^2 \qquad \subseteq \mathbb{R}^2 \qquad \subseteq \mathbb{R}^3$

The <u>Minkowski sum</u> of $P_1,\dots,P_r$ is $P_1+\dots+P_r = \{q_1+\dots+q_r : q_i \in P_i\}$.

<u>Prop 4.3.6</u>: Let $P_1,\dots,P_r$ be lattice polytopes in $\mathbb{R}^n$.
The function $(\lambda_1,\dots,\lambda_r) \mapsto \mathrm{nvol}(\lambda_1P_1+\dots+\lambda_rP_r)$ on $\mathbb{R}^r_{\geq 0}$
agrees with a polynomial in $\lambda_1,\dots,\lambda_r$ that is homogeneous of deg. n with nonnegative integer coefficients.

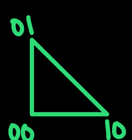
Ex: $P_1 =$ 

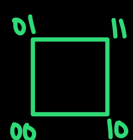
$P_2 =$ 



$\mathrm{nvol}(\lambda_1P_1+\lambda_2P_2)$
$= \lambda_1^2 + 4\lambda_1\lambda_2 + 2\lambda_2^2$

The <u>mixed volume</u> $MV(P_1,\dots,P_n)$ of $P_1,\dots,P_n \subseteq \mathbb{R}^n$
is $\frac{1}{n!}$ times the coeff of $\lambda_1\cdots\lambda_n$ in $\mathrm{nvol}(\lambda_1P_1+\dots+\lambda_nP_n)$.

Ex: $MV($ 

, 

$) = 2$

Equiv def: $MV(P_1,\dots,P_n)$ is the unique function satisfying

(1) $MV(P,\dots,P) = \mathrm{nvol}(P)$

(2) $MV(P_1,\dots,P_n) = MV(P_{\pi(1)},\dots,P_{\pi(n)})$ for $\pi \in S_n$ (symmetry)

(3) $MV(aP_1+bQ_1,P_2,\dots,P_n) = a\,MV(P_1,\dots,P_n) + b\,MV(Q_1,P_2,\dots,P_n)$ (multi-linearity)

Mixed subdivisions

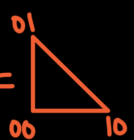
The Cayley polytope of polytopes $P_1,\dots,P_r \subseteq \mathbb{R}^n$ is
$$C(P_1,\dots,P_r) = \mathrm{conv}(e_1\times P_1,\dots,e_r\times P_r) \subseteq \mathbb{R}^{r+n}$$

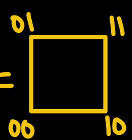
For $\lambda_1,\dots,\lambda_r \in \mathbb{R}_{\geq 0}$, $\lambda_1 P_1 + \dots + \lambda_r P_r$ is affinely isomorphic to

$$C(P_1,\dots,P_n) \cap \left\{ x_i = \frac{\lambda_i}{\sum \lambda_i} \ \forall i = 1,\dots,r \right\}$$

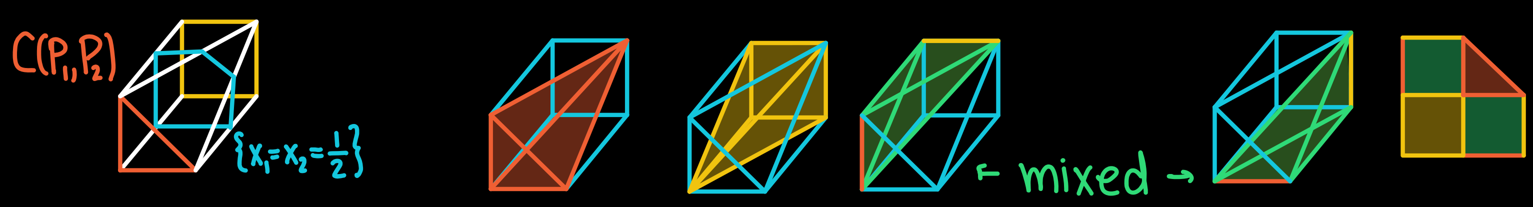
A polyhedral subdivision of $C(P_1,\dots,P_n)$ induces a polyhedral subdivision of $\lambda_1 P_1 + \dots + \lambda_r P_r$, called a <u>mixed subdivision</u> of $\lambda_1 P_1 + \dots + \lambda_r P_r$. For any cell Q of a subdiv. of $C(P_1,\dots,P_r)$, write Q_i for $Q \cap \{x_i = 1\}$ ($=$ conv hull of vertices $e_i \times v \in Q$)

Then $Q \cap \{x_1 = \dots = x_n = \frac{1}{r}\} \cong Q_1 + \dots + Q_r$. This is a <u>mixed cell</u> of the subdiv. of $C(P_1 + \dots + P_r)$ (resp. $P_1 + \dots + P_r$) if $\dim(Q_i) \geq 1 \ \forall i$.

Ex: $P_1 =$ 

$P_2 =$ 

$C(P_1, P_2) = \mathrm{conv}(e_1 \times P_1, e_2 \times P_2) \subseteq \{x_1 + x_2 = 1\} \subseteq \mathbb{R}^4$



Lemma 4.6.6. Given any triangulation of $C(P_1,\dots,P_n)$, the mixed volume $MV(P_1,\dots,P_n)$ is the sum of the volumes of the mixed cells in the induced subdiv. of $P_1 + \dots + P_n$.

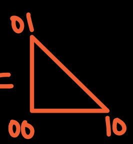
Cor: For lattice polytopes $P_1,\dots,P_n \subseteq \mathbb{R}^n$, $MV(P_1,\dots,P_n) \in \mathbb{Z}_{\geq 0}$.

Idea: Every mixed cell of $P_1,\dots,P_n$ induced by a triangulation of $C(P_1,\dots,P_n)$ has the form $Q_1 + \dots + Q_n$, where Q_i is an edge of P_i.
If $Q_i = \mathrm{conv}(v_i, w_i)$, $\mathrm{vol}(Q_1 + \dots + Q_n) = |\det(w_1 - v_1 \cdots w_n - v_n)| \in \mathbb{Z}^{n}_{\geq 0}$.

Next time:

Thm 4.6.8. (Tropical Bernstein)

Let $P_1, \dots, P_n$ be lattice polytopes and $\Sigma_1, \dots, \Sigma_n$ the associated tropical hypersurfaces (codim-1 part of inner normal fan). Then the stable intersection $\Sigma_1 \cap_{st} \dots \cap_{st} \Sigma_n$ is the origin, with multiplicity $MV(P_1, \dots, P_n)$.

Ex: $P_1 =$ 

$P_2 =$