Tropical Geometry: Tropical linear spaces and the Dressian $Dr(d,n)$

Last time: $\mathrm{trop}(Gr(d,n)) = V(\mathrm{trop}(I_{d,n})) \subseteq \mathbb{R}^{\binom{[n]}{d}}$

$$I_{d,n} = \left\langle \sum_{j \in J} \mathrm{sgn}(j,I,J)\, p_{I \cup j}\, p_{J \setminus j} : I \in \binom{[n]}{d-1},\ J \in \binom{[n]}{d+1} \right\rangle$$

quadratic Plücker relations

Points in $Gr(d,n) \longleftrightarrow$ d-dim'l linear subspaces on K^n

$$p = (p_I) \in Gr(d,n) \longleftrightarrow \left\{ x \in K^n : \sum_{l=1}^{d+1} (-1)^l p_{J \setminus j_l} x_{j_l} = 0 \ \forall J \in \binom{[n]}{d+1} \right\}$$

For $A \in K^{d \times n}$ $p_I = \det(A_I)$, $x \in rs(A) \Leftrightarrow \mathrm{rank}\begin{pmatrix} - x - \\ A \end{pmatrix} \leq d$

$\Leftrightarrow (d+1)\times(d+1)$ minors of $\begin{pmatrix} - x - \\ A \end{pmatrix} = 0$

Points in $\mathrm{trop}(Gr(d,n)) \leftrightarrow$ d-dim'l tropicalized linear spaces (in $\mathbb{R}^n$)

For $w \in \mathrm{trop}(Gr(d,n))$, define $L_w = \bigcap_{J \in \binom{[n]}{d+1}} V\left(\bigoplus_{j \in J} (w_{J \setminus j} \odot u_j) \right)$

$$= \left\{ u \in \mathbb{R}^n : \forall J \in \binom{[n]}{d+1},\ \min_{j \in J} \{ w_{J \setminus j} + u_j \} \text{ is attained} \geq \text{twice} \right\}$$

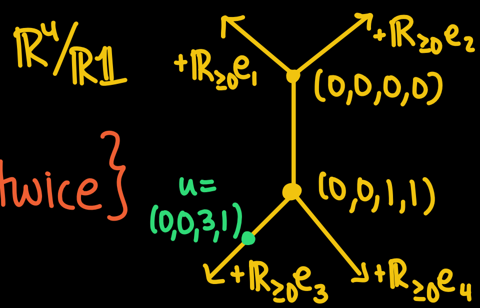
circuits still form a tropical basis

Thm 4.3.17: The map $w \mapsto L_w$ gives a bijection between $\mathrm{trop}(Gr(d,n)) \cap \Gamma_{val}^{\binom{n}{d}-1}$ (in $\mathbb{R}^{\binom{n}{d}}/\mathbb{R}\mathbb{1}$) and the set of tropicalized linear spaces in $\mathbb{R}^n$ of dim d whose underlying matroid is the uniform matroid $\left(\{\text{bases}\} = \binom{[n]}{d} \right)$.

Ex: $A=\begin{pmatrix}1&0&1&1\\0&1&1&1+t\end{pmatrix} \rightsquigarrow (p_{12},p_{13},p_{14},p_{23},p_{24},p_{34})=(1,1,1+t,-1,-1,t)$

$w=(w_{12},w_{13},w_{14},w_{23},w_{24},w_{34})=(0,0,0,0,0,1)$

$L_w=\{u\in\mathbb{R}^4:\ \min\{u_1,u_2,u_3\},\ \min\{u_1,u_2,u_4\},$
$\min\{1+u_1,u_3,u_4\},\ \min\{1+u_2,u_3,u_4\}$ attained $\geq$ twice$\}$



The <u>Dressian</u> $Dr(d,n)$ is the tropical prevariety of the quadratic Plücker relations

$$Dr(d,n)=\bigcap_{I\in\binom{[n]}{d-1},\ J\in\binom{[n]}{d+1}} V\Big(\operatorname{trop}\Big(\sum_{j\in J}\operatorname{sgn}(j,I,J)\,p_{I\cup j}\,p_{J\setminus j}\Big)\Big)$$

Note: $Gr(d,n)\subseteq Dr(d,n)$ but these are not equal in general for $d\geq 2$.
(The quad. Plücker relations are not always a tropical basis!)

For $w\in Dr(d,n)$, define $L_w=\bigcap_{J\in\binom{[n]}{d+1}} V\Big(\bigoplus_{j\in J}(w_{J\setminus j}\odot u_j)\Big)$

$$=\Big\{u\in\mathbb{R}^n:\ \forall J\in\tbinom{[n]}{d+1},\ \min_{j\in J}\{w_{J\setminus j}+u_j\}\text{ is attained}\geq\text{twice}\Big\}$$

We call L_w a <u>tropical linear space</u>.*

<u>Thm 4.4.5</u> For $w\in Dr(d,n)$, L_w is a pure r-dim'l, balanced polyhedral complex in $\mathbb{R}^n$.

* For any matroid M, one can define a Dressian $Dr(M)$. An arbitrary tropical linearspace has the form L_w for w in some $Dr(M)$. See §4.4.

Connection with subdivisions:

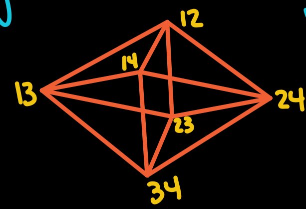
For a matroid $M=([n],\mathcal{C})$, $\mathcal{C}$ = "circuits" = min dependent sets

$\mathcal{B}$ = "bases" = max indep sets = max elt of $\{I \subseteq [n] : C \not\subseteq I \ \forall C \in \mathcal{C}\}$

<u>Thm 4.2.12</u> (Gelfand, Goresky, MacPherson, Serganova '87)

M is a matroid $\iff$ all edges of $\mathrm{conv}\{e_B : B \in \mathcal{B}\}$ are parallel to $e_i - e_j$ for some $i,j \in [n]$.

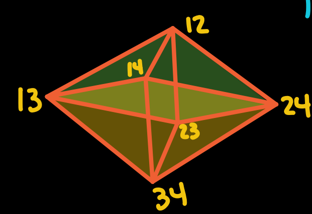
Ex: $M=([4],\mathcal{C})$ $\mathcal{C}=\binom{[4]}{3}$ $\mathcal{B}=\binom{[4]}{2}$



A vector $w \in \mathbb{R}^{\binom{[n]}{d}}$ induces a regular subdivision Δ_w of the hypersimplex $\Delta_{d,n} = \mathrm{conv}\{e_I : I \in \binom{[n]}{d}\}$

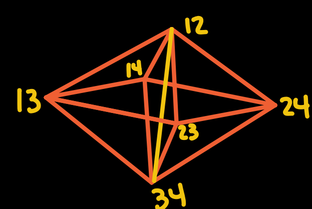
Thm (Lemma 4.4.6) $w \in Dr(d,n) \iff$ every cell of Δ_w is a matroid polytope.

$(d=2, n=4)$ Ex:



$w = (w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}) = (0, 0, 0, 0, 0, 1)$

Non-ex:



$w = (w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}) = (0, 1, 1, 1, 1, 0)$

Elements of $Dr(d,n)$ are also called valuated matroids

For M = uniform matroid, $w \in Dr(d,n)$ and $u \in \mathbb{R}^n$,
define the <u>intial matroid</u> M_u^w to be the matroid with
Circuits $\mathcal{C} = \{ j \in J$ attaining min in $\bigoplus_{j \in J} (w_{J \setminus j} \odot u_j) \}$ for $J \in \binom{[n]}{d+1}$.

For $w \in trop(Gr(d,n))$, $\mathcal{C}$ = supports of $in_u(\ell_J)$ of circuits ℓ_J

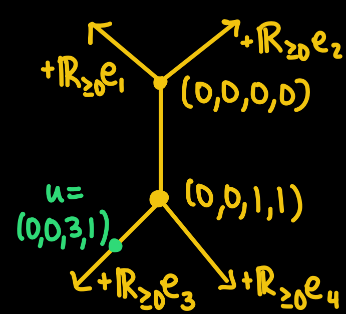
Cor: $L_w = \{ u \in \mathbb{R}^n : M_u^w$ has no loops $\}$
= Circuits of size 1.

The matroid polytope of M_u^w is the cell in Δ_w
minimized by $-u$, i.e.
$$\mathrm{conv}\{ e_B : (-u,1)^T (e_B, w_B) \text{ acheives min over all } (-u,1)^T (e_{B'}, w_{B'}) \}$$

$$\text{Bases of } M_u^w = \{ B \in \binom{[n]}{d} : w_B - \sum_{i \in B} u_i = \bigoplus_{B' \in \binom{[n]}{d}} (w_{B'} - \sum_{i \in B'} u_i) \}$$

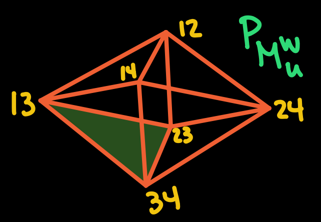
L_w = supp on poly. complex dual to loopless faces of Δ_w

Ex: $w = (w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}) = (0,0,0,0,0,1)$
$u = (0,0,3,1)$ Circuits of M_u^w? $\mathcal{C} = \{12, 14, 24\}$



$L_w = \{ u \in \mathbb{R}^4 : \min\{u_1, u_2, u_3\}, \min\{u_1, u_2, u_4\},$
$\min\{1+u_1, u_3, u_4\}, \min\{1+u_2, u_3, u_4\}$ attained $\geq$ twice $\}$

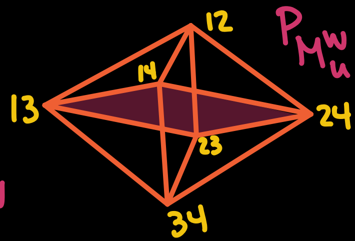
Bases? $\min\{0-u_1-u_2, \underline{0-u_1-u_3}, 0-u_1-u_4, \underline{0-u_2-u_3}, 0-u_2-u_4, \underline{1-u_3-u_4}\}$



$B = \{13, 23, 34\}$

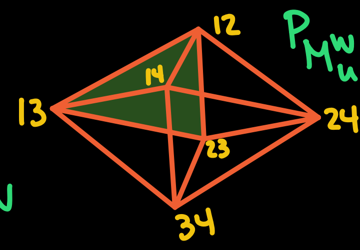
$u = (0, 0, \frac{1}{2}, \frac{1}{2})$

loopless $\rightarrow u \in L_W$



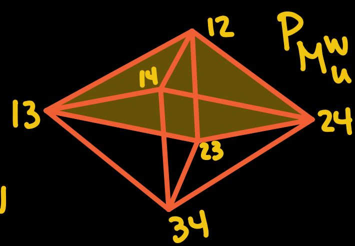
$u = (0, 0, 0, -1)$

$4 = $ loop $\rightarrow u \notin L_W$



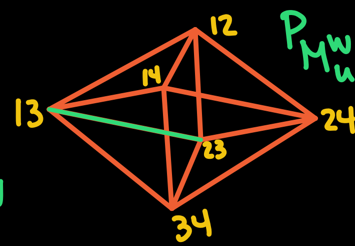
$u = (0, 0, 0, 0)$

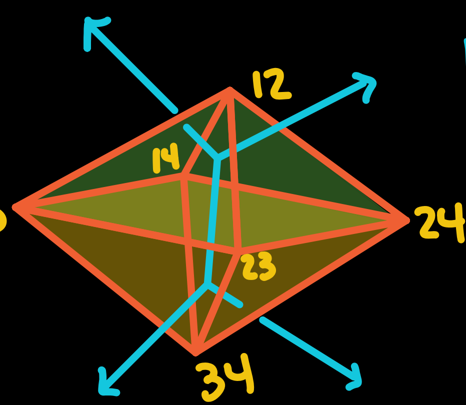
loopless $\rightarrow u \in L_W$



$u = (0, 0, -1, -1)$

$3, 4 = $ loop $\rightarrow u \notin L_W$





L_W

Δ_W