Tropical Geometry: the tropical Grassmannian

The (classical) Grassmannian

The set of d-dim'l linear subspaces of K^n can be realized as a projective variety in $\mathbb{P}^{\binom{n}{d}-1}(K)$: the Grassmannian $Gr(d,n)$.

A subspace $L = \text{rowspan}(A)$ with $A \in K^{d \times n}$ corresponds to its vector of Plücker coordinates $[p_I(L)]_{I \in \binom{[n]}{d}}$ with $p_I(L) = \det(A_I)$.

Note: up to scaling, the Plücker coordinates are independent of the choice of A: $\det((UA)_I) = \det(U)\det(A_I)$ for $U \in K^{d \times d}$ invertible.

Ex $(d=2, n=4)$ $L = rs\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$

$[p_I(L)] = [p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}] = [1 : c : d : -a : -b : ad-bc]$

$Gr(2,4) = V(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) \subseteq \mathbb{P}^{\binom{4}{2}-1}(K)$

$Gr(d,n)$ is a smooth proj. variety of dim $d(n-d)$.

In the affine chart $p_{12\cdots d} \neq 0$ it is parametrized by the rowspan of matrices $A = \begin{pmatrix} 1 & & 0 & * & \cdots & * \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1 & * & \cdots & * \end{pmatrix}$.

It is defined by the prime ideal

quadratic Plücker relations

$$I_{d,n} = \left\langle \sum_{j \in J} \text{sgn}(j, I, J)\, p_{I \cup j}\, p_{J \setminus j} : I \in \binom{[n]}{d-1}, J \in \binom{[n]}{d+1} \right\rangle$$

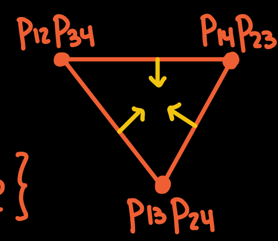
where $\text{sgn}(j, I, J) = (-1)^\ell$ where $\ell = \#\{j' \in J : j < j'\} + \#\{i \in I : i > j\}$.

The <u>tropical Grassmannian</u> $\operatorname{trop}(Gr(d,n))$ is the tropicalization $V(\operatorname{trop}(I_{d,n}))$. This is homogeneous w.r.t. every vector in $L = \operatorname{span}\{\sum_{I \ni i} e_I : i=1,\dots,n\} \subseteq \mathbb{R}^{\binom{[n]}{d}}$

$\uparrow$ dim n

$rs(A_1 \cdots A_n) \to rs(\lambda_1 A_1 \cdots \lambda_n A_n)$

Ex: $\operatorname{trop}(Gr(2,4))$

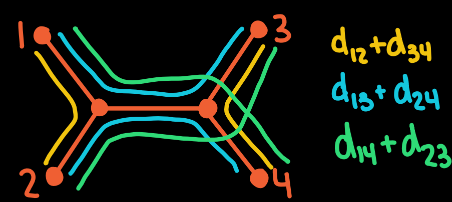


$= \{w \in \mathbb{R}^6 : \min\{w_{12}+w_{34}, w_{13}+w_{24}, w_{14}+w_{23}\}$ attained $\geq$ twice$\}$

$= \underbrace{\left(L+\mathbb{R}_{\geq 0}(e_{12}+e_{34})\right)}_{} \cup \left(L+\mathbb{R}_{\geq 0}(e_{13}+e_{24})\right) \cup \left(L+\mathbb{R}_{\geq 0}(e_{14}+e_{23})\right)$

$w_{12}+w_{34} \geq w_{13}+w_{24} = w_{14}+w_{23}$

For $d_{ij} = -w_{ij}$, (d_{ij}) is a tree metric.



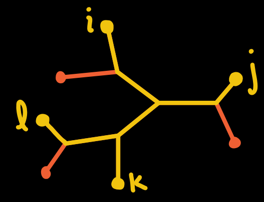
A <u>tree metric</u> is a vector $d = (d_{ij})_{ij \in \binom{[n]}{2}} \in \mathbb{R}^{\binom{n}{2}}$ coming from a tree with n leaves and edge weights $l_e \in \mathbb{R}_{\geq 0}$ in which $d_{ij} = \sum_{e \in P_{ij}} l_e$ where P_{ij} is the unique path $i \to j$.

Lemma (4 pt condition) A metric $d = (d_{ij}) \in \mathbb{R}^n_{\geq 0}$ on $[n]$ is a tree metric iff for every distinct $i,j,k,l \in [n]$

$$\max\{d_{ij}+d_{kl}, d_{ik}+d_{jl}, d_{il}+d_{jk}\}$$

is attained $\geq$ twice.

(Proof ($\Rightarrow$)) the restriction of d to $\{i,j,k,\ell\}$ is a tree metric on $\{i,j,k,\ell\}$.



($\Leftarrow$) How to reconstruct the tree?

Thm 4.3.5: $\operatorname{trop}(Gr(2,n)) = -\Delta + L$

where Δ is the space of tree metrics.

(Proof) ($\subseteq$) Let $w \in \operatorname{trop}(Gr(2,n))$. Note $\mathbb{1} = \sum_{i<j} e_{ij} \in L$.

For $\lambda \gg 0$, $d = -w + \lambda\mathbb{1}$ is a metric ($-w_{ij}+\lambda \geq 0$, $-w_{ij}+\lambda \leq -w_{ik}-w_{kj}+2\lambda$).

For $i<j<k<\ell$, $p_{ij}p_{k\ell} - p_{ik}p_{j\ell} + p_{i\ell}p_{jk} \in I_{2,n}$.

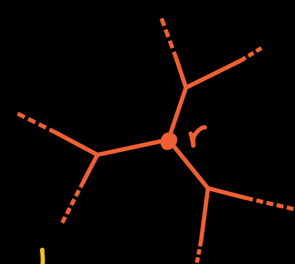
$\Rightarrow \min\{w_{ij}+w_{k\ell},\ w_{ik}+w_{j\ell},\ w_{i\ell}+w_{jk}\}$ attained $\geq$ twice

$\Rightarrow \max\{d_{ij}+d_{k\ell},\ d_{ik}+d_{j\ell},\ d_{i\ell}+d_{jk}\}$ attained $\geq$ twice $\Rightarrow d \in \Delta$.

($\supseteq$) Suppose $d \in \Delta \cap (\Gamma_{val})^n$.

Lemma: Every tree metric d can be written as $d = u + \ell$ where u is an $\underbrace{\text{ultrametric}}$ and $\ell \in L$.

$\max\{u_{ij}, u_{ik}, u_{jk}\}$ attained $\geq$ twice



$\Leftrightarrow$ u is a tree metric in which every leaf is equidistant from some root

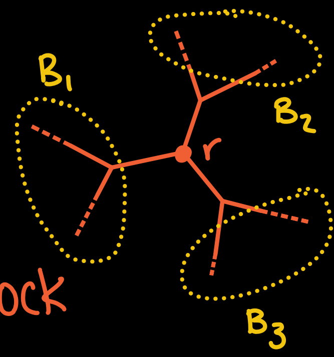
$\Leftrightarrow$ $u \in \operatorname{trop}(M_{K_n})$ where K_n is the graphic matroid of K_n.

$\Rightarrow$ We can assume d is an ultrametric. Let $R = \max\{d_{ij}\}$.

Want to show $\exists\, a_1, \dots, a_n \in K$ s.t. $d_{ij} = -\operatorname{val}(a_i - a_j)\ \forall ij$.

$\Rightarrow d = -\operatorname{val}$(plücker coord of $\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$).

There is a partition of $[n] = B_1 \uplus \dots \uplus B_s$ so that $d_{ij} = R$ when i,j belong to different blocks and $< R$ otherwise.

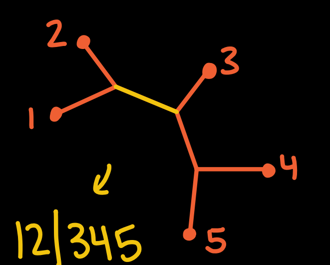


By induction, $\exists\ \mu_1, \dots, \mu_n \in K$ s.t. for i,j in the same block $d_{ij} = -\mathrm{val}(\mu_i - \mu_j) < R$. Choose arbitrary $\lambda_1, \dots, \lambda_s \in K$ s.t. $\mathrm{val}(\lambda_i) = \mathrm{val}(\lambda_i - \lambda_j) = -R\ \forall i$. For $i \in B_k$, take $a_i := \lambda_k + \mu_i$.

Then for $i,j \in B_k$, $\mathrm{val}(a_i - a_j) = \mathrm{val}(\mu_i - \mu_j) = -d_{ij}$.

For $i \in B_k$, $j \in B_\ell$, $\mathrm{val}(a_i - a_j) = \mathrm{val}(\underbrace{\lambda_k - \lambda_\ell}_{\mathrm{val} = -R} + \underbrace{\mu_i - \mu_j}_{\mathrm{val} > -R}) = -R = -d_{ij}$.

What does $\mathrm{trop}(Gr(2,n))$ look like?



Use "splits". Removing an interior edge of a tree splits leaves $I \mid I^c$.

For a tree T, the corresponding cone is

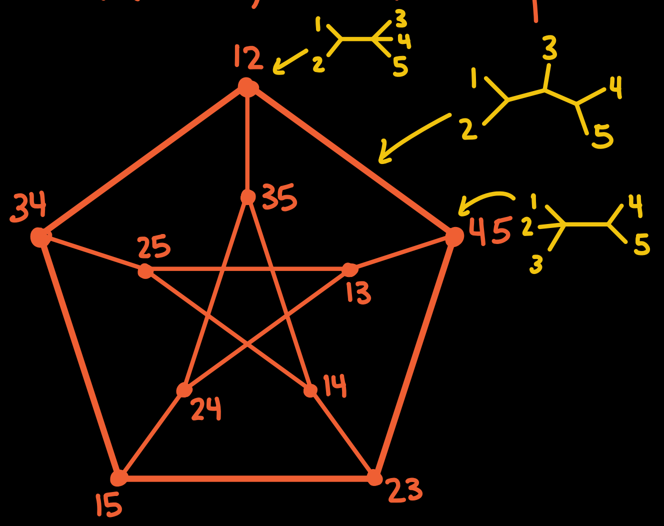
$$C_T = \mathrm{pos}\Big\{ -\sum_{i \in I, j \in I^c} e_{ij} : I \mid I^c \text{ is a split of } T \Big\} + L.$$

Maximal cones $\longleftrightarrow$ trivalent trees ($n-3$ internal edges)

$$C_T \cong \mathbb{R}_{\geq 0}^{n-3} \times \mathbb{R}^n$$

$\# = (2n-5)!! = (2n-5)(2n-7)\cdots 3 \cdot 1.$

Ex ($d=2$, $n=5$) trop($Gr(2,5)$) 7-dim'l cone in $\mathbb{R}^{10}$



$\rightarrow$ 2-dim'l fan in $\mathbb{R}^{10}/L \cong \mathbb{R}^5$

intersecting with sphere $S^4 \subseteq \mathbb{R}^5$

gives a 1-dim'l spherical complex.

$\cong$ the Petersen graph!