Tropical Geometry: tropical & tropicalized linear spaces.

A <u>tropicalized linear space</u> over K is $\mathrm{trop}(L) = V(\mathrm{trop}(I))$ where $I = I(L) \subseteq K[x_1, \dots, x_n]$ is the ideal gen. by linear forms vanishing on the subspace $L \subseteq K^n$.

Ex: $L = \mathrm{rowspan}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$ $I(L) = \langle x_1 - 2x_2 + x_3,\ x_2 - 2x_3 + x_4 \rangle$

A nonempty subset $C \subseteq [n]$ is a <u>circuit</u> of L if there if there is a linear form $\ell = \sum_{i=1}^{n} c_i x_i \in I(L)$ of minimal support with support $\mathrm{supp}(\ell) = \{i : c_i \neq 0\} = C$.

Up to scaling, ℓ is uniquely determined by $C \rightsquigarrow$ denote by ℓ_C

If $L = \mathrm{rowspan}(A)$ then the circuits of L index the minimally dependent sets of columns of $A = [A_1 \dots A_n]$.

Ex: $L = \mathrm{rowspan}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \underline{3} & \underline{2} & 1 & \underline{0} \end{bmatrix} \rightsquigarrow$ Circuits $= \{123, 124, 134, 234\}$

$\ell_{124} = 2x_1 - 3x_2 + x_4 \in I(L)$

Ex: $L = \mathrm{rowspan}\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \rightsquigarrow$ Circuits $= \{124, 135, 2345\}$

Note: If $L \subseteq \{x : x_i = 0\}$, then $C = \{i\}$ is a circuit with $\ell_C = x_i$ and $\mathrm{trop}(L) = \emptyset$.

Prop 4.1.6 If $I = I(L)$ is generated by linear forms over a subfield with trivial valuation, then $\{\ell_C : C \text{ circuit of } L\}$ form a tropical basis for $I(L)$. That is

$$\operatorname{trop}(L) = \bigcap_{C \text{ circuit}} V(\operatorname{trop}(\ell_C)) = \{w \in \mathbb{R}^n : \forall \text{ circuit } C,\ \min\{w_i : i \in C\} \text{ is attained} \geq \text{twice}\}$$

(Proof idea) For $I(L)$, computing a Gröbner basis is equivalent to computing reduced row Echelon form.

$\Rightarrow \{\ell_C : C \text{ circuit}\}$ is a universal Gröbner basis.

Suppose $w \notin \operatorname{trop}(L)$. Then $x^\alpha \in \operatorname{in}_w(I)$ for some $\alpha \in \mathbb{Z}^n_{\geq 0}$.

Since $\operatorname{in}_w I = \langle \operatorname{in}_w(\ell_C) : C \text{ circuit} \rangle$ is linear, it is also prime.

$\Rightarrow x_i \in \operatorname{in}_w I$ for some $i \Rightarrow x_i = \operatorname{in}_w \ell$ for some $\ell \in I(L)$.

For some circuit C, $C \subseteq \operatorname{supp}(\ell)$ and $x_i = \operatorname{in}_w(\ell_C) \Rightarrow w \notin V(\operatorname{trop}(\ell_C))$. $\square$

Ex: Circuits $= \{123, 124, 134, 234\} \rightsquigarrow \operatorname{trop}(L) = \bigcup_{i=1}^{4} (\mathbb{R}_{\geq 0} e_i + \mathbb{R}(1,1,1,1))$

Matroids

A matroid is a pair $M = ([n], \mathcal{C})$ where $\mathcal{C}$ is a collection of nonempty subsets of $[n]$ satisfying

(1) $C_1, C_2 \in \mathcal{C},\ C_1 \subseteq C_2 \Rightarrow C_1 = C_2$ (no proper subset of a circuit is a circuit)

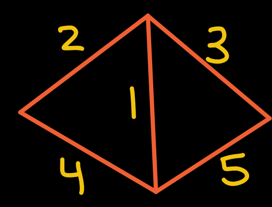
(2) If $C_1 \neq C_2 \in \mathcal{C}$ and $i \in C_1 \cap C_2$, then $\exists\, C_3 \in \mathcal{C}$ with $C_3 \subseteq (C_1 \cup C_2) \setminus \{i\}$

$\mathcal{C}$ = "circuits of M"

Ex: $\mathcal{C} = \{\text{circuits of } L\}$ for some linear space $L \subseteq K^n$

$\Rightarrow$ call M realizable over K

Ex: $\mathcal{C} = \{\text{cycles in a graph with edges } [n]\}$



$\mathcal{C} = \{124, 135, 2345\}$

Independent set of M: $I \subseteq [n]$ s.t. I contains no circuit

Basis of M: maximal independent set

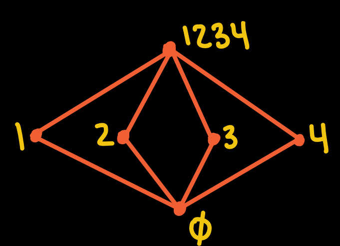
Rank function of M: for $S \subseteq [n]$, $\text{rank}_M(S) = \max\{|I| : I \subseteq S \text{ indep}\}$

Flat of M: $F \subseteq [n]$ s.t. $|C \setminus F| \neq 1$ for any $C \in \mathcal{C}$.

form a poset under inclusion called the lattice of flats

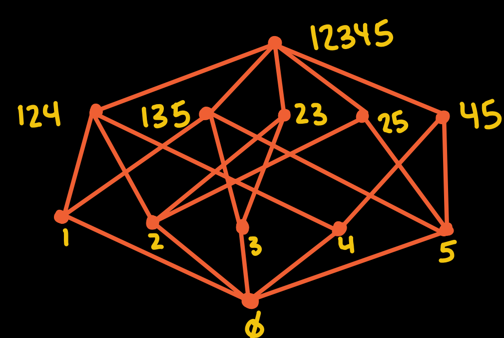
Ex: $\mathcal{C} = \{\text{circuits of } L = \text{rowspan}[A_1 \cdots A_n]\}$

Any flat F has the form $F = \{i : A_i \in L'\}$ for some subspace $L' \subseteq K^d$.



Ex: $L = \text{rowspan}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$

Ex: $L = \text{rowspan}\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$



The <u>tropical linear space</u> of $M=([n],\mathcal{C})$ is

$$\operatorname{trop}(M)=\{w\in\mathbb{R}^n : \forall\, C\in\mathcal{C},\ \min\{w_i : i\in C\} \text{ is attained} \geq \text{twice}\}.$$

If M is realized by $L\subseteq K^n$, this is $\operatorname{trop}(L)$. Also called the "Bergman fan" of M.

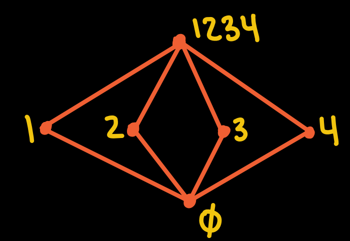
If M has any circuit C with $|C|=1$, then $\operatorname{trop}(M)$ is empty! We assume it doesn't.

Some notation: for $S\subseteq[n]$, $e_S := \sum_{i\in S} e_i$.

For $v_1,\dots,v_r\in\mathbb{R}^n$, $\operatorname{pos}\{v_1,\dots,v_r\} := \left\{\sum_{i=1}^{r}\lambda_i v_i : \lambda_1,\dots,\lambda_r\geq 0\right\}$

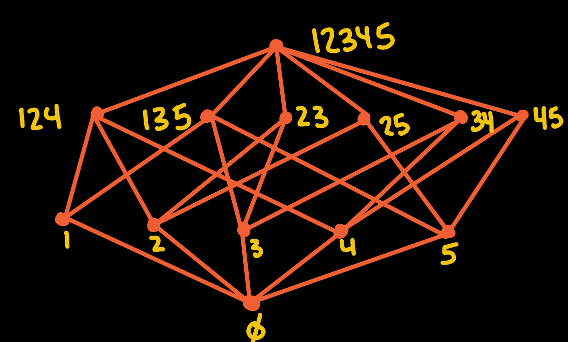
<u>Thm 4.2.6</u> The collection of cones $\operatorname{pos}\{e_{F_1},\dots,e_{F_r}\}+\mathbb{R}\mathbb{1}$ as $\emptyset\subset F_1\subset\dots\subset F_r\subset[n]$ runs over all chains of flats of M forms a pure (simplicial) polyhedral fan of dim = rank(M) whose support is $\operatorname{trop}(M)$.

Ex: $L=\operatorname{rowspan}\begin{bmatrix}1&1&1&1\\3&2&1&0\end{bmatrix}$



$$\operatorname{trop}(L)=\bigcup_{i=1}^{4}\left(\operatorname{pos}(e_i)+\mathbb{R}\mathbb{1}\right)$$

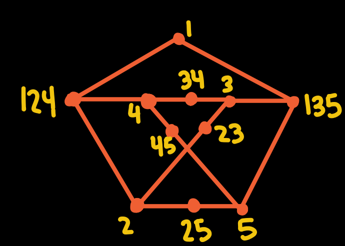
Ex: $L=\operatorname{rowspan}\begin{bmatrix}1&0&0&1&1\\0&1&0&-1&0\\0&0&1&0&-1\end{bmatrix}$



$\operatorname{trop}(L)$ = 3-dim'l fan in $\mathbb{R}^6$

$\operatorname{trop}(L)/\mathbb{R}\mathbb{1}$ = 2-dim'l fan in $\mathbb{R}^5$

intersect with unit sphere S^4 to get 1-dim'l spherical complex



(Proof of $\mathrm{pos}\{e_{F_1},\ldots,e_{F_r}\} + \mathbb{R}\mathbb{1} \subseteq \mathrm{trop}(M)$)

Note: $\mathrm{trop}(M)$ invariant under $+\mathbb{R}\mathbb{1}$. Take a chain of flats $\emptyset \subset F_1 \subset F_2 \subset \ldots \subset F_r \subset [n]$ and $w = \sum_{j=1}^{r} \lambda_j e_{F_j}$ with $\lambda_1, \ldots, \lambda_r \geq 0$.

Let $F_{r+1} := [n]$ and take $C \in \mathcal{C}$. WTS: $\min\{w_i : i \in C\}$ attained $\geq$ twice.

Note: $w_i = \lambda_k + \ldots + \lambda_r$ where where $k = \min\{j : i \in F_j\}$.

Since $\lambda_j \geq 0$, $\min\{w_i : i \in C\} = \lambda_k + \ldots + \lambda_r$ acheived by

by last $i \in C$ to appear in $F_1 \subset \ldots \subset F_r \subset [n]$ (appearing at F_k).

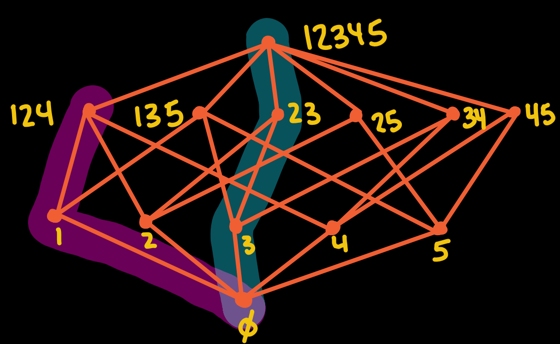
Let $k \in [n]$ be max element with $(F_k \cap C) \setminus F_{k-1}$ nonempty.

Idea: restrict $F_1 \subset F_2 \subset \ldots \subset F_r \subset [n]$, look for last increase in size.

Claim: $|(F_k \cap C) \setminus F_{k-1}| \geq 2$ (If not, $C \subseteq F_k$ and $|C \setminus F_{k-1}| = 1$ $\Rightarrow\Leftarrow$)

Then $\min\{w_i : i \in \mathcal{C}\}$ is attained at $\geq$ two elt of $(F_k \cap C) \setminus F_{k-1}$.

Ex:



$3 \subset 23 \subset \underline{1}23\underline{4}\underline{5}$

$C = 124$ $\quad$ $C = 2345$

$1 \subset 1\underline{2}\underline{4} \subset 12\underline{3}4\underline{5}$