Tropical Geometry : Finishing up orbits, multiplicities, intersections

Orbits/homogenity

trivial valuation!

A polynomial $f=\sum_\alpha c_\alpha x^\alpha \in \mathbb{k}[x_1,\dots,x_n]$ is homog. w.r.t. $v\in\mathbb{R}^n$ of v-degree d if $v^T\alpha = d$ $\forall \alpha$ with $c_\alpha \neq 0$. An ideal $J\subseteq \mathbb{k}[x_1,\dots,x_n]$ is homog. w.r.t. v if $J=\langle f_1,\dots,f_r\rangle$ where each f_i is homog. w.r.t. v.

Prop: If $J\subseteq \mathbb{k}[x_1,\dots,x_n]$ and $\mathrm{in}_v J = J$ for some $v\in\mathbb{R}^n$ then J is homog. w.r.t. v. If $a\in V(J)$ then for any $t\in\mathbb{k}$, $t^v\cdot a = (t^{v_1}a_1,\dots,t^{v_n}a_n)\in V(J)$. $\quad (f(t^v\cdot a) = t^{\deg_v(f)} f(a) = 0)$

Last time: If J doesn't contain a monomial and J is homog. w.r.t. all $v\in L$ where L is a d-dim'l $\mathbb{Q}$-linear space then $\dim(V(J))\geq d$. ← applied to $J=\mathrm{in}_w I$

Take a basis $v^{(1)},\dots,v^{(d)}\in\mathbb{Z}^n$ for L and $a\in V(J)\cap(\mathbb{k}^*)^n$.
For $t_1,\dots,t_d\in\mathbb{k}^*$, $t_1^{v(1)}\cdots t_d^{v(d)}\cdot a$ belongs to $V(J)$.
Map $(\mathbb{k}^*)^d \mapsto V(J)$ finite-to-one $\Rightarrow \dim(V(J))\geq d$.

Ex: $f=x^2y-z^3$ homog w.r.t. $v\in L=\mathrm{span}\{\overset{v^{(1)}}{(1,1,1)},\overset{v^{(2)}}{(0,3,1)}\}$

$a=(1,1,1)\in V(f) \Rightarrow t_1^{v(1)}t_2^{v(2)}\cdot a = (t_1, t_1t_2^3, t_1t_2)\in V(f)$.

Cor: If $w\in V(\mathrm{trop}(I))$ and $\mathrm{in}_v(\mathrm{in}_w I) = \mathrm{in}_w I$ for all v in a d-dim'l subspace L then $\dim V(\mathrm{in}_w I)\geq d$.

Recall: The multiplicity $m(\sigma)$ of a maximal cell on $V(\mathrm{trop}(I))$ is $\sum_P \mathrm{mult}(P, \mathrm{in}_w I)$ where P ranges over the minimal primes of $\mathrm{in}_w I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

<u>Claim</u>: For $I = \langle f \rangle$, the multiplicity $m(\sigma)$ of maximal cone σ of $V(\mathrm{trop}(f))$ is the lattice length of the edge of $\mathrm{Newt}(f)$ dual to σ.

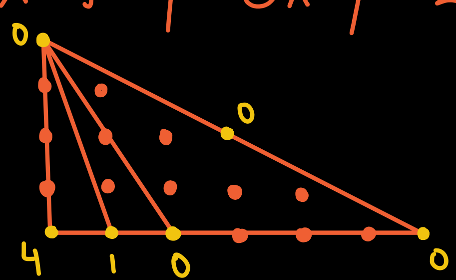
(Proof) The lattice points of the edge dual to σ are $\alpha, \alpha+\beta, \dots, \alpha+\ell\beta$ for some $\alpha, \beta \in \mathbb{Z}^n$ and $\ell \in \mathbb{Z}_{\geq 1}$ (ℓ = lattice length).

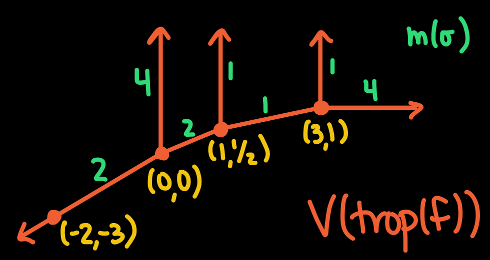
For $w \in \mathrm{relint}(\sigma)$,

$$\mathrm{in}_w f = \sum_{j=0}^{\ell} c_j x^{\alpha + j\beta} = x^{\alpha} \sum_{j=0}^{\ell} c_j (x^{\beta})^j \quad \text{for some } c_j \in \Bbbk.$$

Minimal primes of $V(\mathrm{in}_w f)$ and multiplicities correspond to roots of the univariate poly. $p(t) = \sum_{j=0}^{\ell} c_j t^j$.

Ex: $f = y^4 + 3x^3y^2 + 2x^6 + x^2 + tx + t^4 \in \mathbb{C}\{\{t\}\}[x_1, x_2]$





$\sigma = \mathbb{R}_{\geq 0} w \quad w = (-2,-3)$ $\quad \mathrm{in}_w f = y^4 + 3y^2x^3 + x^6 = x^6\left(\left(\frac{y^2}{x^3}\right)^2 + 3\left(\frac{y^2}{x^3}\right) + 2\right)$

$$= x^6\left(\frac{y^2}{x^3} + 1\right)\left(\frac{y^2}{x^3} + 2\right).$$

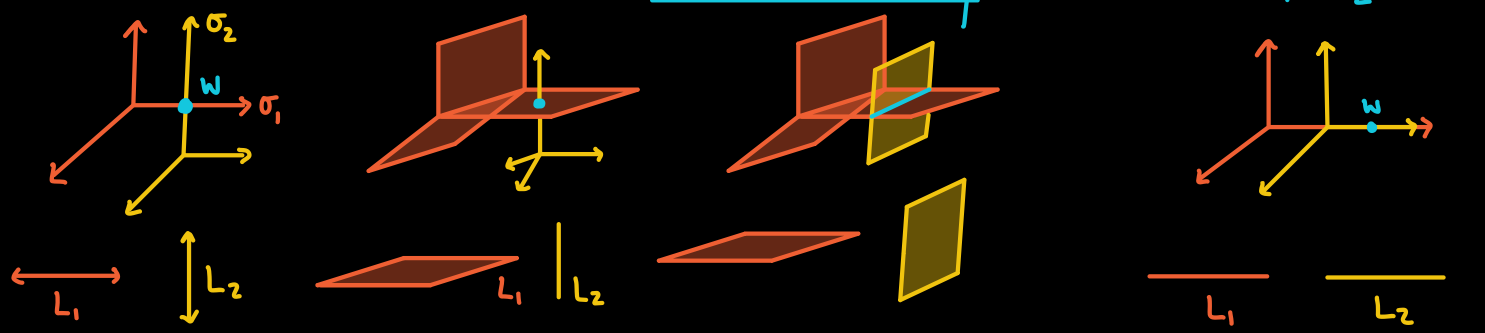
Intersections (transverse, stable, multiplicities)

Σ_1, Σ_2 pure, weighted balanced Γ_{val}-rat. poly. complexes.

$w \in \operatorname{supp}(\Sigma_1) \cap \operatorname{supp}(\Sigma_2)$, $w \in \operatorname{relint}(\sigma_i)$ for $\sigma_i \in \Sigma_i$

$\operatorname{aff\,span}(\sigma_i) = w + L_i$ for some linear subspace L_i

Σ_1 and Σ_2 intersect <u>transversely</u> at w if $L_1 + L_2 = \mathbb{R}^n$.



transverse

not transverse

The <u>tropical multiplicity</u> of the intersection at w is

$\operatorname{mult}_{\Sigma_1}(\sigma_1) \cdot \operatorname{mult}_{\Sigma_2}(\sigma_2) \cdot [\mathbb{Z}^n : N_{\sigma_1} + N_{\sigma_2}]$ where $N_{\sigma_i} = \mathbb{Z}^n \cap L_i$.

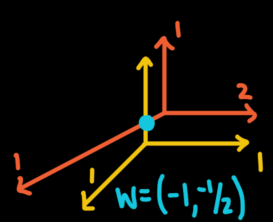
<u>Thm</u>: If $V(\operatorname{trop}(I))$ and $V(\operatorname{trop}(J))$ intersect transversely at w, then $w \in V(\operatorname{trop}(I+J))$ and $\operatorname{in}_w(I+J) = \operatorname{in}_w I + \operatorname{in}_w J$.

If $V(\operatorname{trop}(I))$ and $V(\operatorname{trop}(J))$ meet transversely at every point of their intersection, then

$$V(\operatorname{trop}(I+J)) = V(\operatorname{trop}(I)) \cap V(\operatorname{trop}(J)).$$

Ex: $I = \langle 1 + x + y^2 \rangle$

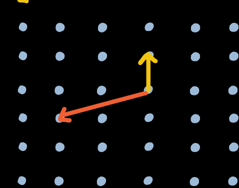
$J = \langle 1 + tx + ty \rangle$



$N_{\sigma_1} = \mathbb{Z}(-2,-1)$ $\quad N_{\sigma_2} = \mathbb{Z}(0,1)$

$[\mathbb{Z}^2 : N_{\sigma_1} + N_{\sigma_2}]$

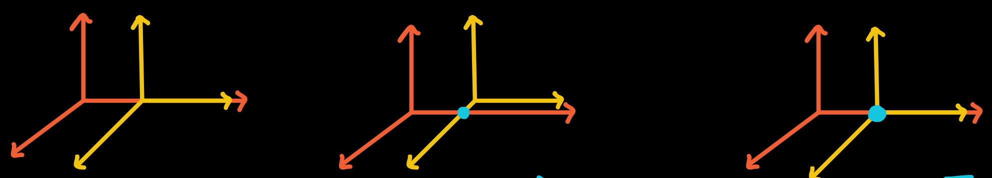
$= \left|\det\begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix}\right| = 2$



$\mathrm{in}_w(I+J) = \mathrm{in}_w I + \mathrm{in}_w J = \langle x+y^2, 1+x\rangle = \langle 1+y, 1+x\rangle \cap \langle 1-y, 1+x\rangle$

If the intersection $\Sigma_1 \cap \Sigma_2$ is not transverse, we can define the stable intersection of Σ_1 and Σ_2:

$$\Sigma_1 \cap_{st} \Sigma_2 = \lim_{\varepsilon \to 0} \left(\Sigma_1 \cap (\Sigma_2 + \varepsilon v) \right) \text{ for } v \in \mathbb{R}^n \text{ generic.}$$



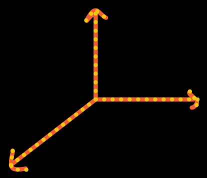
Remark: any two tropical lines in the plane stably intersect in a unique point.

<u>Thm</u> If Σ_1, Σ_2 are pure, weighted balanced Γ_{val}-rat. poly. complexes of codim d and e, resp., then $\Sigma_1 \cap_{st} \Sigma_2$ is either empty or a pure weighted balanced poly. complex of codim $d+e$.

<u>Thm 3.6.1</u> Let X_1, X_2 be subvarieties of $(K^*)^n$. There is a Zariski dense set U of points $t = (t_1, \ldots, t_n) \in (K^*)^n$ with $\mathrm{val}(t) = (0, \ldots, 0)$ for which

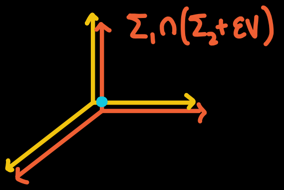
$$\mathrm{trop}(X_1 \cap t \cdot X_2) = \mathrm{trop}(X_1) \cap_{st} \mathrm{trop}(X_2).$$

Ex: $X_1 = X_2 = V(1-x-y)$ $\quad t\cdot X_2 = V(1-t_1^{-1}x - t_2^{-1}y)$ for $t_1, t_2 \in \mathbb{C}\{\{t\}\}^*$



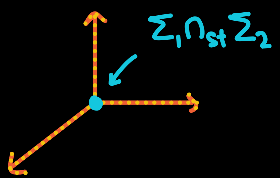
Take $t_1 = (a+t)^{-1}$, $t_2 = (b+t)^{-1}$ with $a \neq b \in \mathbb{C}\setminus\{0,1\}$.

Then $\mathrm{val}(t_1,t_2) = (0,0)$, $\det\begin{pmatrix} t_1^{-1} & t_2^{-1} \\ 1 & 1 \end{pmatrix} = \det\begin{pmatrix} a+t & b+t \\ 1 & 1 \end{pmatrix} = a-b \neq 0$



Unique pt in $X_1 \cap t\cdot X_2$: $(x,y) = \frac{1}{a-b}(1-b-t,\ a-1+t)$

$\mathrm{val}(x,y) = (0,0)$



$X_1 \cap_{st} X_2$ is a single pt of $\mathrm{val} = (0,0)$